Canad. Math. Bull. Vol. **29** (4), 1986

THE FRACTAL DIMENSION OF SETS DERIVED FROM COMPLEX BASES

BY
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ABSTRACT. For each positive integer n, the radix representation of the complex numbers in the base $-n + i$ gives rise to a tiling of the plane. Each tile consists of all the complex numbers representable in the base $-n + i$ with a fixed integer part. We show that the fractal dimension of the boundary of each tile is $2 \log \lambda_n / \log(n^2 + 1)$, where λ_n is the positive root of $\lambda^3 - (2n - 1)\lambda^2 - (n - 1)^2\lambda - (n^2 + 1)$.

1. **Introduction**. For each positive integer n, the complex numbers can be represented in the base $-n + i$ using the digits $0, 1, 2, \ldots, n^2$. Each of these representations gives rise to a tiling of the plane into a set of interlocking tiles of unit area with the boundary of each tile having infinite length. A single tile consists of all the complex numbers representable in the base $-n + i$ with a fixed integer part. For a given base, there is one tile corresponding to each Gaussian integer. In this paper we investigate the geometry of the boundaries of these regions.

The base $-1 + i$ yields Davis and Knuth's space-filling twin dragon curve. Mandelbrot [6; pp. 66–67] has calculated the fractal (Hausdorff) dimension of the dragon's "skin" to be 1.5236 to four places. We prove that the fractal dimension of the tile derived from the base $-n$ + i is $2 \log \lambda_n / \log(n^2 + 1)$ where λ_n is the positive root of $\lambda^3 - (2n - 1)\lambda^2 - (n - 1)^2\lambda - (n^2 + 1)$. This agrees with Mandelbrot's calculation for the case $n = 1$.

These results were first announced in [3]. The author would like to thank F. M. Dekking for his helpful comments.

2. **The Construction of the Tiles**. A complex number is said to be written in *base* $-n + 1$ if it is expressed in the form $\Sigma_{j=-\infty}^{q} r_j(-n + i)^j$, where the *digits* $r_j \in \{0, 1, 2, \ldots, n^2\}$. We represent this number as $(r_q r_{q-1} \ldots r_0 \cdot r_{-1}\, r_{-2} \ldots)_{-n+i}$. The digits to the left of the radix point make up the *integer part* of the expression. Kátai and Szabó [5] showed that, if n is a positive integer, every complex number could be expressed in the base $-n + i$. We mentioned in [4] that some numbers have two or even three different expansions. The boundary of the tiles that we construct are the complex numbers having two expansions with different integer parts.

Received by the editors May 13, 1985.
AMS Subject Classification (1980) Primary 11K55, Secondary 51M20

The tiling of the complex plane derived from the base $-n + i$ is constructed as follows. Successive approximations to each tile will consist of the set of points representable in the base $-n + i$ with a given integer part and using a fixed number of negative powers of that base. Each Gaussian integer is uniquely representable in the base $-n + i$ by means of an expression of the form $(r_q r_{q-1} \ldots r_0)_{-n+i}$. Divide up the plane into unit squares whose centres are the Gaussian integers. These squares are the initial approximations to the tiles. The kth approximation consists of squares of side $(n^2 + 1)^{-k/2}$ whose centres are numbers of the form $(r_q r_{q-1} \ldots r_0 \cdot r_{-l} \ldots r_{-k})_{-n+i}$. The squares whose centres have the same integer part are coloured the same colour. As k tends to infinity, the limits of these coloured regions yield the tiling of the plane derived from the base $-n + i$.

Let us fix our attention on the particular tile consisting of those numbers with zero integer part. The kth approximation consists of the union of squares with centres $(0 \cdot r_{-1} \ldots r_{-k})_{-n+i}$ and can be obtained from the $(n^2 + 1)^k$ unit squares, whose centres are the Gaussian integers $(r_{-1} \ldots r_{-k})_{-n+i}$, by dividing by $(-n + i)^k$. Figure 1 shows the beginning of the construction of the tile derived from the base $-2 + i$.

