CAPUT V.

DE
INTEGRATIONE FORMULARUM ANGULOS SINUSVE ANGULORUM IMPLICANTIUM.

Problema 23.

234.

Proposita formula differentiali $X\partial x$ Ang. sin. x, ejus integrale investigare.

Solutio.

Cum sit ∂. Ang. sin. $x = \frac{\partial x}{\sqrt{(1-xx)}}$, formula proposita ita in factores discerpatur: Ang.sin. $x \times X\partial x$. Si jam $X\partial x$ integrationem patiatur, sitque $\int X\partial x = P$, erit nostrum integrale $\int X\partial x$ Ang.sin. $x =$ P Ang. sin. $x - \int \frac{P\partial x}{\sqrt{(1-xx)}}$; itaque opus reductum est ad integrationem formulae algebraicae, pro qua supra praecepta sunt tradita.

Caeterum si fuerit $X = \frac{1}{\sqrt{(1-xx)}}$, manifestum est integrale fore $\int \frac{\partial x}{\sqrt{(1-xx)}}$ Ang. sin. $x = \frac{1}{2}$ (Ang. sin. $x)^2$; quo solo casu quadratum anguli in integrale ingreditur.

Exemplum 1.

235. *Hanc formulam* $\partial y = x^n \partial x$ Ang. sin. x *integrare.*

Cum sit $P = \int x^n \partial x = \frac{x^{n+1}}{n+1}$ habebimus

$$y = \frac{x^{n+1}}{n+1} \text{ Ang. sin. } x - \frac{1}{n+1}\int \frac{x^{n+1}\partial x}{\sqrt{(1-xx)}}.$$

Hinc pro variis valoribus ipsius n erunt integralia ope §. 120. eruta, ut sequentur:

$$\int \partial x \text{ Ang. sin. } x = x \text{ Ang. sin. } x + \sqrt{(1-xx)} - 1;$$

$$\int x \partial x \text{ Ang. sin. } x = \tfrac{1}{2} xx \text{ Ang. sin. } x + \tfrac{1}{4} x \sqrt{(1-xx)} - \tfrac{1}{4} \text{ Ang. sin. } x;$$

$$\int x^2 \partial x \text{ Ang. sin. } x = \tfrac{1}{3} x^3 \text{ Ang. sin. } x + \tfrac{1}{3}\left(\tfrac{1}{3} x^2 + \tfrac{2}{3}\right) \sqrt{(1-xx)} - \tfrac{1}{3} \cdot \tfrac{2}{3};$$

$$\int x^3 \partial x \text{ Ang. sin. } x = \tfrac{1}{4} x^4 \text{ Ang. sin. } x + \tfrac{1}{4}\left(\tfrac{1}{4} x^3 + \tfrac{1.3}{2.4} x\right) \sqrt{(1-xx)} - \tfrac{1}{4} \cdot \tfrac{1.3}{2.4} \text{ Ang. sin. } x;$$

quae ita sunt sumta, ut evanescant posito $x = 0$.

Exemplum 2.

236. *Hanc formulam* $\partial y = \frac{x \partial x}{\sqrt{(1-xx)}}$ Ang. sin. x *integrare.*

Cum sit $\int \frac{x \partial x}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)} = \mathrm{P}$, erit integrale quaesitum $y = \mathrm{C} - \sqrt{(1-xx)}$ Ang. sin. $x + \int \frac{\partial x \sqrt{(1-xx)}}{\sqrt{(1-xx)}}$, sicque habebitur:

$$y = \int \frac{x \partial x}{\sqrt{(1-xx)}} \text{ Ang. sin. } x = \mathrm{C} - \sqrt{(1-xx)} \text{ Ang. sin } x + x.$$

Exemplum 3.

237. *Hanc formulam* $\partial y = \frac{\partial x}{(1-xx)^{\frac{3}{2}}}$ Ang. sin. x *integrare.*

Hic est $\mathrm{P} = \int \frac{\partial x}{(1-xx)^{\frac{3}{2}}} = \frac{x}{\sqrt{(1-xx)}}$; unde fit

$$y = \frac{x}{\sqrt{(1-xx)}} \text{ Ang. sin. } x - \int \frac{x \partial x}{1-xx}, \text{ seu}$$

$$y = \int \frac{\partial x}{(1-xx)^{\frac{3}{2}}} \text{ Ang. sin. } x = \frac{x}{\sqrt{(1-xx)}} \text{ Ang. sin. } x + l\sqrt{(1-xx)},$$

quod integrale evanescit posito $x = 0$.

Scholion.

