BRIDGES
Mathematical Connections
in Art, Music, and Science

Uniform Polychora

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Abstract

Like polyhedra, polychora are beautiful aesthetic structures - with one difference - polychora are four dimensional. Although they are beyond human comprehension to visualize, one can look at various projections or cross sections which are three dimensional and usually very intricate, these make outstanding pieces of art both in model form or in computer graphics. Polygons and polyhedra have been known since ancient times, but little study has gone into the next dimension - until recently.

Definitions

A polychoron is basically a four dimensional "polyhedron" in the same since that a polyhedron is a three dimensional "polygon". To be more precise - a **polychoron** is a 4-dimensional "solid" bounded by cells with the following criteria: 1) each cell is adjacent to only one other cell for each face, 2) no subset of cells fits criteria 1, 3) no two adjacent cells are corealmic. If criteria 1 fails, then the figure is **degenerate**. The word "polychoron" was invented by George Olshevsky with the following construction: poly = many and choron = rooms or cells. A polytope (polyhedron, polychoron, etc.) is **uniform** if it is vertex transitive and it's facets are uniform (a uniform polygon is a regular polygon). Degenerate figures can also be uniform under the same conditions. A **vertex figure** is the figure representing the shape and "solid" angle of the vertices, ex: the vertex figure of a cube is a triangle with edge length of the square root of 2. A **regiment** is a set of polytopes that have the same vertex and edge arrangements. Polytopes in the same regiment will have vertex figures with the same corners.

Shorter Names

As anyone who studies polyhedra would know - polyhedron names can get very long - writing down "great quasitruncated icosidodecahedron" fifty times can get wearisome. I would usually write an abbreviation, so instead of "great quasitruncated icosidodecahedron", I would write GQTID, and eventually I started to pronounce the abbreviations. Now I call the GQTID - gaquatid (pronounced GAH qua tid). Shorter polyhedron names - like dodecahedron - were simply shortened - doe. Following are the short and long names of the 9 regular polyhedra as well as a few other uniforms:

Tet - tetrahedron, Cube - cube, Oct - octahedron, Doe - dodecahedron, Ike - icosahedron, Gad - great dodecahedron, Sissid - small stellated dodecahedron, Gissid - great stellated dodecahedron, Gike - great icosahedron, Co - cuboctahedron, Oho - octahemioctahedron, Cho - cubihemioctahedron, Did - dodecadodecahedron, Quit Sissid - quasitruncated small stellated dodecahedron, Sirsid - small inverted retrosnub icosicosidodecahedron.

Just for fun - here is a sampling of polychoron short names: Gogishi, quit gashi, gun hidy, gixhinxhi, ican phix, frogfix, gnappoth, gaquapac, thaquitpath, sidhiquit paddy, irp xanady, madtapachi hiccup, gotpodady, quicdady, sirc, gixhi fohixhi, paquerfix, gaqrigagishi, wavhiddix, dod honho, and spiddit.

Discovery of the 8186 Uniform Polychora

More like "known" polychora - there may still be undiscovered ones. As been mentioned above, there are 75 uniform polyhedra as well as the infinite series of prisms and antiprisms, these polyhedra are pictured in Magnus J. Wenninger's book Polyhedron Models [1]. Many of these polyhedra are very interesting expecially the monster sirsid (#118) which has 3060 pieces. Now the question is - what happens in four dimensions? - what are the uniform polychora like? In the early 1990's I wondered the same thing - also how many were there? So I started to search every book or article related to the subject, and found very little except for the list of regular polychora and the grand antiprism [2] as well as nice color illustrations of the convex regulars [3]. Earlier discoveries were made by Alicia Boole Stott using generative processes she discovered to visualize new polytopes from known ones, as well as John Conway who completed the discovery of convex uniform polychora, and Norman Johnson who found 4-D antiprisms. As it turns out, before 1990, only 130 or so were known, these included the 16 regular polychora, the 41 "Archemedian" polychora, the 74 polyhedral prisms, and a few extra concave ones - mainly antiprisms and those with ditrigonal polyhedra as vertex figures. So I started the search and several years later the number of uniform polychora grew to 8186 - where eight of the latest ten were discovered by George Olshevsky this past year, these latest finds are the swirlprisms. George and I have been corresponding quite regularly recently, George is the author of the primary polychoron web site [4]. Some of the methods that I used to find polychora included Coxeter's graphics notation [5] as well as "digging" into the vertex figure for more polychora - basically this involves finding all polychora within the regiment and then digging into the cells to find cells in deeper regiments.

