# Archimedes and the Baths: Not Only One Eureka

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*Abstract*

*In the light of recent archeological studies of the North Baths at Morgantina, we discuss Archimedes' treatment of cross vaults and of the unusual cylindrical shape called the 'hoof' in his work usually referred to as the Method. Our conclusion is that Archimedes' study of these solids may have been inspired by some real edifice in the Syracuse of his time. The presumed vaulting in the Hellenistic baths at Syracuse could have provided this inspiration; however, because the archaeological evidence for this presumed vaulting no longer exists, the dome and barrel vaults of the comparable bath complex at Morgantina provide important information that helps to elucidate the approach taken by Archimedes in his exploration of particular shapes. We conclude with the observation that the question of the possible role of Archimedes in the invention of these early domes and vaults must remain open, because of the current lack of relevant documentation.*

Recent excavations at Morgantina have revealed baths with dome and barrel vaults that were built in the 3rd century BCE and featured an innovative construction technique.[1] This archaeological discovery opens interesting perspectives in interpretation of what is perhaps the most famous work of Archimedes: *The Method on Mechanical Theorems*, addressed to Eratosthenes, the librarian of Alexandria.

## *The Method*

In 1906, the Danish philologist and historian Johan Ludvig Heiberg found in Istanbul a palimpsest containing a series of Archimedean texts. Among other previously known works, a hitherto unknown text was discovered in this palimpsest, which caused a huge sensation. The work bears the title: The approach (*éphodos*) of mechanical theorems, for Eratosthenes (*Peri tō̃n mēchanikō̃n theōrēmátōn pros Eratosthénēn éphodos*).[2]

This discovery also caused a sensation because in *The Method*, Archimedes reveals to Eratosthenes the heuristic 'approach': the *éphodos* of the title, or better the *trópos* (way, manner), since this is the word that appears repeatedly in the text. This was the approach that had enabled him to arrive at many of his amazing results: the quadrature of the parabola, the ratio of the circumscribed sphere to the cylinder, the determination of (the volume of) 'conoids' (hyperboloid and paraboloid) and of 'spheroids' (ellipsoids) and their centers of gravity.

Heiberg himself and the entire community of historians of mathematics interpreted this heuristic approach in the ultimate terms: as a brilliant and visionary anticipation of modern integral calculus, as it is still considered by many even today. The work was renamed *The Mechanical Method* in the translation, though the Greek word *méthodos* never appears in this work. It is not the place to enter here into a detailed discussion of these issues. Suffice it to mention that *The Method* (we continue to use this title for the sake of convenience) contains a description of how to use an 'ideal' balance in order to obtain conjectures about how to determine the ratio between two figures (the parabola and the triangle inscribed in it; the sphere and the circumscribed cylinder etc.), or to reduce the determination of the center of gravity of a figure to that of another figure whose center of gravity is known. This was the Archimedean *trópos*: however, this was not the reason (or at least not the only reason) why Archimedes wrote to Eratosthenes. Indeed, he writes:[3]

> Archimedes to Eratosthenes greeting: I sent you on a former occasion some of the theorems discovered by me, merely writing out the enunciations and inviting you to discover the proofs, which at the moment I did not give. The enunciations of the theorems which I sent were as follows.

The two theorems of which Archimedes is speaking here are about two 'strange' solids. Let us call this solid the 'hoof', whose definition we can gather from what Archimedes writes (*fig. 1*):

> If in a right prism with a parallelogrammic base a cylinder be inscribed which has its bases in the opposite parallelograms, and its sides on the remaining faces of the prism, and if through the center of the circle which is the base of the