238. Simili modo integratur formula $\partial y = X\partial x$ Ang. cos. x. Cum enim sit ∂. Ang. cos. $x = \frac{-\partial x}{\sqrt{(1-xx)}}$, si ponamus $\int X\partial x = P$, erit $y = P$ Ang. cos $x + \int\frac{P\partial x}{\sqrt{(1-xx)}}$. Quin etiam si proponatur formula $\partial y = X\partial x$ Ang. tang. x, quia est ∂. Ang. tang. $x = \frac{\partial x}{1+xx}$, posito $\int X\partial x = P$, erit hoc integrale:

$$y = \int X\partial x \text{ Ang. tang. } x = P \text{ Ang. tang. } x - \int\frac{P\partial x}{1+xx}.$$

Quoties ergo $\int X\partial x$ algebraice dari potest, toties integratio reducitur ad formulam algebraicam, sicque negotium confectum est habendum. Cum igitur in his formulis angulus, cujus sinus, cosinus, vel tangens erat $= x$, inesset, consideremus etiam ejusmodi formulas, in quas quadratum hujus anguli, altiorve potestas ingreditur.

Problema 24.

239. Denotet Φ angulum, cujus sinus tangensve est functio quaedam ipsius x, unde fiat $\partial\Phi = u\partial x$, propositaque sit haec formula $\partial y = X\partial x \,.\, \Phi^n$ quam integrare oporteat.

Solutio.

Sit $\int X\partial x = P$, ut habeamus $\partial y = \Phi^n\partial P$, eritque integrando $y = \Phi^n P - n\int\Phi^{n-1} P u\partial x$. Jam simili modo sit $\int Pu\partial x = Q$, erit

$$\int\Phi^{n-1} P u\partial x = \Phi^{n-1} Q - (n-1)\int\Phi^{n-2} Q u\partial x,$$

tum posito $\int Q u\partial x = R$, erit

$$\int\Phi^{n-2} Q u\partial x = \Phi^{n-2} R - (n-2)\int\Phi^{n-3} R u\partial x.$$

Hocque modo potestas anguli Φ continuo deprimitur, donec tandem ad formulam ab angulo Φ liberam perveniatur: id quod semper eveniet, dummodo n sit numerus integer positivus, et haec integralia continuo sumere liceat $\int X\partial x = P$, $\int Pu\partial x = Q$, $\int Qu\partial x = R$, etc. quae integrationes, si non succedant, frustra integratio suscipitur.

Exemplum.

240. *Sit Φ angulus cujus sinus $= x$, ut sit $\partial\Phi = \frac{\partial x}{\sqrt{(1-xx)}}$, integrare formulam $\partial y = \Phi^n \partial x$.*

Erit ergo $X = 1$; $P = x$;

$Q = \int \frac{P\partial x}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)}$; $R = \int \frac{Q\partial x}{\sqrt{(1-xx)}} = -x$

$S = \int \frac{R\partial x}{\sqrt{(1-xx)}} = \sqrt{(1-xx)}$; $T = x$ etc.

quibus valoribus inventis reperietur:

$$y = \int \Phi^n \partial x = \Phi^n x + n\Phi^{n-1}\sqrt{(1-xx)} - n(n-1)\Phi^{n-2}x$$
$$- n(n-1)(n-2)\Phi^{n-3}\sqrt{(1-xx)} + \text{etc.}$$

Pro variis ergo valoribus exponentis n habebimus:

$$\int\Phi\partial x = \Phi x + \sqrt{(1-xx)} - 1;$$
$$\int\Phi^2\partial x = \Phi^2 x + 2\Phi\sqrt{(1-xx)} - 2.1x;$$
$$\int\Phi^3\partial x = \Phi^3 x + 3\Phi^2\sqrt{(1-xx)} - 3.2\Phi x - 3.2.1\sqrt{(1-xx)} + 6;$$
etc.

integralibus ita determinatis, ut evanescant posito $x = 0$.

Scholion.

241. Si sit $X\partial x = u\partial x = \partial\Phi$, formulae $\Phi^n\partial\Phi$ integrale est $\frac{1}{n+1}\Phi^{n+1}$; similique modo, si fuerit Φ functio quaecunque anguli Φ, formulae $\Phi u\partial x = \Phi\partial\Phi$ integratio nihil habet difficultatis. Multo latius patent formulae sinus, cosinusve angulorum et tangentes implicantes, quarum integratio per inversam Analysin amplissimum habet usum; cum praecipue Theoria Astronomiae ad hujusmodi formulas sit reducta. Prima autem fundamenta peti debent ex calculo differentiali, unde cum sit:

$$\partial.\sin.n\Phi = n\partial\Phi\cos.n\Phi;\quad \partial.\cos.n\Phi = -n\partial\Phi\sin.n\Phi;$$
$$\partial.\text{tang}.n\Phi = \frac{n\partial\Phi}{\cos.n\Phi^2};\quad \partial.\cot.n\Phi = \frac{-n\partial\Phi}{\sin.n\Phi^2};$$
$$\partial.\frac{1}{\sin.n\Phi} = \frac{-n\partial\Phi\cos.n\Phi}{\sin.n\Phi^2};\quad \partial.\frac{1}{\cos.n\Phi} = \frac{n\partial\Phi\sin.n\Phi}{\cos.n\Phi^2};$$