Many of the vertex figures look bizarre, some have holes, atleast two instances have vertex figures with a complete tunneling system inside, some look like encased spindles, another appears to have a small tetrahedron dangling by corners inside a strange encasing, one looks like a football trophy, another looks like a geometric bird skull (I nick named it "Phoenix"), and several appear to have faces - not quite the nice simple bowtie, and trapezoid shapes of the polyhedron vertex figures!

Uniform Polychoron Categories

Following are the categories of the uniform polychora:

1. The 16 regulars and 3 miscellaneous facetings (19)
2. The truncates and quasitruncates (21)
3. The triangular rectates (7 regiments of 3) - vertex figures are triangular prisms and facetings.
4. The ico regiment, 14 in regiment, ico is the regular 24-cell and has been counted (13)
5. The pentagonal rectates (4 regiments of 22) - vertex figures are pentagonal prisms and facetings.
6. The sphenoverts which include small rhombates (24 regiments of 7, three already counted in category 3) - vertex figures are wedges and facetings.
7. The bitruncates (11) - vertex figures are tetrahedra with 2 pairs of isosceles triangles, two have tripled up vertices.
8. The great rhombates and related (23) - vertex figures are tetrahedra with bilateral symmetry.
9. The maximized polychora (includes great prismates) (22) - v.f's are irregular tetrahedra.
10. Prismatorhombates and related (30 regiments of 3) - v.f's are trapezoid pyramids and facetings.
11. Triangular small prismates and related (1 regiment of 5 and 5 regiments of 7) - v.f's are triangular antiprisms and facetings.
12. Frustrumverts (4 regiments of 7 and 2 more from a regiment in category 3) - v.f's are triangular frustra and facetings.

13. Spic and giddic regiments (2 regiments of 22) - v.f's are square antiprisms and facetings.
14. Skewverts (4 regiments of 15) - v.f's are skewed wedges and facetings.
15. The afdec regiment (52) - afdec has a v.f. with 6 faces, 2 parallel rectangles in orthogonal positions seperated by 4 alternating trapezoids. Cells are 48 coes and 48 goccoes (great cubicuboctahedra).
16. The affixthi regiment (96) - affixthi's v.f is similar to afdec's, except one rectangle is a square and the other long. Cells are 600 octs, 120 dids (dodecadodecahedra), 120 gaddids (great dodekicosidodecahedra), and 120 gidditdids (great ditrigonal dodekicosidodecahedra).
17. The sishi regiment - sishi is the small stellated 120 cell and has a dodecahedron v.f. (34) - there are actually 44 in this regiment, 2 are regulars and been counted, the remaining 8 are swirlprisms.
18. The ditetrahedrals (3 regiments of 71) - v.f's are truncated tetrahedral for 2 regiments and "cuboctahedral" in the other (this "cuboctahedron" has rectangles replacing the squares) plus the facetings, many are very intricate.
19. Prisms (74) - prisms of uniform polyhedra, cube prism has been counted as the regular tesseract.
20. The 6 antiprisms, 6 normal snubs, and the 10 swirlprisms (22). Swirlprisms have a strange swirly symmetry.
21. The padohi super-regiment (354) - padohi's v.f is a pentagonal antiprism with a larger base. Cells are 120 sissids, 120 ikes, 720 5/2Ps, and 1200 3Ps.
22. The gidipthi super-regiment (354) - gidipthi's v.f is a pentagonal frustrum, its cells are 120 sissids, 120 ikes, and 120 gaddids.
23. The rissidtixhi super-regiment (316) - its v.f is a "ditrigon" prism - a ditrigon is a semiregular hexagon. Its cells are 120 gids (great icosidodecahedra), 600 octs, and 120 sidtids (small ditrigonal icosidodecahedra).
24. The stut phiddix super-regiment (238) - it's v.f. has a triangle top and a ditrigon base, it's cells are 120 sidtids, 600 tets, 720 5/2Ps, and 600 coes.
25. The getit xethi super-regiment (238) - it's v.f. is like stut phiddix's but squattier, it's cells are 120 sidtids, 120 gaddids, 120 quit gissids, and 600 tets.
26. The blends (16) - this regiment is formed by the fact that 3 regiments have the same convex hull and can blend together, the v.f's resemble a "3-D pentagon" and facetings.
27. Baby monster snubs (2 regiments of 17) - the "sidtaps" are formed by compounding 5 polychoron A's and 5 polychoron B's where A and B are in the rox regiment (the rectified 600-cell which has a 5P v.f), both need octs for cells so this conglomerate compound can "blend" together to form a true polychoron - if A and B are different - the result is a non-Wythoffian chiral snub. One of the sidtaps (formed by blending 10 roxes together) is hollow inside - it's v.f looks like 2 pentagonal prisms attached to a removed square which both have in common. The "gidtaps" are formed the same way but with the raggix regiment (rectified grand 600-cell).
28. Idcossids (2749) - one of the ultimate monster snub regiments formed by the blending compound of 5 padohi A's and 5 padohi B's. Nearly all of these are chiral.
29. Dircospids (2749) - the other ultimate monster snub regiment, formed by the blending compound of 5 gidipthi A's and 5 gidipthi B's. Nearly all of these are chiral.

