The Hasse-Weil Zeta Function of a Quotient Variety

1. Introduction

Let $V/\mathbb{Q}$ be a smooth projective variety,

$G \le \mathrm{Aut}_{\mathbb{Q}}(X)$ be a finite group of automorphisms,

$W = G\backslash V$ the quotient variety.

Note: W is usually a singular variety (if $\dim V > 1$).

Questions: 1) How is the Hasse-Weil zeta-function $\zeta_W(s)$ of W related to $\zeta_V(s)$?

2) How can we determine $\zeta_W(s)$ (if $\zeta_V(s)$ is "known")?

3) What properties does $\zeta_W(s)$ have? Meromorphic continuation? Tate Conjecture?

Motivating Example: Let $V = X_N \times X_N$ product surface and $G = \Delta_{G_N} \le G_N \times G_N$ diagonal subgroup, where:

X_N is the modular curve classifying level N structures,

$G_N = \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\} \le \mathrm{Aut}_{\mathbb{Q}}(X_N)$,

$\Delta_{G_N} = \{(g,g) : g \in \Gamma_N\}$ the diagonal subgroup of G_N.

Thus: $W = Z_N := \Delta_{G_N}\backslash V$ is the modular diagonal quotient surface of level N.

Remark: For $N = p$ (prime), $\zeta_{Z_p}(s)$ was studied by S. Mohit in his thesis (Queen's, 2001).

2. The Hasse-Weil Zeta Function

Let $X/\mathbb{Q}$ be a projective variety of dimension d, and $\mathcal{X}/\mathbb{Z}$ a projective model of $X/\mathbb{Q}$.

Then its zeta function is defined by the Euler product:

$$\zeta_{\mathcal{X}}(s) := \prod_{x \in |\mathcal{X}|} (1 - N(x)^{-s})^{-1} = \prod_p \zeta_{\mathcal{X}_p}(s),$$

which converges absolutely for $\Re(s) > \dim \mathcal{X} = d + 1$. Here $\mathcal{X}_p = \mathcal{X} \otimes \mathbb{F}_p$ is the fibre of $\mathcal{X}$ over p, and $\zeta_{\mathcal{X}_p}(s)$ is the usual zeta function of the projective variety $\mathcal{X}_p/\mathbb{F}_p$.

If $\mathcal{X}'/\mathbb{Z}$ is another projective model of $X/\mathbb{Q}$, then $\zeta_{\mathcal{X}'}(s)$ agrees with $\zeta_{\mathcal{X}}(s)$ up to finitely many Euler factors, i.e.

$$\zeta_{\mathcal{X}}(s) \sim \zeta_{\mathcal{X}'}(s).$$

Thus we can define the zeta function of $X/\mathbb{Q}$ up to finitely many Euler factors by

$$\zeta_X(s) \sim \zeta_{\mathcal{X}}(s),$$

where $\mathcal{X}/\mathbb{Z}$ is any projective model of $X/\mathbb{Q}$.

Remark. To study the analytic properties of $\zeta_X(s)$, it is useful to factor it into finitely many factors which have (conjecturally) a functional equation. More precisely, we expect that

$$\zeta_X(s) \sim \prod_{m=0}^{2d} L_m(s)^{(-1)^m},$$

where each $L_m(s)$ is the L-function of a suitable rational Galois representation, but this is known only if $X/\mathbb{Q}$ is smooth.

3. Rational Galois Representations

Definition. A Galois representation (of degree n) is a system $\rho = \{\rho_\ell\}_{\ell \in P}$ of ℓ-adic representations

$$\rho_\ell : G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \overset{\text{cont}}{\to} \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell), \quad \dim_{\mathbb{Q}_\ell} V_\ell = n.$$

Following Taniyama, such a representation is called rational if:

$(*)$ $\exists$ finite set $S \subset P$ (= set of all primes) such that if $p \in P \setminus S$ and if $\ell \neq p$, then

1) ρ_ℓ is unramified with respect to p;

2) the characteristic poly. $\chi_p(T) = \det((1 - T\rho_\ell(Frob_p)^{-1})|V_\ell)$ has coefficients in $\mathbb{Q}$ and is independent of ℓ.