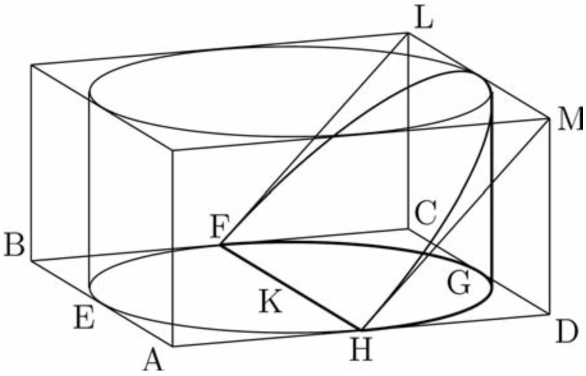


*Fig. 1. Diagram showing the 'hoof' (drawing P.D. Napolitani and K. Saito).*

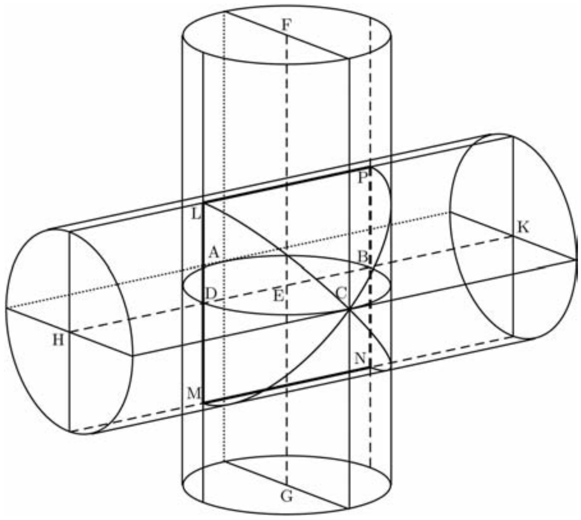


*Fig. 2. Diagram of two intersecting cylinders (drawing P.D. Napolitani and K. Saito).*

> cylinder and [through] one side of the square in the plane opposite to it a plane be drawn, the plane so drawn will cut off from the cylinder a segment which is bounded by two planes and the surface of the cylinder, one of the two planes being the plane which has been drawn and the other the plane in which the base of the cylinder is, and the surface being that which is between the said planes; and the segment cut off from the cylinder is one sixth part of the whole prism.

The 'hoof' *H* (the solid figure placed above *HKFG*) is therefore $\frac{1}{6}$ of the prism *ABCDLM*.[4]

The other solid is obtained by the intersection of two cylinders:

> If in a cube a cylinder be inscribed which has its bases in the opposite parallelograms and touches with its surface the remaining four faces, and if there also be inscribed in the same cube another cylinder which has its bases in other parallelograms and touches with its surface the remaining four faces, then the figure bounded by the surfaces of the cylinders, which is within both cylinders, is two-thirds of the whole cube (*fig. 2*).[5]

That is, if we denote by *W* the solid resulting from the intersection of two cylinders, and with *C* the cube on which the bases are built, we have

$$W = \frac{2}{3}C$$

Note that Archimedes was particularly proud of the results:

> Now, these theorems differ in character from those communicated before; for we compared the figures then in question, conoids and spheroids and segments of them, in respect of size, with figures of cones and cylinders: but none of those figures have yet been found to be equal to a solid figure bounded by planes; whereas each of the present figures bounded by two planes and surfaces of cylinders is found to be equal to one of the solid figures which are bounded by planes.

It is only at this point that Archimedes says to Eratosthenes that before passing to the proofs of these facts he wants to explain to him the way (*trópos*) he arrived at these and the other results shown in his earlier works. In short, if not the purpose, at least the opportunity to write this text was provided to Archimedes by these two solids. But, why these two solids?

At first glance they appear to be strange, 'unnatural' objects, different from those which the Greek mathematicians normally dealt with - or, at least different from those that we *think* the Greek geometers were studying. Indeed, why study the 'hoof' and why study the intersection of cylinders? However, this latter solid immediately turns out to be less fantastic and abstract than it seems at first sight; it can be thought of as a double groin vault: a groin vault, in fact, is obtained by the intersection of two vaults, or two half-cylinders (*fig. 3*).

