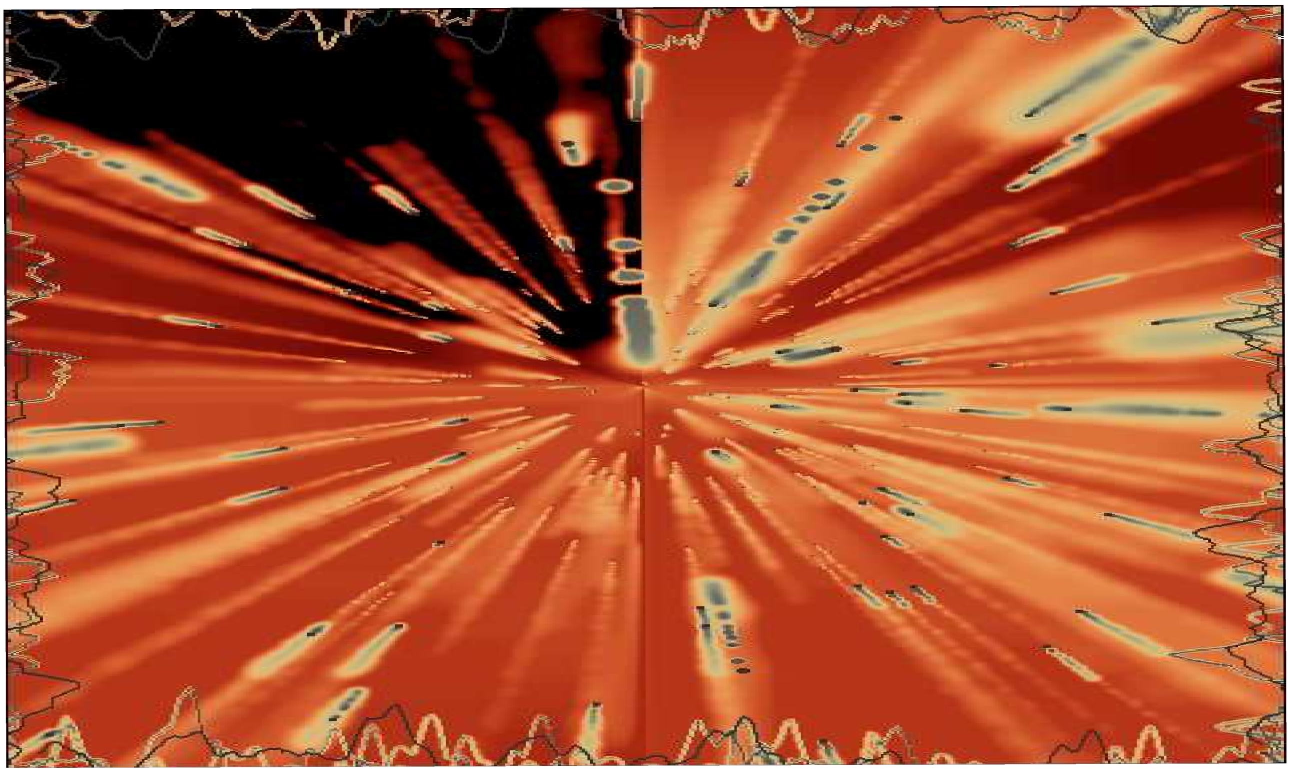
Has the Continuum Hypothesis been settled?



Matt Foreman

University of California, Irvine

The Continuum Problem

- Continuum Hypothesis (CH)

  If $A \subset P(\mathbb{N})$ then $|A| \leq |\mathbb{N}|$ or $|A| = |P(\mathbb{N})|$

- Generalized Continuum Hypothesis (GCH)

  For all infinite sets X, and all sets $A \subset P(X)$ either $|A| \leq |X|$ or $|A| = |P(X)|$.

Celebrated Problem in Set Theory

- Posed by Cantor in 1890's

- Was Problem 1 on Hilbert's famous list of problems given to the International Congress of Mathematics in 1900.

- Recent work of Woodin has raised hope that there is an imminent solution.

- "Growing Consensus that Woodin has solved the Continuum Problem". (Well known set theorest.)