There are also two infinite categories not counted in the main count - these are the duoprisms and the antiprism prisms (also called antiduoprisms).

Figure 1 shows various vertex figures of the polychora. Even the vertex figures make great models to build, in which I have constructed over 160 of them.

Just for the record, here are the short names of the 16 regular polychora - I use Coxeter's notation for identification: Pen {3,3,3}, tes {4,3,3}, hex {3,3,4}, ico {3,4,3}, hi {5,3,3}, ex {3,3,5}, fix {3,5,5/2}, gohi {5,5/2,5}, gahi {5,3,5/2}, sishi {5/2,5,3}, gaghi {5,5/2,3}, gishi {5/2,3,5}, gashi {5/2,5,5/2}, gofix

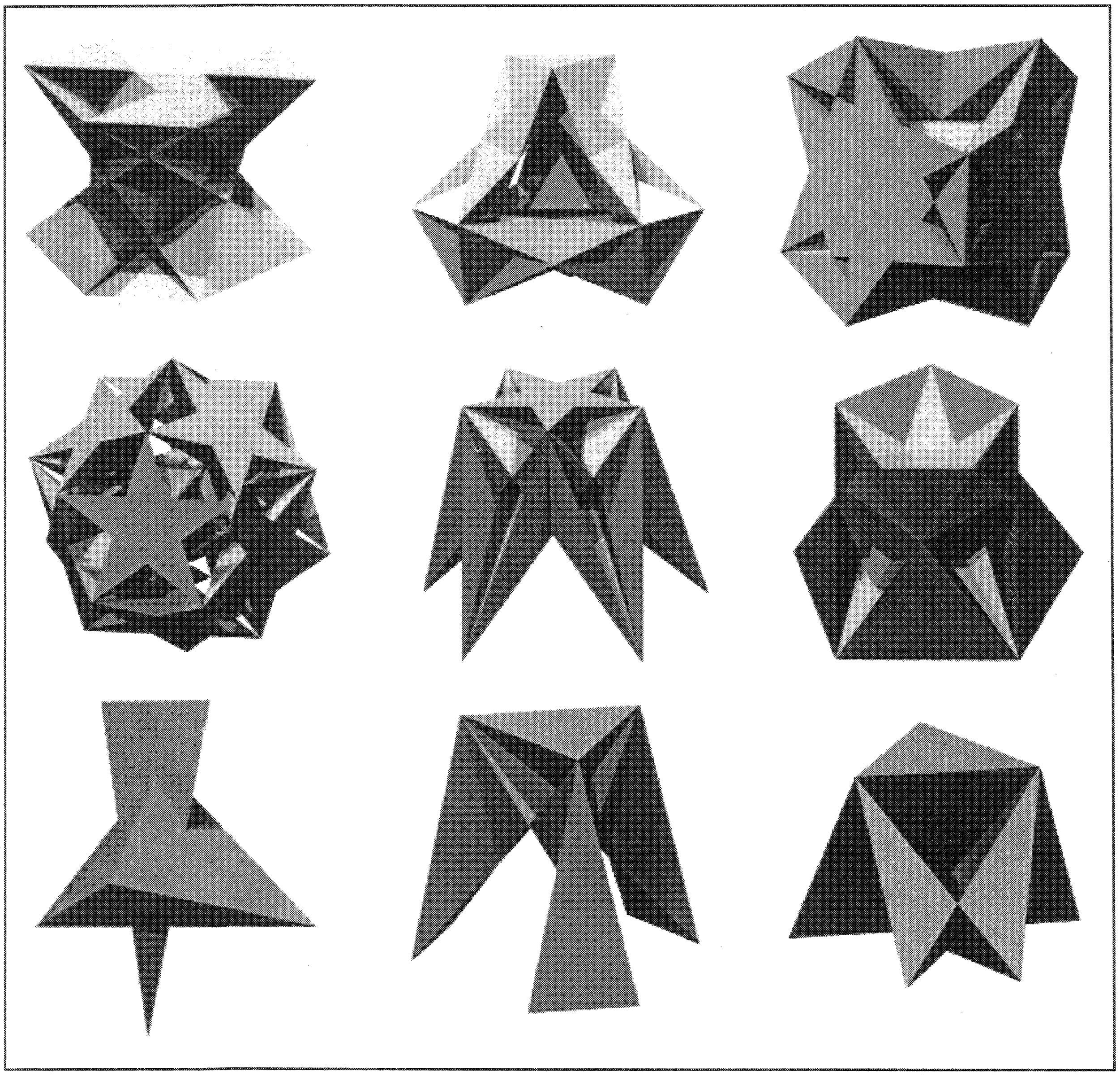

Figure 1: *Vertex figures of various polychora, the top left one belongs to the rissidtixhi regiment, the next two are ditetrahedrals, left middle is in the sishi regiment and contains many holes, the central one belongs to the padohi regiment, next is a gidipthi regiment member, bottom left is the vertex figure of one of the blends, the next two belong to the stut phiddix and getit xethi regiments respectively. As you can see, even the vertex figures alone are very interesting.*