Piero della Francesca confirms that the solid can be seen in this way in his *Libellus de quinque corporibus regularibus*, where he examines it, though

*Fig. 3. Vaulted porch of courtyard at the Palazzo Ducale in Urbino (photo Francesco Serafini).*

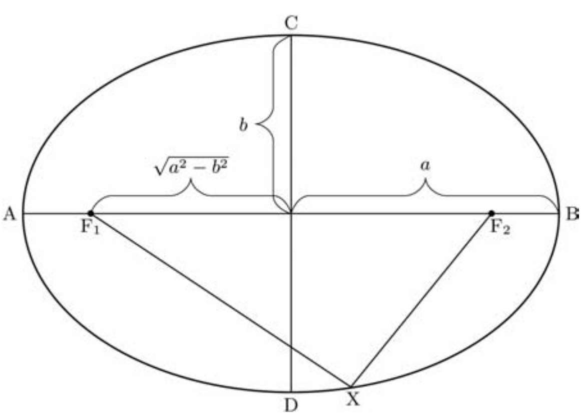


*Fig. 4. Diagram of ellipse.*

not in a very rigorous way.[6] Piero, however, is an artist-mathematician, and in the context of Italian culture in the 15th century, it is quite natural that he addressed a problem related to architecture which made extensive use of arches and vaults. In figure 3 is shown the porch of the courtyard in the Palazzo Ducale in Urbino, one of the places where Piero worked and developed so many important artistic and intellectual relationships.

It seems less natural, however, at least at first glance, that a Greek mathematician like Archimedes was interested in vaults. First, until very recently, it seemed that Greek architectural culture of the 3rd century BCE did not include the construction of freestanding, above ground arches or vaults and domes. Moreover, Archimedean works, like those of Euclid, Apollonius, or the other Greek geometers, are often regarded as examples of Platonic detachment from practical things, all aimed at exploring the arcana of a hyperuranic world of ideas.[7]

## Some Features of Greek Geometry

To understand how and why Archimedes came to study the two figures discussed in the preface of *The Method*, one has to be aware of some features of Greek mathematics, and in particular that of Archimedes, as they have emerged from recent studies - although we limit ourselves here to an extremely synthetic and somewhat dogmatic presentation of defining characteristics:[8]

- a clear separation between geometry and arithmetic;
- lack of a symbolic language similar to our abstract algebra;
- objects are mathematical formalizations of effective solution procedures and of 'concrete' objects.

Let us illustrate this last point. For us, an ellipse is the locus of zeros of a polynomial of second degree with two variables:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where $B^2 - 4AC < 0$, or it is defined as the locus of points such that the sum of their distances from the foci is constant (for every point X of the ellipse, $XF_1+XF_2=AB$ (*fig. 4*).

In other words, the curve is defined by an abstract property of algebraic nature, pre-existing the object 'ellipse'. For the Greeks, however, the ellipse is the curve obtained by cutting a cone with a plane that intersects all of its generatrices (for example, the yellow and the red curve in fig. 5, while the blue and green are respectively a parabola and hyperbole; *fig. 5*).

The curve is defined by a specific procedure that precedes its properties. Greek mathematics is thus a mathematics of individual objects, each generated from a suitable constructive process. If we assume this point of view, some consequences