Note: If ρ is a rational Galois representation, then its L-function is

$$L(\rho, s) = \prod_{p \notin S} \chi_p(p^{-s})^{-1}.$$

Example 1) If $\rho = 1_{G_{\mathbb{Q}}}$ is the trivial representation, then ρ is rational and $L(1_{G_{\mathbb{Q}}}, s) = \zeta(s)$ is the usual Riemann ζ-function.

2) If $X/\mathbb{Q}$ is any projective variety, then for any $m \geq 0$ we have the Galois representation $\rho_{X,m} = \{\rho_{X,m,\ell}\}$ which is defined by the Galois action on the m-th ℓ-adic etale cohomology group $V_\ell = H^m_{et}(X \otimes \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$. Moreover, we have:

Fact: If $X/\mathbb{Q}$ is smooth, then

1) $\rho_{X,m}$ is rational, $\forall m \geq 0$ (Deligne, 1974);

2) $\zeta_X(s) \sim \prod_{m=0}^{2d} L(\rho_{X,m}, s)^{(-1)^m}$ (Grothendieck)

4. Rational A-Module Structures

Observation: The Galois representations $\rho_{X,m}$ come equipped with extra structure which can be used to construct other Galois representations.

Definition. Let A be a ring. A Galois representation $\rho = \{\rho_\ell\}$ is said to have an A-module structure if each V_ℓ has a right $A \otimes \mathbb{Q}_\ell$-module structure such that

$$(\rho_\ell(\sigma)v)a = \rho_\ell(\sigma)(va), \quad \forall \sigma \in G_{\mathbb{Q}}, a \in A.$$

Thus, each V_ℓ has a $(\mathbb{Q}_\ell[G_{\mathbb{Q}}], A \otimes \mathbb{Q}_\ell)$-bimodule structure.

Then: for any (f.g.) left A-module $M \in {}_A Mod$, each $V_\ell \otimes_A M := V_\ell \otimes_{A \otimes \mathbb{Q}_\ell} (M \otimes \mathbb{Q}_\ell)$ is a $G_{\mathbb{Q}}$-module and hence defines an ℓ-adic representation $\rho_\ell \otimes_A M$. We therefore obtain a Galois representation $\rho \otimes_A M = \{\rho_\ell \otimes_A M\}$.

Examples: 1) For any projective variety $X/\mathbb{Q}$ with group $G \leq \mathrm{Aut}_{\mathbb{Q}}(X)$, each etale Galois representation $\rho_{X,m}$ has a $A = \mathbb{Q}[G]$-module structure because G acts on V_ℓ by functoriality.

2) If $X/\mathbb{Q}$ is smooth, then the Chow group $C(X) := A^d(X \times X)$ has a ring structure which induces (by functoriality) a $C(X)$-module structure on each $\rho_{X,m}$.

Definition. An A-module structure on ρ is called rational if $\rho \otimes_A M$ is a rational Galois representation, for all $M \in {}_A Mod$.

Examples: 1) If $X/\mathbb{Q}$ is a smooth curve, then $C(X) = \mathrm{End}(J_X)$, and the $C(X)$-module structure on $\rho_{X,m}$ is rational.

2) It is conjectured that the $C(X)$-module structure on $\rho_{X,m}$ is rational for any smooth $X/\mathbb{Q}$.

Indeed, this conjecture is a basic assumption in any discussion of motivic L-functions; cf. Deligne, 1979.