Purposes of the talk

- Review the current state of the problem.

- Present various alternatives to the Woodin work which I think should be seriously considered.

The Problem

- Godel: The GCH is consistent with ZFC

  (It holds in the canonical model L.)

- Cohen: The C.H. and most instances of GCH are independent of ZFC.

  (They fail in models created by the method of *forcing*.)

- Shelah: The phenomenon is mostly a phenomenon involving regular cardinals.

  (There is deep combinatorial structure at singular cardinals that relates the behaviour at singular cardinals to the behaviour at regular cardinals.)

What could it mean to settle the problem?

We will begin by covering some familiar ground.

Mathematics = 1^{st} Order Logic + Axioms

- We understand 1^{st} order logic very well. (complete recursive proof system, L-S Theorem etc.)

- Isolates the "variable" in the foundational study as "which Axioms to take."

- Other alternatives "squeeze the balloon" to put essentially equivalent foundational issues elsewhere than the axioms. This tends to make their study more difficult, while not providing more insite into problems like the CH.

- For example, second order logic "settles" the CH, but doesn't give any tools for finding the answer.

What is an Axiomatization?

Why are Axioms adopted/accepted?

- Intuitive Appeal

  (e.g. Axiom of Choice or Union Axiom.)

- Pragmatic or Utilitarian Reasons

  (e.g. Axiom of Regularity.)

Godel and other proposed strengthening the axioms to settle independent questions.

Very Best Situation

The realization of simple, clearly stated axioms that settle all “interesting” problems and appeal directly to the intuition.

Minimal Adequate Solution

(From a realist point of view.)

Recursive collection of Axioms that cohere with known facts and appear to give an accurate description of the set theoretic universe.

Problem of Weather Prediction

The mathematical universe may not admit a simple description.

Euclid's Axioms, as a simple, poignant axiomatization may be misleading. The mathematical universe may be as complicated and have as deep a structure as the physical universe.

Indeed Shelah seems to have proposed such a universe: more or less every possible behaviour occurs.

(Stuff Happens!)

Life's Compromises

Search for some solution that is in between the "Very Best Solution" and the "Minimal adequate Solution"

Evaluation of Axiom Systems

1. Primary Considerations

   - Content of the Axioms

     What do they say?

   - Educated intuitions

2. Secondary/Tertiary Considerations

   - "Predictions," True statements φ that are first proved using strong axioms that are later "verified" in ZFC (or have analogues that are verifiable, e.g. by counting on your fingers.)

   - Effectiveness in answering interesting questions, "completeness"

Secondary/Tertiary Considerations cont.

- Coherence

- Hierarchy in Consistency Strength (reafica-tion)

<u>Case Study: "New Foundations"</u>

Almost all mathematicians working on Foundational issues find Quine's "New Foundations" sufficiently unintuitive that is not a serious contender for an axiomatization of mathematics.

Case Study:
Large Cardinals a successful axiom system

This example is important because it does seem like there is a near consensus among set theorests to adopt large cardinal axioms.

- Primary Considerations:
  - Intuitive Appeal
  - An appealing and coherent body of consequences, especially in Descriptive Set Theory. (Facts about Lebesgue Measurability, etc.)

Case Study: Large Cardinals

- Secondary Considerations:
  - Form an (almost) linear hierarchy of consistency strengths which is very useful for calibrating the consistency strength of "all interesting propositions"

    (Not Calibrated $\Rightarrow$ Not Interesting??)
  - "Complete" for $ThL(\mathbb{R})$. In the presence of large cardinals, the theory of $L(\mathbb{R})$ cannot be changed by forcing.

Main Weakness of Large Cardinals

Large Cardinals can say (almost) nothing about the questions "low down", i.e. in the vicinity of the $\aleph_n$'s that are not effective or that involve the Axiom of Choice.

In particular, large cardinals say nothing at all about the CH.

(Theorems of Levy and Solovay make this assertion rigorous.)

A typical Large Cardinal

There is an elementary embedding

$$j : V \to M$$

such that

1. j moves certain ordinals certain places
2. M is a sufficiently robust transitive subclass of the universe V.