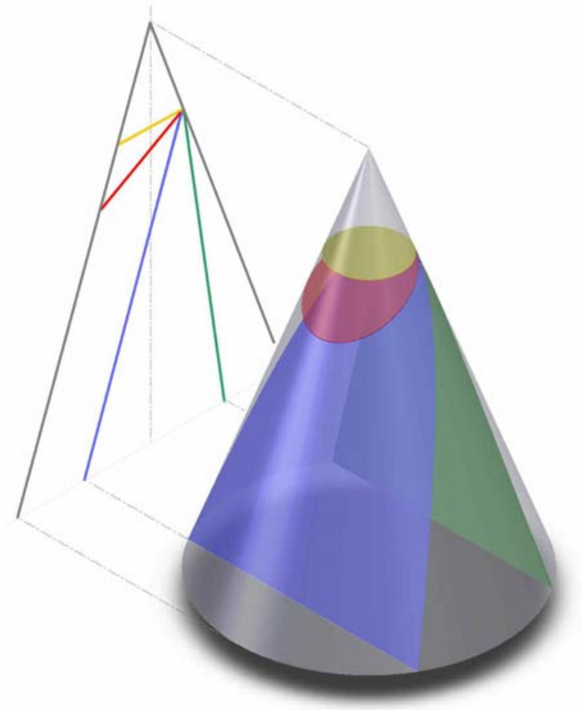

*Fig. 5. Diagram of cones (SergV; released into public domain by its author. http://ru.wikipedia.org/wiki/Изображение:Conic_sections.png).*

immediately follow:

- Greek mathematics is not a general mathematics, such as post-Cartesian mathematics.
- There are no general objects, nor general methods.
- The procedure of measuring an object is a formalization of a concrete process; indirect confrontations are applied only if the direct one is proved impossible.

## Some Features of Archimedean Mathematics

Archimedes' works fit well the general characteristics stated above. The extant texts are concerned with two major themes: geometry of measure and mechanics.

In the works of geometry (*On the Sphere and the Cylinder, Measurement of the Circle, Quadrature of the Parabola, Spiral Lines, Conoids and Spheroids*) Archimedes deals with the problem of measuring, through direct comparison between an 'unknown' figure (e.g. the sphere) and a better known one (the cylinder), and shows, for example, that the sphere is of the circumscribed cylinder, or that the paraboloid is half of the cylinder circumscribed to it and so on. This is why Archimedes was so proud to tell Erathosthenes that he had succeeded in demonstrating for the first time the equivalence between a solid curved figure and a 'straight' one (a parallelepiped), for the quadrature (or cubature in this case) of a figure was not the result of finding a formula, like $S = \frac{4}{3}\pi r^3$, but of finding the simplest known figure equal to it.

The works known as 'mechanical' *(Plane Equilibria, Floating Bodies, parts of Quadrature of the Parabola and of Spiral Lines*) investigate real situations elaborating geometrical models or, conversely, apply those models to solve geometrical problems. In the first book of the *Plane Equilibria* he thus studies abstract balance by modeling the real steelyards; in the first book of the *Floating Bodies* he lays down the general laws of buoyancy (the famous 'Archimedes' principle') and applies them to the study of the flotation of a segment of a sphere or paraboloid: a clear attempt to model the behavior of ships at sea. In the *Quadrature of the Parabola* the laws of equilibrium are used to compare the area of a segment of a parabola with an inscribed triangle; in the *Spiral Lines* the laws of uniform motion are used to define the spiral (called today the 'Archimedean spiral') and to study its properties.

This aspect of theoretical modeling fits well with the historical anecdotes and stories passed down concerning the life and the figure of Archimedes of Syracuse.

## Was Archimedes a Technologist?

The figure of Archimedes is in fact surrounded by a legendary halo, maintained primarily by reports of Polybius, Livy and Plutarch, and a number of other minor sources (generally less reliable than these three historians).[9] Archimedes, since Polybius, is the great defender of Syracuse against the Roman siege: the great builder of war machines capable of seizing the Roman ships and dashing them into the water. To these accounts there has been added the famous legend of burning mirrors, related only by late Byzantine sources, such as Anthemius (6th century CE), Tzetzes and Zonaras (12th century CE).[10]

Chris Rorres writes in his splendid website dedicated to Archimedes:

> The legend of Archimedes' burning mirrors is too good to die, regardless of how much evidence is presented to refute it. The legend is ever captivating - even after 2300 years: a city under attack, a solitary genius who rescues the

city, a spectacular military weapon, and the reduction of the attacking forces to ashes.[11]

Archimedes' reputation as a brilliant inventor is also fuelled by other accounts of inconsistent reliability: the invention of the *cochlea*, a hydraulic tool that was used to remove water from the Spanish silver mines (but Archimedes may have simply improved an already existing tool); the launching of the ship *Syracosia*, which was so massive in size and weight that it was impossible for anyone else to launch - he is said to have done it easily using a lever. But the story which concerns us most is that of the golden crown which King Hieron II of Syracuse had dedicated to the gods.