{3,5/2,5}, gax {3,3,5/2}, and gogishi {5/2,3,3}. This notation works as follows: the cells of {a,b,c} are {a,b}'s and its vertex figure is a {b,c}, the faces of {a,b} are a-gons, and the v.f is a b-gon. Ex. {4,3} is a cube, and {5,5/2} is a gad.

Rapsady

The most interesting normal snub is "rapsady" = retroantiprismatosnub dishecatonicosachoron. Rapsady is the only known polychoron other than the prisms that contain snub polyhedra as cells - and guess what they are, it has 120 "sesides" (the small snub icosicosidodecahedron) and 120 sirsids! as well as 1440 pentagonal antiprisms as snub cells. The sirsids are on the surcell and connect to each other by ten vertices, the vertex figure has triangular symmetry - containing three sirsid vertex figures on the sides. Rapsady probably has over a million pieces. Rapsady, like sirsid is not chiral. Figure 2 shows rapsady's vertex figure as well as one of it's cells - sirsid.

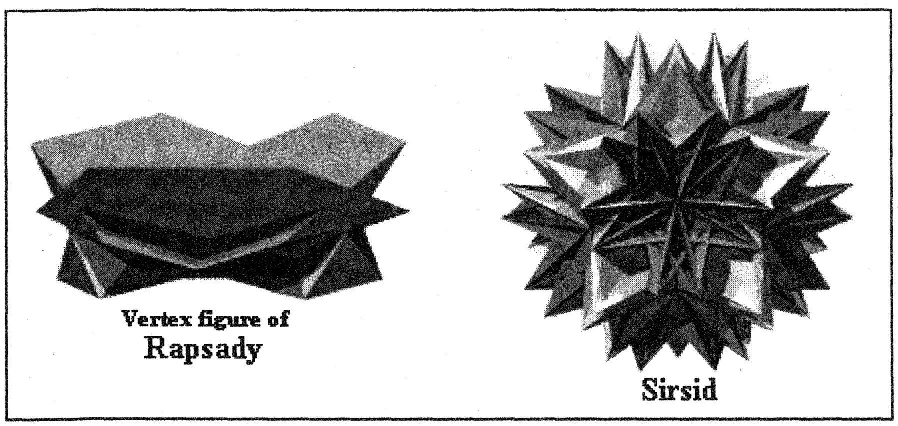

Figure 2: *The vertex figure of rapsady and it's most intricate type of cell - sirsid (also known as Yog-Sothoth by George O.)*