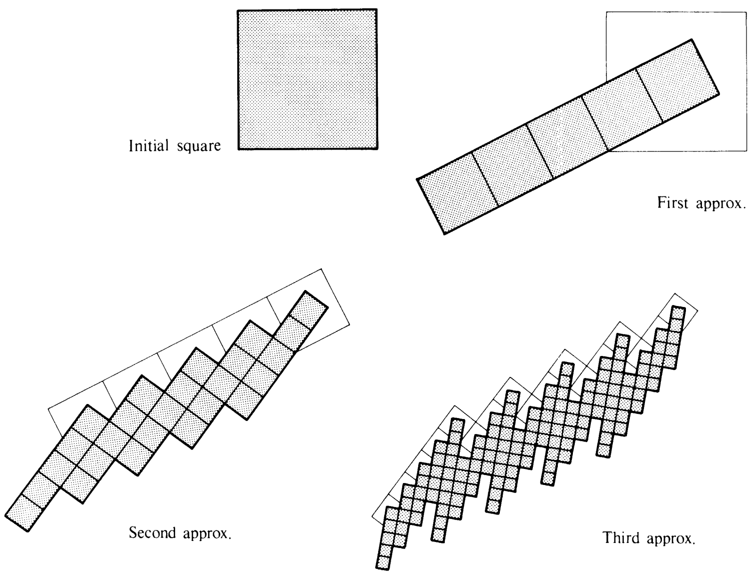

FIG. 1. The first four stages of the construction of the tile derived from the base $-2 + i$.

The kth approximation contains $(n^2 + 1)^k$ squares, each of side $(n^2 + 1)^{-k/2}$, so its total area is unity. However, the length of the boundary increases indefinitely as k increases and its rate of increase will determine its fractal dimension. Figures 6 to 9 illustrate some examples of these regions.

In each base $-n + i$, there is an identically shaped tile of unit area for each Gaussian integer. Points on the boundary of two tiles have two expressions in base $-n + i$ with different integer parts. Because the plane is two dimensional, there must be some points lying on the boundary of three areas which have three expressions in the base $-n + i$, all with different integer parts. For example, $(1 + i)/2 = (0.\overline{041})_{-2+i} = (13.\overline{104})_{-2+i} = (14.\overline{410})_{-2+i}$, where the bar over a sequence of digits indicates that they are to be repeated indefinitely.

These approximations to the tiles are natural examples of replicating superfigures as constructed in [2]. In fact [2; Figure 1] shows the first three approximations to the tiling derived from the base $-2 + i$, using the digit set $\{0, \pm 1, \pm i\}$.

3. **The Boundary of the Tiles.** We now determine a formula for the length of the boundary of the kth approximation of each tile. This kth approximation, $\mathcal{S}_k$, has a similar shape to the union $\mathcal{G}_k$, of unit squares whose centres are the Gaussian integers that require at most k digits in their base $-n + i$ representation. Hence it will be sufficient to find the length of the boundary of $\mathcal{G}_k$.

The region $\mathcal{G}_k$ can be considered as being built up of rectangles of length $n^2 + 1$ and height 1 corresponding to the $n^2 + 1$ Gaussian integers whose base $-n + i$ representations differ only in the unit place. These rectangles lie in the framework shown in Figure 2. The left end of each tile is a multiple of the base $-n + i$. The framework is constructed as shown in Figure 3. There are three basic edges, which we call A, B and C, used in this construction. The boundary of $\mathcal{G}_k$ is built up of a certain number of these three edges, while the boundary of $\mathcal{S}_k$ is built up of the same number of shrunken edges.

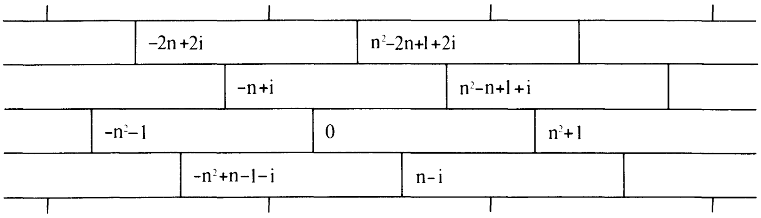

FIG. 2. The framework of the Gaussian integers in the base $-n + i$.

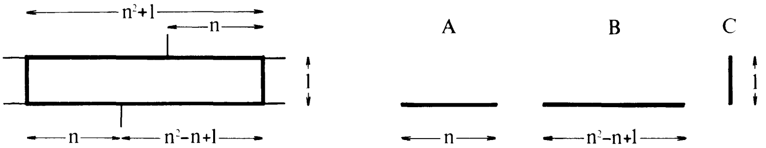

FIG. 3. Details of the construction of the framework.