## Archimedes at the Baths: the Problem of the Crown

The story, very famous and one of the most interesting anecdotes about Archimedes, is told by Vitruvius.[12] King Hieron II of Syracuse had given a certain amount of gold to make a wreath to be dedicated to the gods. When the work was finished, however, the king suspected that the crown was not made of gold, but of an alloy, even though the weight of the crown was equal to that of the gold Hieron had given to the goldsmith. He therefore asked Archimedes to solve the problem of discovering the theft without damaging the crown, now sacred. Archimedes went to the baths, trying to solve the problem put to him by Hieron, and while he was bathing he suddenly realized that the more his body was immersed in water, the more water was escaping from the tub.

Then suddenly he jumped out, naked, from the (public) bath shouting 'Eureka, Eureka', 'I found it! I found it!' and ran back home. We do not attempt to discuss here the possible methods Archimedes may have used in order to know if the wreath was made of pure gold or if it contained silver, nor the various attempts to improve those methods from late antiquity until Galileo.

The point of interest to us here is that Archimedes is connected to frequenting the public baths in Syracuse. Plutarch also depicts an Archimedes at the baths:

> His servants used to drag Archimedes away from his diagrams by force to give him his rubbing down with oil; and as they rubbed him he used to draw the figures on his belly with the scraper; and at the bath, as the story goes, when he discovered from the overflow how to measure the crown, as if possessed or inspired, he leapt out shouting 'I have it' and went off saying this over and over.[13]

But what did the Syracusan baths look like? Those baths that Archimedes frequented not only to immerse his body in a bathtub, but also to be immersed in mathematical meditations?

## The barrel vaults: another Eureka?

Excavations at Morgantina have revealed the existence of two barrel vaults which encounter each other at a right angle without intersecting (*fig. 6*).[14] It should also be noted that in the thirties of the last century the Italian archaeologist, Giuseppe Cultrera, discovered in Syracuse the remains of a 'hydraulic establishment' in which the same construction technique as that found at Morgantina was used. Unfortunately, of the Syracusan excavations we have only a preliminary report and some photos, in addition to the partially pre-

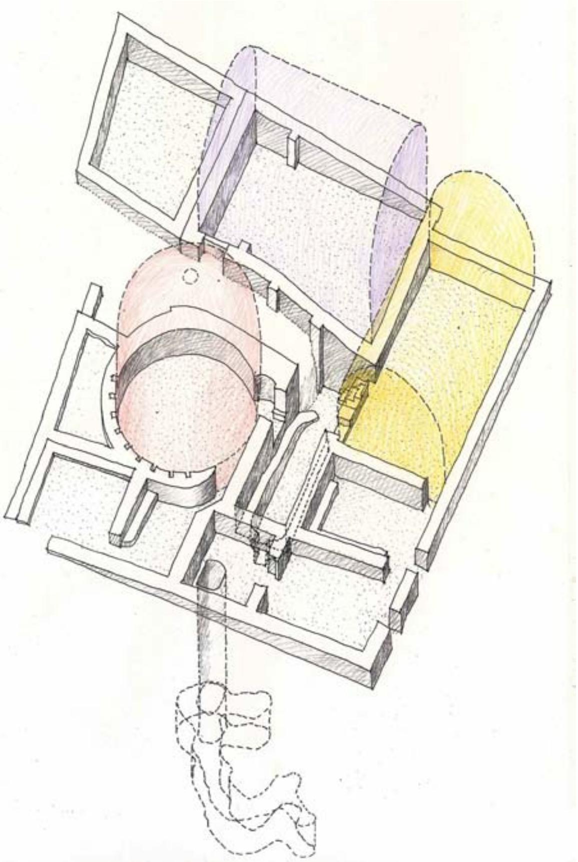

*Fig. 6. Morgantina North Baths: axonometric reconstruction of dome and vaults (E. Thorkildsen; American Excavations at Morgantina).*