Idcossids and Dircospids

The ultimate polychora are the extremely intricate great monster snubs - these are the idcossids and dircospids (these names are shortened forms of the short names for the first idcossid and dircospid named). There is a uniform compound of 5 padohis as well as 10 padohis, same for gidipthi - the compound of 5 is chiral. The compound of 10 has vertices that double up, and this is what forms the idcossids (from padohi compound) and the dircospids (from gidipthi compound). The vertex figure of an idcossid looks like a padohi member's vertex figure, blended with another member which is flipped 180 degrees - they blend about the vertex figure of co, oho, or cho which are cells that each component needs in order to be a true polychoron. Same thing with dircospids, except using the gidipthi members. Each idcossid and dircospid has 7200 vertices. They also have some of the most horrendous vertex figures known, making these the ultimate monstrosities in the fourth dimension. There are 2749 of each, and 2732 of each are chiral! So there are 5538 snubs where 5492 are chiral - that means over 2/3 of the uniform polychora are monster non-Wythoffian chiral snubs! Just to show how bad these snubs get, they can have not just one - but as many as five different kinds of snub cells - where one kind shows up in two sets of orientations. These snubs could have tens of millions if not hundreds of millions of pieces!

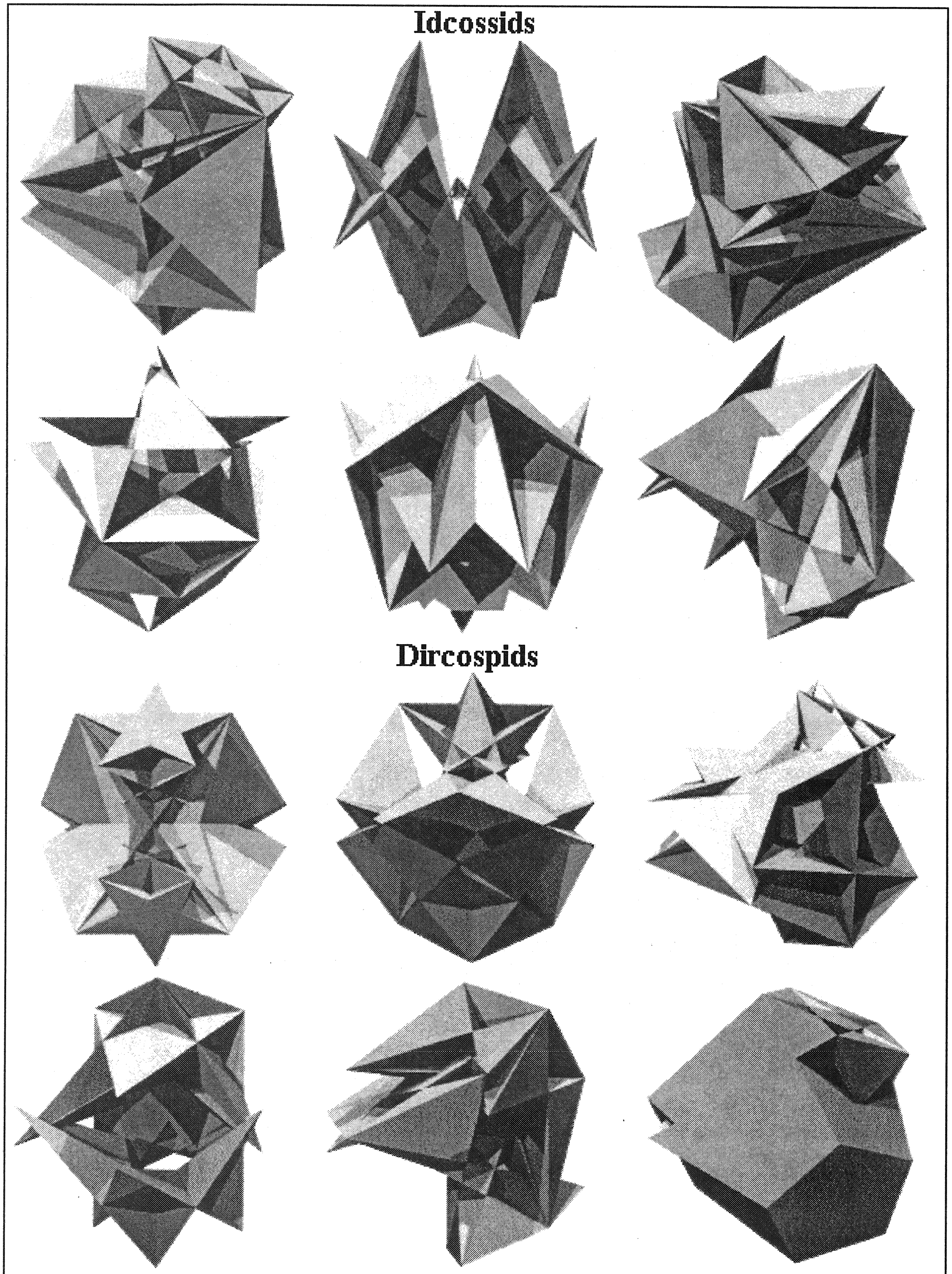

Figure 3: *Vertex figures of various idcossids and dircospids. Phoenix is on bottom left.*