Figure 4 depicts the frameworks of $\mathcal{S}_k$ and $\mathcal{S}_{k+1}$ in the passage from one approximation to the next. In this passage, each tile of $\mathcal{S}_k$ is replaced by a region of the same area in $\mathcal{S}_{k+1}$ consisting of $n^2 + 1$ tiles such as those shaded in Figure 4. This corresponds to adding an extra digit in the base $-n + i$ representation. The three edges A, B, C in Figure 4 are the common boundaries of the tile I with the tiles II, III and

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IV respectively. These are transformed into the configurations in $\mathcal{S}_{k+1}$ shown in Figure 5. In particular, each A edge is transformed into $(2n - 1)$ A edges and $2n$ C edges, each B edge is transformed into $(n^2 - 2n + 2)$ A edges and $(n - 1)^2$ C edges, while each C edge is transformed into one B edge. Hence, if a_k, b_k and c_k are the numbers of A, B and C edges, respectively, on the boundaries of $\mathcal{G}_k$ or $\mathcal{S}_k$, then

$$\begin{bmatrix} a_{k+1} \\ b_{k+1} \\ c_{k+1} \end{bmatrix} = \begin{bmatrix} 2n-1 & 0 & 2n \\ n^2 - 2n + 2 & 0 & (n-1)^2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_k \\ b_k \\ c_k \end{bmatrix}.$$

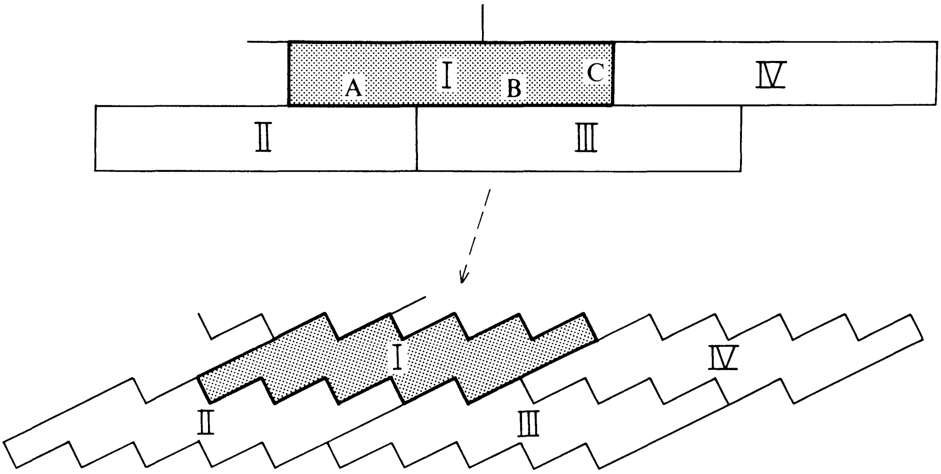

FIG. 4. Successive approximations in the shrinking of the framework.

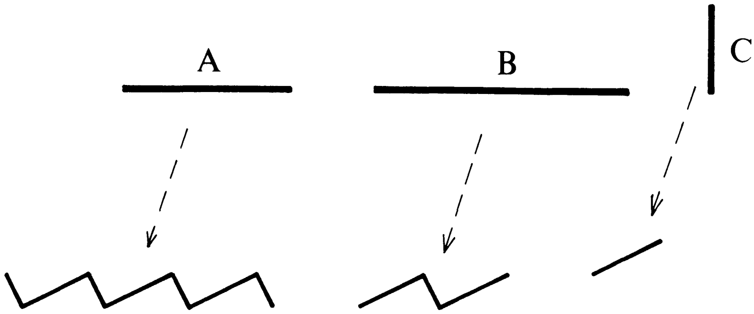

FIG. 5. How the edges change in successive approximations.

The initial conditions for this difference equation are $a_1 = b_1 = c_1 = 2$. Denote the vector $(a_k, b_k, c_k)^t$ by v_k and the transition matrix by T. Then the length of the boundary

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of $\mathcal{G}_k$ is $g_k = (n, n^2 - n + 1, 1)v_k = (n, n^2 - n + 1, 1)T^{k-1} v_1$, while the length of the boundary of $\mathcal{S}_k$ is $(n^2 + 1)^{-k/2}$ times this.

The eigenvalues of this transition matrix T are the roots of the characteristic polynomial $\lambda^3 - (2n - 1)\lambda^2 - (n - 1)^2\lambda - (n^2 + 1)$. Since T^4 is a positive matrix, it follows from the Perron-Frobenius Theorem that T has one real positive dominant eigenvalue; call it λ_n. Since $\lambda_n > (n^2 + 1)^{1/2}$, the length of the boundary of $\mathcal{S}_k$ tends to infinity as k tends to infinity.

4. **The Fractal Dimension of the Boundary**. In [6] Mandelbrot calls the Hausdorff dimension of a space its *fractal dimension*. This dimension is a metrical, not a topological concept. It may take non-integral values, but yields the usual dimension for the most ordinary spaces. A *fractal set* is one whose Hausdorff dimension is strictly greater than its topological dimension. Examples of fractal sets are the Cantor set and the boundary of Koch's snowflake.