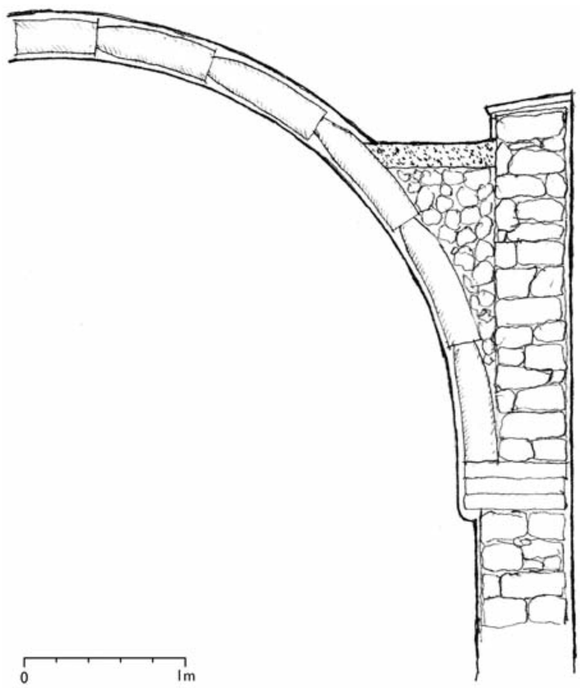


*Fig. 7. Morgantina North Baths: reconstruction of vault (E. Thorkildsen; American Excavations at Morgantina).*

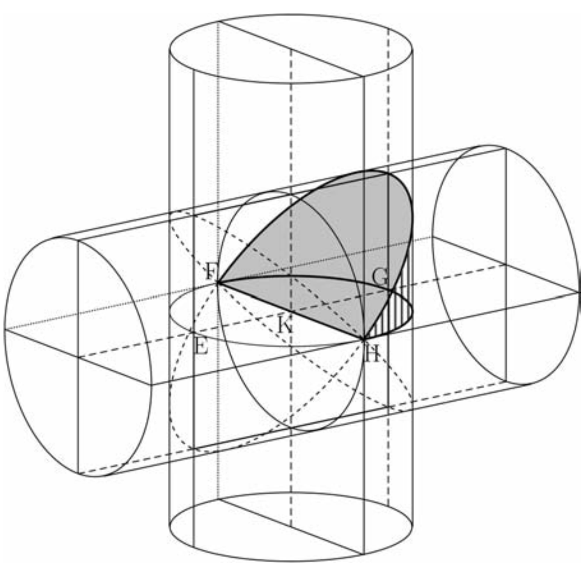


*Fig. 8. Diagram of intersecting cylinders, showing 'hoofs' (drawing P.D. Napolitani and K. Saito).*

served remains of the building itself.[15]

In any case, the existence of this type of construction at Morgantina - at that time part of the Syracusan kingdom of Hieron II - and the existence of at least one similar building at Syracuse itself, allow us to imagine Archimedes, lying in his bath or having a massage, asking himself the question: What if those two vaults were to intersect? What kind of shape would result?

It is noteworthy to observe that it is by no means obvious to imagine what kind of curves are obtained by the intersecting of two barrel vaults or, if you prefer, two cylinders. Moreover, the construction techniques used at Morgantina (and most likely at Syracuse) were not such that would easily have allowed the construction of a groin vault. The Morgantina barrel vaults (as well as the dome discovered in the early excavations of the North Baths, *fig. 6*) are in fact built by constructing arch segments by placing hollow terracotta tubes, which are shaped like truncated cones, front to end vertically (*fig. 7*). Ultimately, the resulting surface was coated with plaster and painted.