Proposition 1. Let $A/\mathbb{Q}$ be an abelian variety and let $\mathbb{E} = \mathrm{End}^0(A)$. Then for every $\mathbb{E}$-module $M \in {}_{\mathbb{E}}Mod$ there is an abelian variety $A_M/\mathbb{Q}$ such that

$$\rho_{A,m} \otimes_{\mathbb{E}} M \simeq \rho_{A_M,m}, \quad \forall m \geq 0;$$

in particular, the $\mathbb{E}$-module structure on $\rho_{A,m}$ is rational.

Proof (Sketch). Since $\mathbb{E}$ is semi-simple, we can reduce to the case that $M = \mathbb{E}\varepsilon$ is an ideal (generated by an idempotent). If $r_{\mathbb{E}}(M) = (1-\varepsilon)\mathbb{E}$ is the right annihilator of M, then the above identity holds with $A_M = A/r_{\mathbb{E}}(M)A$.

By Deligne (or by Weil for $m = 1$), $\rho_{A_M,m}$ is rational, so the assertion follows.

5. Quotient Varieties

Theorem 2. Let $X/\mathbb{Q}$ be a smooth projective variety, and $Y = G\backslash X$ the quotient of X by a finite group $G \leq \mathrm{Aut}_{\mathbb{Q}}(X)$.

(a) For every $m \geq 0$ we have

$$\rho_{Y,m} \simeq \rho^G_{X,m} := \rho_{X,m} \otimes_{\mathbb{Q}[G]} \mathbb{Q}[G]\varepsilon_G, \text{ where } \varepsilon_G = \frac{1}{|G|}\sum_{g\in G} g.$$

(b) Each $\rho_{Y,m}$ is a rational Galois representation and we have

$$\zeta_Y(s) \sim \prod_{m=0}^{2d} L(\rho_{Y,m}, s)^{(-1)^m} = \prod_{m=0}^{2d} L(\rho^G_{X,m}, s)^{(-1)^m}.$$

Proof (Sketch). (a) This follows from the fact that

$$H^m_{et}(Y \otimes \overline{\mathbb{Q}}, \mathbb{Q}_\ell) \simeq H^m_{et}(X \otimes \overline{\mathbb{Q}}, \mathbb{Q}_\ell)^G = H^m_{et}(X \otimes \overline{\mathbb{Q}}, \mathbb{Q}_\ell)\varepsilon_G.$$

(b) Pick a model $\mathcal{X}/\mathbb{Z}$ on which G acts, and put $\mathcal{Y} = G\backslash\mathcal{X}$. Then for almost all p the fibre $\mathcal{X}_p$ is smooth and $\mathcal{Y}_p = G\backslash\mathcal{X}_p$. If also $p \neq \ell$, then we have analogously:

$$H^m_{et}(\mathcal{Y}_p \otimes \overline{\mathbb{F}_p}, \mathbb{Q}_\ell) \simeq H^m_{et}(\mathcal{X}_p \otimes \overline{\mathbb{F}_p}, \mathbb{Q}_\ell)^G = H^m_{et}(\mathcal{X}_p \otimes \overline{\mathbb{F}_p}, \mathbb{Q}_\ell)\varepsilon_G.$$

Thus, by Deligne's result (applied to $\mathcal{X}_p$), all (reciprocal) eigenvalues of Frobenius acting on $H^m_{et}(\mathcal{Y}_p \otimes \overline{\mathbb{F}_p}, \mathbb{Q}_\ell)$ have absolute value $p^{m/2}$. Thus, by an argument similar to the smooth case, one concludes from Grothendieck's formula that (b) holds.

Thus: $\rho_{Y,m}$ is an explicit rational subrepresentation of $\rho_{X,m}$. How can we relate its L-function to that of $\rho_{X,m}$?