Note that there are two parameters that determine the strength of the embedding: where ordinals go and the closure properties of M.

(Variations: $\forall f : \kappa \to \kappa \exists j \ldots$)

There are various technical obstacles to overcome with this definition: it is clearly not first order.

The definition given does, however, give a sufficiently precise heuristic to allow first order definitions that can be shown equivalent using metamathematical means.

A typical statement would be:

There is a normal, fine, κ-complete ultrafilter on $[\lambda]^{\kappa}$.

Case Study: The axiom "$V = L$"

This axiom is interesting because there is a near consensus among set theorests rejecting this axiom.

Advantages:

- The theory of L is completely forcing absolute; in fact it is absolute between models of set theory with the same ordinals, a much stronger property.

- The structure of L is well understood and we know the answers to essentially all set thoeretic questions.

- The study of L has given rise to many techniques that are important in set theory, such as $\Diamond$ and $\Box$. (methadological predictions, see below)

Advantages of L, continued

- The theory $V = L$ has made predictions later "verified" by ZFC: In L there are combinatorial "morasses" on all regular cardinals. It was later shown in ZFC that there are morasses on ω_1.

Why was "$V = L$" rejected?

The exact reasons vary, but all seem to have to do with objections to the consequences of L. Some typical examples:

- An intuition that L is somehow "too small"; that the axiom $V = L$ is intolerably limiting to the notion of "set".

- Consequences of "$V = L$", particularly in descriptive set theory are counterintuitive. For example, in L it is easy to define sets that are not Lebesgue measurable.

- There is a competing, more intuitively succesful theory that contradicts $V = L$. (Large Cardinals)

The Moral:

The considerations of completeness and absoluteness are secondary when considering axioms.

The main criterion is what the axioms **SAY**.

Woodin's Work

Woodin's work is exposited extensively in two articles in the Notices of the American Math Society. (For a correct and more complete explanation, the reader is referred to this primary source.)

Very Roughly:

New logic, termed Ω-logic. This "logic" makes sense in the presence of large cardinals. It is not a logic in the sense that it has a syntax, or that the validities are enumerable. "Proofs" are witnessed by weakly homogeneous trees.

<u>The results</u>

- There is a theory that is Ω-complete for Σ_2 statements in $H(\omega_2)$. Morever this Σ_2 theory of $H(\omega_2)$ cannot be changed by forcing.

  This is viewed as the correct analogue of the invariance of the theory of $L(\mathbb{R})$ under forcing (in the presence of large cardinals.)

- Any such theory implies that the continuum in ω_2.

Forcing Axioms:

- (Foreman/Magidor/Shelah '84)

  Martin's Maximum $\vdash 2^{\aleph_0} = \omega_2$

- (Todorcevic)

  Proper Forcing Axiom $\vdash 2^{\aleph_0} = \omega_2$

- (Todorcevic)

  Bounded Martin's Maximum $\vdash 2^{\aleph_0} = \omega_2$.

What does Woodin's canonical theory say?

The example of a canonical Σ_2 theory of $H(\omega_2)$ given by Woodin includes Bounded Martin's Maximum.

As far as we know this is the combinatorial content of what any canonical theory "says".

Moral:
An evaluation of the theory on "primary considerations" should be of BMM.

To summarize

Woodin's solution of the continuum hypothesis is a very sophisticated utilitarian argument, based more on the desirability of generic absoluteness than on what the content of the theory is.

For the rest of the talk I outline what I view as a potential alternative to Woodin's theory that is much closer in content to conventional large cardinals.

Generic Elementary Embeddings

Typical Statement:

There is an elementary embedding

$$j : V \to M$$

where:

1. j moves certain ordinals certain places
2. M is a sufficiently robust transitve subclass of a generic extension $V[G]$.

The only difference with large cardinals is that model M and j are defined an generic extension.

3 parameters control the strength

1. Where ordinals go.

2. the closure properties of M.

3. The nature of the forcing $\mathbb{P}$ that produces G.

First Order Equivalents

These typically take the form:

There is a normal fine κ-complete ideal on $Z \subset P(X)$ which is:

- Precipitous or
- Saturated or
- has a small dense subset
- etc.