These problems - developing a geometric model, studying its properties and, if possible, its practical feasibility - fit well with what we know about the character of Archimedes. Above all, they go very well with the apparently 'strange', 'unnatural' solid described in the letter to Eratosthenes and can explain how Archimedes came to conceive of it: he would have basically followed the same path followed by Piero 18 centuries later.[16] But even accepting the hypothesis that Archimedes was interested in modeling vaults, a question remains: does the cylindrical 'hoof' have anything to do with all this? And why did Archimedes send to Eratosthenes these two results together?

## Eight 'Hoofs' for a Double Vault

In this article, we shall call the intersection of two cylinders described by Archimedes in his letter to Eratosthenes a 'double (groin) vault', and we shall denote it by *W*. We shall denote the (cylindrical) 'hoof' by *H*, and the cube in which *W* is built with *C*, while P shall be the prism cut from the cube by the same plane that cuts the 'hoof' *H* from the cylinder.

In the text that we have of *The Method* (do not forget that the codex that has come down to us is a palimpsest and part of the original Archimedean codex has been lost). there is no proof of the asserted result $W = \frac{2}{3} C$. However, by certain stringent codicological arguments - based on the number of sheets of lost parchment that existed between the last sentence of *The Method* we have today and the first of the next treatise - it is shown that the proof had to be rather short and cannot have been much more than a rather simple reasoning. This is in fact quite possible, since the double vault can be decomposed into eight cylindrical 'hoofs' equal to each other, as shown in figure 8.

Moreover, this is a very natural decomposition, because it is made by cutting $W$ along the elliptical 'ribs' of the vault. It should also be noted that Greek mathematics, as we noted earlier, preferred methods of direct comparison: if you have a figure whose area or volume is unknown, the first thing you try to do is to decompose it into simpler ones. A similar way is followed, for example, by Hippocrates of Chios (second half of the 6th century BCE) in the investigation of quadrable 'lunes', i.e. figures contained by two circle arcs convex in the same direction.[17]

Nothing, therefore, prevents us from thinking that Archimedes tried this route. However, this was the route that required facing a much harder problem: how can one determine the volume of the 'hoof'? The investigation of the 'hoof' was for him, in all likelihood, a very hard nut to crack. As matter of fact, the extant text of *The Method* presents three different demonstrations: two heuristic approaches and a rigorous proof based on the second heuristic approach. In the first heuristic approach (propositions 12 and 13), Archimedes uses his 'ideal' balance in a unique way, completely different from previous propositions in *The Method*, where he explains its use to find other results. In this unique use of balance, Archimedes had to cut the half cylinder, which is introduced as a solid balancing the hoof on the ideal balance, by different parallel planes. This device of a new way of cutting the solid is important for the following proposition 14, which introduces another heuristic approach without ideal balance and applies this way of cutting the half cylinder in proposition 13 to the hoof itself. The novel approach in proposition 14 in turn seems to have led Archimedes to a profound methodological rethinking and the invention of a new *trópos*.

At this point, the volume of a double vault is reduced to a simple calculation, which could easily be contained in the five or six columns of text missing today in the codex, space that would have been absolutely insufficient for a complex proof. In these missing pages there must have been a figure showing the decomposition of a double vault into eight 'hoofs', along with its explanation, and the following calculation expounded in words, as was customary in Greek mathematics.

Since $H = \frac{1}{6}P$ and $P = \frac{1}{2}C$, we have:

$$W = 8H = 8 \times \frac{1}{6}P = 8 \times \frac{1}{6} \times \frac{1}{2}C = \frac{8}{12}C = \frac{2}{3}C$$

i.e. the double groin vault is two-thirds of the cube in which it is built: as Archimedes had stated in his letter to Eratosthenes.