6. Modular Curves

Let $X = X_\Gamma/\mathbb{Q}$ be a modular curve of level N

$\Rightarrow (X_\Gamma \otimes \mathbb{C})^{an} = \Gamma\backslash\mathfrak{H}^*$,

$\Omega = \Omega_\Gamma := H^0(X, \Omega^1_{X_\Gamma/\mathbb{Q}}) \simeq S_2(\Gamma, \mathbb{Q})$,

$\mathbb{E} = \mathrm{End}^0(J_X) \subset \mathrm{End}_{\mathbb{Q}}(\Omega)^{op}$

$\mathbb{T}' = \mathbb{Q}[\{T_n : (n, N) = 1\}] \subset \mathbb{E}$, the Hecke algebra

Recall: Atkin-Lehner theory $\Rightarrow$ every (f.g.) $\mathbb{T}' \otimes \mathbb{C}$-module M has the form

$$M \simeq \bigoplus_{f\in\mathcal{N}(\Gamma)} (\mathbb{C}f)^{m_f(M)},$$

where $\mathcal{N}(\Gamma)$ is the set of normalized newforms of weight 2 of all levels $M|N$.

Notation. If M is a $\mathbb{T}' \otimes \mathbb{C}$-module, then put

$$L(M, s) = \prod_{f\in\mathcal{N}(\Gamma)} L(f, s)^{m_f(M)},$$

where (as usual) $L(f, s) = \sum a_n(f)n^{-s}$, if f has Fourier expansion $f = \sum a_n(f)q^n$.

Recall: If $f \in \mathcal{N}(\Gamma)$, then by Shimura (1971) there is an abelian variety A_f such that

$$L(A_f, s) := L(\rho_{A_f,1}, s) \sim L(M_f, s), \text{ where } M_f = \sum_\sigma \mathbb{C}f^\sigma.$$

The following theorem may be viewed as an extension of the above result.

Theorem 3. If $M \in {}_E Mod$, then $M' := \Omega \otimes_{\mathbb{E}} M \in {}_{\mathbb{T}'}Mod$ and

$$(1) \qquad L(\rho_{X,1} \otimes_{\mathbb{E}} M, s) \sim L(M' \otimes \mathbb{C}, s).$$

Conversely, if $M' \in {}_{\mathbb{T}'}Mod$, then $M = \mathrm{Hom}_{\mathbb{T}'}(\mathbb{T}', M') \in {}_{\mathbb{E}}Mod$ and (1) holds.

Key Point: $\mathbb{E}$ is the centralizer of $\mathbb{T}'$ in Ω, i.e. $C_\Omega(\mathbb{T}') = \mathbb{E}$.

Note: The proof of this fact uses results of Ribet (1980); cf. my CMS lecture (Winter 2004).

Application to the modular curve X_N :

Let $X_N = X(N) \otimes \mathbb{Q}(\zeta_N)$ (viewed as a curve/$\mathbb{Q}$)

$\Omega := H^0(X, \Omega^1_{X_N/\mathbb{Q}}) = \Omega_{\Gamma(N)} \otimes \mathbb{Q}(\zeta_N)$

$J_{X_N} = \mathrm{Res}_{\mathbb{Q}(\zeta_N)/\mathbb{Q}}(J_{X(N)} \otimes \mathbb{Q}(\zeta_N))$, the Weil restriction

$\mathbb{E} = \mathrm{End}^0(J_{X_N}) \subset \mathrm{End}_{\mathbb{Q}}(\Omega)^{op}$

Lemma. There exists an embedding $\mathbb{T}' := \mathbb{T}'_{X(N)} \hookrightarrow \mathbb{E}$ such that $Z(\mathbb{E}) = \mathbb{T}'$ and $C_\Omega(\mathbb{T}') = \mathbb{E}$.

