The Beckman-Quarles theorem via the triangle inequality*

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Abstract

A short, elementary and non-computational proof is given for the classical Beckman-Quarles theorem asserting that a map of a Euclidean space into itself that preserves distance 1 must be an isometry.

One of the gems of elementary Euclidean geometry is the Beckman-Quarles theorem [1]:

Theorem 1 *If $n \geq 2$ and $\tau : \mathbf{R}^n \to \mathbf{R}^n$ maps points of distance 1 into points of distance 1, then τ is an isometry.*

In other words, if a mapping of $\mathbf{R}^n$ into itself preserves distance 1, then it preserves all distances.

Note that injectivity[1] of τ is not required.

The theorem has been independently discovered later (see [3],[9]), and was the starting point of a number of similar results in various settings (see e.g. [4], [7], [10], and particularly the survey paper [8], just to name a few). Several proofs are known (see e.g. [1], [2], [5] or [6]).

In this note we give a short and elementary proof that uses no computation whatsoever, only the triangle inequality.

Let $d(\cdot, \cdot)$ denote the Euclidean distance in $\mathbf{R}^n$, Recall the triangle inequality: if $P, Q, R \in \mathbf{R}^n$, then $d(P, R) \leq d(P, Q) + d(Q, R)$, with strict inequality unless Q lies on the segment connecting P and R. Simple iteration gives that if $P_0, P_1, \ldots, P_l \in \mathbf{R}^n$, then $d(P, P_l) \leq \sum_{j=0}^{l-1} d(P_j, P_{j+1})$.

As in [5], we write P' for $\tau(P)$. Let F be the set of those $r > 0$ for which τ preserves r-distance (i.e. points of distance r are mapped into points of distance r). By assumption $1 \in F$. We shall repeatedly use the following

Observation. *If $r_j \in F$ and $d(P, Q) \leq \sum_1^l r_j$, then $d(P', Q') \leq \sum_1^l r_j$.*

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[1] In [1] actually the statement was for multi-valued mappings, but that can be easily reduced to Theorem 1.

This follows from the fact that P and Q can be joined by a sequence $P_0 = P,\ P_1, \ldots, P_{l-1},\ P_l = Q$ of points with $d(P_j, P_{j+1}) = r_{j+1}$, which implies $d(P'_j, P'_{j+1}) = r_{j+1}$, and the claim follows from the triangle inequality.

Next, we show that if $\alpha/2$ is the length of the height of a regular tetrahedron of side-length 1, then $\alpha \in F$. Indeed, let $V_0, \ldots, V_n$ be the vertices of a regular tetrahedron with side-length 1 and let V_0^* be the reflection of V_0 onto the hyperplane spanned by $V_1, \ldots, V_n$. Then the distance of V_0 and V_0^* is twice the length of the height, hence $\alpha = d(V_0, V_0^*)$. Since (the vertices of) regular tetrahedra of side-length 1 are mapped into (the vertices of) regular tetrahedra of side-length 1, it follows that the image of $\{V_0, V_1, \ldots, V_n, V_0^*\}$ is congruent to $\{V_0, V_1, \ldots, V_n, V_0^*\}$ itself,[2] therefore $d(V'_0, (V_0^*)') = \alpha$. However, that implies $\alpha \in F$ by building the above configuration for any P, Q with $d(P,Q) = \alpha$ so that $V_0 = P$ and $V_0^* = Q$.

The same argument gives that if $r \in F$, then $\alpha r \in F$. Therefore, the numbers $1, \alpha, \alpha^2, \alpha^3, \ldots$ are all in F. About α the only information we need is that $1 < \alpha < 2$. Indeed, $\alpha < 2$ follows by applying the triangle inequality in the triangle $V_0V_1V_0^*$, and we must have $\alpha > 1$, otherwise the distance $d(V_0, M)$ from V_0 to the center of mass M of $\{V_0, \ldots, V_n\}$ (which lies on the segment $V_0V_0^*$) would be smaller than $1/2$, which contradicts the triangle inequality in the triangle V_0V_1M (note that $d(V_1, M) = d(V_0, M)$ by symmetry).

The theorem claims that $d(P', Q') = d(P, Q)$ for all points $P, Q \in \mathbf{R}^n$. First we prove $d(P', Q') \geq d(P, Q)$ for all such P, Q. Suppose to the contrary that for some P, Q and $\delta \leq 1/2$ we have $d(P,Q) =: \Delta$ but $d(P', Q') \leq \Delta - \delta$. We claim that there are natural numbers s_0, r_0 such that $\{r_0\alpha^{s_0}\} \in (\delta/2, \delta)$, where $\{\cdot\}$ denotes fractional part. If α is irrational,[3] then this follows with $s_0 = 1$ and some r_0 since then the numbers $\{r\alpha\}$, $r = 1, 2, \ldots$, are dense in $[0, 1]$. On the other hand, if $\alpha = p/q$ with relative prime p, q, then choose s_0 so that $1/q^{s_0} < \delta/2$, then r_0^* so that $\{r_0^*(p^{s_0}/q^{s_0})\} = 1/q^{s_0}$,[4] and finally an r_0^{**} so that $r_0^{**}(1/q^{s_0}) \in (\delta/2, \delta)$. Clearly, $r_0 = r_0^* r_0^{**}$ and s_0 are appropriate. Since, by the choice of r_0, any interval of length δ contains modulo 1 one of the points $jr_0\alpha^{s_0}$, $1 \leq j \leq 3/\delta$, for any $x \in \mathbf{R}$ there is an $1 \leq i \leq 3r_0/\delta$ and an integer m such