## Archaeology and Archimedean Studies

Consideration of the excavations at Morgantina has led us to a quite unexpected realization of a synergy, between the archaeological research and the interpretation of the heuristics and methodology of one of the greatest mathematicians of antiquity.

This result is necessarily based on the assumption that Archimedes was inspired by the invention of an architectural technique that allowed the construction of vaults and domes, surely an important technical novelty for its time. Despite this limitation, this result nonetheless allows us to have a better appreciation of the Archimedean work in general and a better understanding of the connection between Archimedes' reputation as inventor - as handed down by the historical tradition - and his mathematical works we have today. Above all, it allows us to place *The Method* fully in the context of what is now known about Greek mathematics, and not to see the various *trópoi* used by him as an anticipation of modern calculus, or even, as someone has proposed, the idea of infinite in Cantor's set theory.

Yet one question remains: was it Archimedes who invented the construction technique based on truncated cones in clay? Of course, it is possible; if Archimedes was consulted by Hieron concerning how to launch the famous *Syracosia*, and if he was Hieron's main advisor for the construction of war and defense machines, it is quite probable that he was, at the very least, aware of what was being built in Syracuse, and how - baths included.

Still, however attractive may seem the idea that he was the inventor of the vaults and domes, because of the lack of witnesses and documents, this question - we are afraid - will remain unanswered.

## Notes

1 We thank Sandra Lucore, director of the renewed Morgantina North Baths excavations, for information on this project. For the history of the U.S. excavations of this thermal complex, see Lucore 2009a, and in this volume. For a detailed discussion of the vaulted construction (dome and barrel vaults) that characterizes the North Baths, see especially Lucore 2009a.

[2] To have an idea of the vicissitudes of this codex before and after its rediscovery, the popular book Netz/Noel 2007 is useful. For detailed academic descriptions, see Netz/Noel/Tchernetska/wilson (eds) 2011.

[3] Quotations from Heath 1953, supplement p. 12.

[4] Note that the manuscript does not contain a perspective drawing, which we provide here for the convenience of the readers.

[5] The manuscript has no figure for this solid. Here we draw half of the solid; the other half exists behind the square LMNP.

[6] Piero considered the perpendicular (CE in *fig. 2*) to the axes of both cylinders, erected from the point where the axes meet (point E), then cut the intersection of the cylinders by various planes through this straight line (CE), and considered the section of the solid and of the circumscribed cube, ellipse and rectangle respectively. For a detailed examination of Piero's argument, see Gamba et al. 2006.

[7] This idea of Archimedes' 'platonic' detachment, sustained by Plutarch's portrayal, is a misleading stereotype that only in recent decades has begun to be seriously re-evaluated.

[8] For discussion of Greek mathematics, see Giusti 1999; Napolitani 2001; see also the introduction in Fried and Unguru 2001, 1-15.

[9] For bibliographic information on these sources and discussion of their credibility, see Dijksterhuis 1956, 28-29. More complete and critical examination can be found in Acerbi 2010, 157-163. There is no doubt that the story of the burning mirror of Archimedes is a later invention.

[10] See previous note.

[11] http://math.nyu.edu/crorres/Archimedes/Mirrors/legend/legend.html.

[12] Vitr. *De arch*. 9.9-12.

[13] Plut. Mor. 1094 B-C; *Non Posse suaviter vivi secundum Epicurum* (That Epicurus actually makes a pleasant life impossible); trans. B. Einarson/Ph. H. de Lacy, Cambridge Mass. 1967.

[14] See supra n. 1 on the excavations of the North Baths at Morgantina.

[15] Cultrera 1938. See Lucore 2009a for discussion of the evidence of the use of tubular vaulting at Syracuse.

[16] See supra n. 6.

[17] For Hippocrates' quadrature of lunes reported mainly by Simplicius, see Knorr 1986, 25-39.