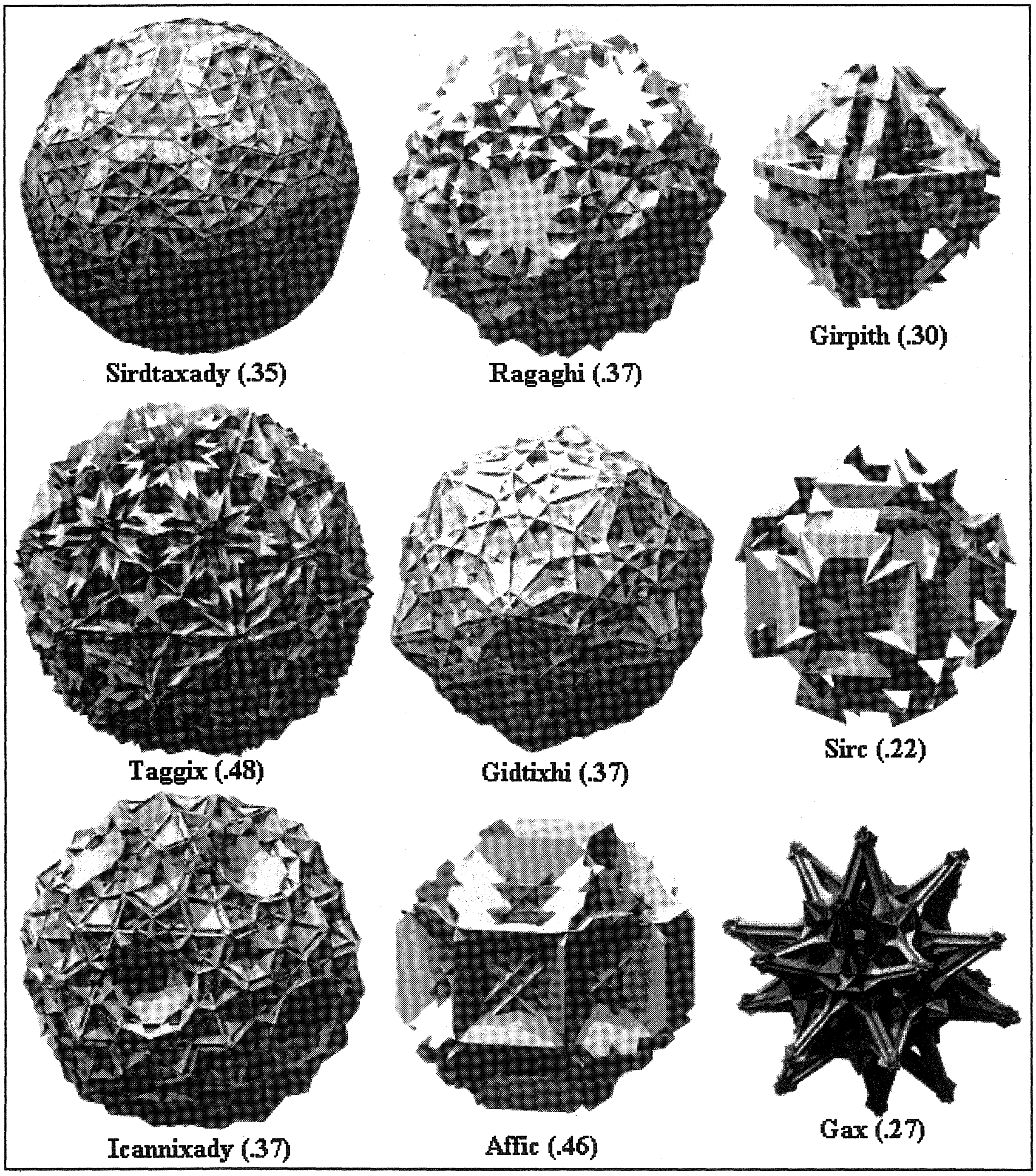

Figure 4: *Cross sections of nine various polychora, the numbers in parenthesis represent the slicing realm (0 at top, 1 at bottom, .5 halfway). Sirdtaxady is a ditetrahedral, ragaghi is a triangular rectate, girpith goes with category 10, taggix is a truncate, gidtixhi is in the sishi regiment, sirc is in the spic regiment, icannixady is in category 5, affic is in the afdec regiment, and finally is gax – a regular polychoron.*

Certainly the idcossids and dircospids are the grand-daddies of all regiments! I discovered these non-Wythoffian snubs in 1997. Figure 3 shows vertex figures of various idcossids and dircospids.

Cross Sections

The best way to visualize polychora are by using cross sections - however even this would be quite puny compared to actually seeing a polychoron in full 4-D splendor. Presently I have wrote a program to section polychora and regularly add new cells and polychoron code, so far over 140 polychora are sectioned including all 98 with tesseractic symmetry, all 16 regulars and many others. They also have been animated by John Cranmer. Figure 4 shows a cross section of several different polychora.

Further Searches

There may yet be more uniform polychora - however I do believe that all of the "normal" ones are found, any more will most likely be bizarre cases - the search continues!

Meanwhile in the fifth dimension, I began the search for the uniform polytetra. There are only 3 regulars here and we lose the larger symmetries, however the regiments get fairly large - one of the hixic regiments (hix = hexatetron the 5-D symplex) has 42 polytetra - surprisingly