(important names involved in discovering these are Ulam, Jech, Solovay, Kunen and many others)

Consider these axioms in the light of the criteria put forth.

Axioms fall into two incompatible classes

- Generalized Large Cardinals: This is a large coherent family of Axioms including such axioms as generic huge embeddings with critical point ω_1.

- An apparently isolated example: The statement "the non-stationary ideal on ω_1 is ω_2-saturated" (and close variations).

  As far as we know, the latter class is is limited to results about sets of hereditary size ω_1. It does not have analogues at accessible cardinals such as ω_2 or ω_3 or in conventional large cardinals.

Generalized Large Cardinals

The exact properties are far from being extensively worked out, but the following results are known:

- Form a directed 3-parameter family of axioms under the ordering of implication.

- Contain various hierarchies of consistency strength, e.g.

Theorem(Foreman) The statements that "ω_1 is generically n-huge when forcing with $Col(\omega, \omega_1)$" is a strict hierarchy of consistency strength.

Properties of Generalized Large Cardinals

- Generalized large cardinals intertwine with large large cardinals in consistency strength, e.g.

Theorem(Foreman) The statement: "There is a cardinal κ that is κ^+-supercompact" is equiconsistenty with the statement that "ω_1 is the critical point of a well determined generic ω_2-supercompact embedding."

Similar statements hold for your favorite large cardinal: generic huge cardinals, or ...

<u>The range of Strength</u>

At the weaker end, the Generalized Large Cardinals take the form of stationary set reflection properties; at the stronger end various Chang's Conjectures.

Primary Criterion:

Content

I contend that “Generalized Large Cardinals” are straightforward generalizations of conventional large cardinals. Moreover, that whatever the direct or indirect evidence for large cardinals is, when suitably viewed does not distinguish between conventional large cardinals and generic large cardinals.

This includes “educated intuitions”.

Note that arguments for cardinals such as inaccessible or Mahlo cardinals, that are based on the height or magnitude of the ordinals don’t seem to apply to generic large cardinals. However they do not seem to suffice for even moderately strong large cardinals either.

Secondary Criterion:

Effectiveness in answering questions

Theorem(Foreman early 1980's)
If there is a $j : V \to M \subset V[G]$ where:

a.) $crit(j) = \omega_1$

b.) $M^{\omega_1} \cap V[G] \subset M$

c.) $G \subset Col(\omega, \omega_1)$ is generic

Then the CH holds and $2^{\omega_1} = \omega_2$.

Note: Jech showed that if there is a saturated ideal on ω_1 then the CH implies $2^{\omega_1} = \omega_2$.

Woodin improved this result:

Theorem: If there is a countably complete, uniform ω_1-dense ideal on ω_2 then the CH holds and $2^{\omega_1} = \omega_2$.

A major strength of these axioms is that they generalize to to any cardinal. Thus e.g. Woodin's Theorem easily generalizes to show:

Theorem Suppose that $\kappa = \lambda^+$, λ regular and there is a λ-complete, uniform ideal on κ^+ such that $P(\kappa^+)/I$ has a dense set isomorphic to $Col(\lambda, \kappa)$. Then $2^\kappa = \kappa^+$.

or one can appeal to the following theorem to deduce the GCH from the CH and $2^{\omega_1} = \omega_2$:

Theorem (Foreman) Suppose that $2^\kappa = \kappa^+$ and there is a generic elementary embedding $j : V \to M$ such that:

a.) $crit(j) = \kappa, j(\kappa) = \lambda, j(\kappa^+) = (\lambda^+)^V$

b.) $j``\lambda^+ \in M$.

Then $2^\lambda = \lambda^+$.

The theory determined by Generalized Large Cardinals

- GCH: Yes
- Suslin Trees on Successors of regular cardinals: Yes
- Kurepa Trees: No
- $\square_\kappa$: No
- Chang's Conjectures: $(\kappa^+, \kappa) \twoheadrightarrow (\lambda, \lambda')$: Yes
- Stationary set reflection at regular cardinals (in the various possible forms): Yes
- Strong partition properties at successor cardinals such as ω_2: mostly open, but significant results in the "Yes" direction.