Theorem 4. The analogue of Theorem 3 holds for $X = X_N$: if $M \in {}_{\mathbb{E}}Mod$, then $M' := \Omega \otimes_{\mathbb{E}} M \in {}_{\mathbb{T}'}Mod$ and

$$(2) \qquad L(\rho_{X_N,1} \otimes_{\mathbb{E}} M, s) \sim L(M' \otimes \mathbb{C}, s).$$

Corollary. For any subgroup $G \leq G_N := \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$ we have

$$L(\rho_{G\backslash X_N}, s) \sim L(\Omega^G, s).$$

7. Quotients of the Product Surface $X_N \times X_N$

Observation: If X, Y are smooth/$\mathbb{Q}$ and $G \leq \mathrm{Aut}_{\mathbb{Q}}(X \times Y)$, then the zeta function of the quotient surface $Z = G\backslash(X \times Y)$ is a product/quotient of L-functions of the form

$$L((\rho_{X,r} \otimes \rho_{Y,s})^G, s),$$

for by the Künneth formula and Theorem 2 we have that

$$\rho_{G\backslash(X\times Y),m} \simeq \bigoplus_{r+s=m} (\rho_{X,r} \otimes \rho_{Y,s})^G.$$

Note that if X and Y are curves, then the only really new term is $L((\rho_{X,1} \otimes \rho_{Y,1})^G, s)$.

Assume: from now on that $X = Y = X_N$.

Notation. Let $\mathbb{T}' = \mathbb{T}'_{X(N)}$ and write $\mathbb{T}'_{\mathbb{C}} = \mathbb{T}' \otimes \mathbb{C}$.

If M is any $\mathbb{T}'_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{T}'_{\mathbb{C}}$-module, then for $f, g \in \mathcal{N} = \mathcal{N}(\Gamma(N))$, let $m_{f,g}(M)$ denote the multiplicity of the $\mathbb{T}'_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{T}'_{\mathbb{C}}$-module $\mathbb{C}(f \otimes g)$ in M. Moreover, put

$$L(M, s) = \prod_{f,g\in\mathcal{N}} L(f \otimes g, s)^{m_{f,g}(M)},$$

where $L(f \otimes g, s)$ denotes the tensor product (or Rankin convolution) of f and g:

$$L(f \otimes g, s) = L(\pi_f \times \pi_g, s) = L(2s, \chi_f\chi_g) \sum a_n(f)a_n(g)n^{-(s+1)}$$

where χ_f and χ_g denote the Nebentypus characters of f, g.

Theorem 5. In the situation of Theorem 4, let M be an $\mathbb{E}\otimes\mathbb{E}$-module. Then $M' := (\Omega\otimes\Omega)\otimes_{\mathbb{E}\otimes\mathbb{E}} M$ is a $\mathbb{T}'\otimes\mathbb{T}'$-module and we have

$$L((\rho_{X_N,1}\otimes\rho_{X_N,1})\otimes_{\mathbb{E}\otimes\mathbb{E}} M, s) \sim L(M'\otimes\mathbb{C}, s).$$

In particular, for any $G \leq G_N \times G_N$ we have

$$L((\rho_{X_N,1}\otimes\rho_{X_N,1})^G, s) \sim L((\Omega\otimes\Omega)^G\otimes\mathbb{C}, s).$$

Note: There are other interesting subrepresentations of $\rho_{X_N,1}^{\otimes 2}$ which are not of the above form. For example, the symmetric square $Sym^2(\rho_{X_N,1})$ cannot be obtained by this method since it is not an $\mathbb{E}\otimes\mathbb{E}$-module.

Corollary. The zeta-function of the modular diagonal quotient surface $Z_N = \Delta_{G_N}\backslash(X_N\times X_N)$ is given by

$$\zeta_{Z_N}(s) \sim [\zeta(s)\zeta(s-1)^2\zeta(s-2)]^{\phi(N)} L((\Omega\otimes\Omega)^{\Delta_{G_N}}, s).$$

In particular, $\zeta_{Z_N}(s)$ has a meromorphic continuation to the whole complex plane.

Remark. Since Z_N is a regular surface, i.e. $b_1(Z_N) = 0$, it follows that $L(\rho_{\mathbb{Z}_N,1}, s) = L(\rho_{X_N,3}, s) = 1$. This is the reason that $\zeta_{Z_N}(s)$ has no "denominators".