[2]Since both V'_0 and $(V_0^*)'$ are vertices of regular tetrahedra with common face $\{V'_1, \ldots, V'_n\}$, and since we do not assume τ to be injective, theoretically there are two possibilities for the distance $d(V'_0, (V_0^*)')$: either $d(V'_0, (V_0^*)') = 0$ (when $V'_0 = (V_0^*)'$) or $d(V'_0, (V_0^*)') = \alpha$ (when $V'_0 \neq (V_0^*)'$, i.e. when $(V_0^*)'$ is the reflection of V'_0 onto the hyperplane spanned by $\{V'_1, \ldots, V'_n\}$). But the first one is impossible, for otherwise if $(\tilde{V}_0, \ldots, \tilde{V}_n, \tilde{V}_0^*)$ is obtained by a rotation of $(V_0, \ldots, V_n, V_0^*)$ about V_0 so that $d(V_0^*, \tilde{V}_0^*) = 1$, then the image $(\tilde{V}_0^*)'$ cannot be of unit distance from $(V_0^*)' = V'_0$—as is required by the assumption of the theorem—, since, as we have just observed, it is of distance either 0 or α from $(\tilde{V}_0)' = V'_0$, and here $\alpha > 1$ (see below). This reasoning was taken from [1].

[3]We do not need the exact value of α nor the information if it is rational or irrational. But for completeness let us state that $\alpha = \sqrt{2(n+1)/n}$, and it can be rational or irrational depending on n: for $n = 2$ it is irrational, while for $n = 8$ it is rational.

[4]That is possible since there are integers $r_0^* > 0$, t_0^* for which $r_0^* p^{s_0} + t_0^* q^{s_0} = 1$ because p^{s_0} and q^{s_0} are relative primes.

that

$$x \leq i\alpha^{s_0} + \Delta + m < x + \delta,$$

and if here $x > (3r_0/\delta)\alpha^{s_0} + \Delta + 1$, then the m is positive. We apply this with $x = \alpha^k$ with a large integer k for which the previous inequality holds. Then

$$\alpha^k \leq i\alpha^{s_0} + \Delta + m < \alpha^k + \delta,$$

and m is a positive integer. On the half-line $\overrightarrow{PQ}$ let R be the point for which $d(P,R) = \alpha^k$. Then $d(Q,R) = \alpha^k - \Delta \leq i\alpha^{s_0} + m$, so by our Observation $d(Q',R') \leq i\alpha^{s_0} + m$. But this, $d(P',Q') \leq \Delta - \delta$ and $d(P',R') = \alpha^k$ contradicts the triangle inequality because $(i\alpha^{s_0} + m) + (\Delta - \delta) < \alpha^k$, implying $d(Q',R') + d(P',Q') < d(P',R')$. This contradiction proves that, indeed, $d(P',Q') \geq d(P,Q)$.

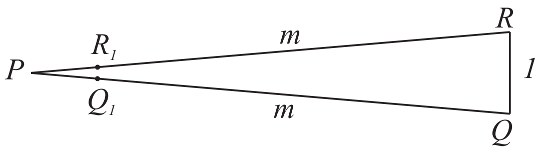

Figure 1: The distance of Q_1, R_1 is $1/m$

After these we can easily complete the proof of the theorem. Indeed, if $d(P,Q) = m$ is an integer, then $d(P',Q') \geq d(P,Q) \geq m$. On the other hand, by our Observation we have $d(P',Q') \leq m$, so actually $d(P',Q') = m$, which means that all natural numbers belong to F. This immediately implies that there are arbitrarily small numbers in F: consider a triangle PQR of side-lengths $d(P,Q) = d(P,R) = m$, $d(Q,R) = 1$, with some large natural number m, and let Q_1, R_1 be the points on the sides PQ, PR that lie of distance 1 from P (hence $d(Q_1,R_1) = 1/m$, see Figure 1). Then

$$d(P',Q_1') + d(Q_1',Q') = 1 + (m-1) = d(P',Q'),$$

so (again by triangle inequality) Q_1' lies on the segment connecting P' and Q'. Similarly, R_1' lies on the segment connecting P' and R'. But that means that

$$d(Q_1',R_1') = (1/m)d(Q',R') = 1/m = d(Q_1,R_1),$$

so all $1/m$-distances are preserved (i.e. $1/m \in F$ for all natural number m).

Finally, we verify $d(P',Q') \leq d(P,Q)$ for all P,Q, which, with the inequality $d(P,Q) \leq d(P',Q')$ proven before, completes the proof of the theorem. Let $\varepsilon \in F$ be small, and let l be the smallest number for which $d(P,Q) \leq l\varepsilon$. By our Observation then $d(P',Q') \leq l\varepsilon < d(P,Q) + \varepsilon$, and upon letting $\varepsilon \to 0$ we obtain $d(P',Q') \leq d(P,Q)$.

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