large for a "small" regiment. There are 164 known polytetra with hixic symmetry - I have yet to give a count on the penteractic and hemi-penteractic ones. There are also 8111 known prisms and 75 infinite waves of duoprisms which are the cross-product of the uniform polyhedra and uniform polygons. Bizarre things start happening in six dimensions. There is a symmetry which is not at all related to the simplex and the hexeract symmetries - it is the sporadic "yaz" symmetry (yaz = heptacontidiicosiheptapenton (72 + 27 "pentons" - I use y and z to represent 72 and 27 respectively). Yaz was discovered by H.S.M. Coxeter [6] as 2_{21}. It has 27 vertices and 27 "tacs" and 72 hixes as its pentons (5-D version of cell), tac = tricontiditetron and is the 5-D version of oct. Yaz resembles some sort of 6-D "pentagon". Seven and eight dimensions also have sporadics, but they have even number of vertices. Beyond eight dimensions - the sporadics are gone, however with that much space - bizarre symmetries bound to show up elsewhere, but thats .. another dimension.

References

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[1] Magnus J. Wenninger, *Polyhedron Models,* Cambridge University Press, 1971

[2] H.S.M. Coxeter, *Regular Complex Polytopes,* Cambridge University Press, 1974, p.167

[3] Thomas Banchoff, *Beyond the Third Dimension: Geometry, Computer Graphics, and Higher Dimensions,* Scientific American Library, 1990

[4] http://members.aol.com/Polycell/uniform.html

[5] H.S.M. Coxeter, *Regular Complex Polytopes,* pp. 14-18

[6] H.S.M. Coxeter, *Regular and Semi-Regular Polytopes. III,* Mathematische Zeitschrift **200**, p. 22, 1988.