The fractal dimension of the boundary of the tiles can be found as follows. The d dimensional measure of the boundary can be computed from the sequence of grids obtained by dividing the integer grid by $(-n + i)^k$ as in [1]. This d dimensional measure is a constant times $\lim_{k\to\infty} \lambda_n^k (n^2 + 1)^{-kd/2}$. Hence, if $\lambda_n (n^2 + 1)^{-d/2} > 1$, the measure will be infinite while, if $\lambda_n (n^2 + 1)^{-d/2} < 1$, the measure will be zero. Hence the *fractal dimension of the boundary of the tiles* derived from the base $-n + i$ is D_n where $(n^2 + 1)^{D_n/2} = \lambda_n$ and λ_n is the positive root of $\lambda^3 - (2n - 1)\lambda^2 - (n - 1)^2\lambda - (n^2 + 1)$. That is, $D_n = 2 \log \lambda_n / \log (n^2 + 1)$. Since D_n is always greater than one, these boundaries are examples of fractal curves. Figures 6 to 9 illustrate the first four cases.

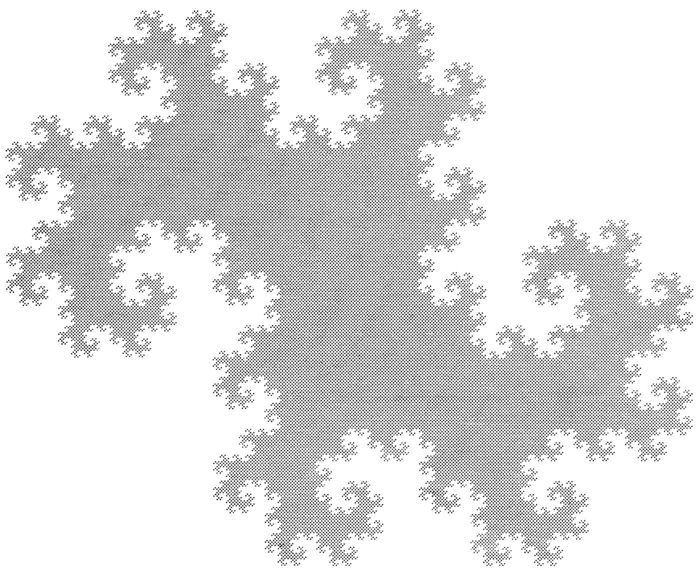

FIG. 6. The base $-1 + i$ tile whose boundary has dimension approximately 1.5236.

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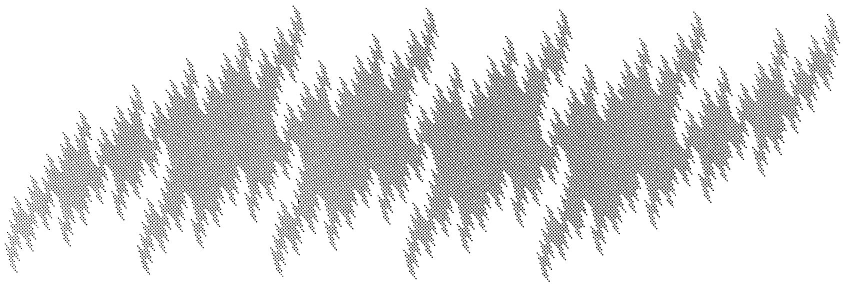

FIG. 7. The base $-2 + i$ tile whose boundary has dimension approximately 1.6087.

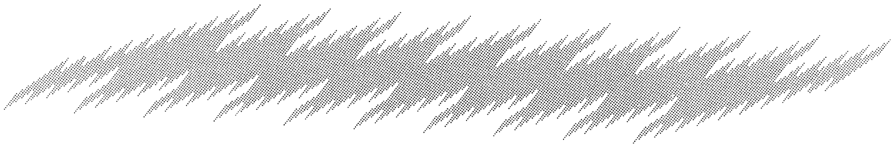

FIG. 8. The base $-3 + i$ tile whose boundary has dimension approximately 1.5496.

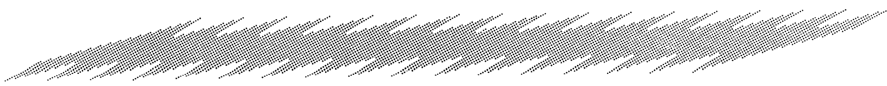

FIG. 9. The base $-4 + i$ tile whose boundary has dimension approximately 1.4961.

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