Descriptive Set Theory

Since the generalized large cardinals include conventional large cardinals (taking the forcing to be trivial), all of the descriptive set theoretic consequences of Large Cardinals remain.

Many consequences are implied immediately by generalized large cardinals axioms just involving ideals on ω_1 or ω_2.

<u>Metaconjecture</u>

All "standard" set theoretic problems are settled by generalized large cardinals.

Secondary Criterion:
Predictions

Godel's suggestion of predictive capacity of axioms is fraught with various kinds of dangers. In particular, the criterion is inherently sociological, rather than mathematical because it seems to necessarily involve a temporal element. I'll consider three kinds of "predictions".

Version 1

A prediction is when there are theories $\Sigma_0 \supset \Sigma_1$ a proposition φ such that $\Sigma_1 \vdash \varphi$, but a proof that $\Sigma_0 \vdash \varphi$ was found first.

Standard example given for large cardinals:

The Wadge Hierarchy for Borel Sets is Well-ordered.

Examples From Generalized Large Cardinals

Silver's Theorem about the Singular Cardinals Hypothesis:

Magidor proved that if there is a precipitous ideal on ω_1 and the GCH holds below a singular cardinal λ of cofinality ω_1 then $2^\lambda = \lambda^+$.

A short time later, Silver, originally using similar techniques, eliminated the assumption of a precipitous ideal on ω_1.

There is a technical example of a metrizability property for Moore Spaces that was shown by Tall using generic huge embeddings and later proved in ZFC by Dow.

A more doubtful example:

The proof of

- "A normal fine $\aleph_1$-dense ideal on $[\aleph_2]^{\aleph_1}$ implies the CH"

  preceded Woodin's Theorem that

- "A uniform, countably complete $\aleph_1$-dense ideal on ω_2 implies the CH."

A still more dubious example of "prediction":

- (Woodin) The existence of an $\aleph_n$-saturated normal fine ideal on $[\aleph_\omega]^{\aleph_\omega}$ implies that the CH fails.

- (Foreman) There is no $\aleph_n$-saturated normal fine ideal on $[\aleph_\omega]^{\aleph_\omega}$

Version 2: Methadolical Predictions

Techniques that arise from the use of an Axiom collection $\Sigma_0 \supset \Sigma_1$ have natural analogues that are discovered to work in Σ_1.

A clear example of this

- (Foreman) If there is a $j : V \to M \subset V[G]$ where

  a.) $crit(j) = \omega_1, j(\omega_1) = |\mathbb{R}|$

  b.) $j``|\mathbb{R}| \in M$

  c.) $G \subset Col(\omega, \omega_1)$ is generic.

  Then every set of reals in $L(\mathbb{R})$ is Lebesgue Measurable, has the Property of Baire etc.

- Large Cardinals imply that every set is LM, has POB etc.

The latter was proved by technique heavily involving generic elementary embeddings. (That there are now proofs that don't use generic elementary embeddings may also be viewed as a "prediction".)

<u>Version 3: Gradation of Consequences</u>

Stronger Axioms have stronger natural consequences.

This appears to be less like a “prediction” but has the advantage of not being temporal.

Example of gradations

- (Erdos-Rado) CH implies that

$$\omega_2 \to (\omega_1 + 1, \omega_2)$$

- (Laver, later Kanamori) CH + there is an $(\omega_2, \omega_2, \omega)$-saturated ideal on ω_1 imples that

$$\omega_2 \to (\omega_1 \times 2 + 1, \omega_2)$$

- (Foreman and Hajnal) CH + there is an $\aleph_1$-dense ideal on ω_1 implies:

$$\omega_2 \to (\omega_1^2 + 1, \omega_2)$$

Another example

- (Woodin) If there is an ω_2-saturated uniform ideal on ω_2 then $(\Theta)^{L(\mathbb{R})} < \omega_2$

- (Woodin) If there is an $\aleph_1$-dense, uniform ideal on ω_2 then the CH holds.

Here it is easy to see that the first hypothesis is not sufficient for CH.

Conclusion

There are viable alternatives to the Woodin "Solution" of the CH and these should be considered and explored before we rush to celebrate.

The End