nanciscimur has integrationes elementares:

$$\int\partial\Phi\cos.n\Phi = \frac{1}{n}\sin.n\Phi;\ \int\partial\Phi\sin.n\Phi = -\frac{1}{n}\cos.n\Phi;$$
$$\int\frac{\partial\Phi}{\cos.n\Phi^2} = \frac{1}{n}\text{tang.}\,n\Phi;\ \int\frac{\partial\Phi}{\sin.n\Phi^2} = -\frac{1}{n}\cot.n\Phi;$$
$$\int\frac{\partial\Phi\cos.n\Phi}{\sin.n\Phi^2} = -\frac{1}{n\sin.n\Phi};\ \int\frac{\partial\Phi\sin.n\Phi}{\cos.n\Phi^2} = \frac{1}{n\cos.n\Phi};$$

unde statim hujusmodi formularum differentialium integratio

$$\partial\Phi(A + B\cos.\Phi + C\cos.2\Phi + D\cos.3\Phi + E\cos.4\Phi + \text{etc.})$$

consequitur, cum integrale manifesto sit

$$A\Phi + B\sin.\Phi + \tfrac{1}{2}C\sin.2\Phi + \tfrac{1}{3}D\sin.3\Phi + \tfrac{1}{4}E\sin.4\Phi + \text{etc.}$$

Deinde etiam in subsidium vocari convenit, quae in elementis de angulorum compositione traduntur: scilicet

$$\sin.\alpha.\sin.\beta = \tfrac{1}{2}\cos.(\alpha-\beta) - \tfrac{1}{2}\cos.(\alpha+\beta);$$
$$\cos.\alpha.\cos.\beta = \tfrac{1}{2}\cos.(\alpha-\beta) + \tfrac{1}{2}\cos.(\alpha+\beta);$$
$$\sin.\alpha.\cos.\beta = \tfrac{1}{2}\sin.(\alpha+\beta) + \tfrac{1}{2}\sin.(\alpha-\beta) = \tfrac{1}{2}\sin.(\alpha+\beta) - \tfrac{1}{2}\sin.(\beta-\alpha);$$

unde producta plurium sinuum et cosinuum in simplices sinus cosinusve resolvuntur.

Problema 25.

242. Formulae $\partial\Phi\sin.\Phi^n$ integrale investigare.

Solutio.

Repraesentetur in hos factores resoluta $\sin.\Phi^{n-1}.\ \partial\Phi\sin.\Phi$, et quia $\int\partial\Phi\sin.\Phi = -\cos.\Phi$, erit

$$\int\partial\Phi\sin.\Phi^n = -\sin.\Phi^{n-1}\cos.\Phi + (n-1)\int\partial\Phi\sin.\Phi^{n-2}\cos.\Phi^2.$$

Hinc ob $\cos.\Phi^2 = 1 - \sin.\Phi^2$, habebitur

$$\int\partial\Phi\sin.\Phi^n = -\sin.\Phi^{n-1}\cos.\Phi + (n-1)\int\partial\Phi\sin.\Phi^{n-2} - (n-1)\int\partial\Phi\sin.\Phi^n:$$

ubi cum postrema formula ipsi propositae sit similis, hinc colligitur ista reductio

$$\int \partial \phi \sin. \phi^n = -\tfrac{1}{n} \sin. \phi^{n-1} \cos. \phi + \tfrac{n-1}{n} \int \partial \phi \sin. \phi^{n-2},$$

qua integratio ad hanc formulam simpliciorem $\partial \phi \sin. \phi^{n-2}$ revocatur. Cum igitur casus simplissimi constent,

$$\int \partial \phi \sin. \phi^0 = \phi \text{ et } \int \partial \phi \sin. \phi = -\cos. \phi,$$

hinc via ad continuo majores exponentes n paratur:

$$\int \partial \phi \sin. \phi^0 = \phi$$
$$\int \partial \phi \sin. \phi = -\cos. \phi$$
$$\int \partial \phi \sin. \phi^2 = -\tfrac{1}{2} \sin. \phi \cos. \phi + \tfrac{1}{2} \phi$$
$$\int \partial \phi \sin. \phi^3 = -\tfrac{1}{3} \sin. \phi^2 \cos. \phi - \tfrac{2}{3} \cos. \phi$$
$$\int \partial \phi \sin. \phi^4 = -\tfrac{1}{4} \sin. \phi^3 \cos. \phi - \tfrac{1.3}{2.4} \sin. \phi \cos. \phi + \tfrac{1.3}{2.4} \phi$$
$$\int \partial \phi \sin. \phi^5 = -\tfrac{1}{5} \sin. \phi^4 \cos. \phi - \tfrac{1.4}{3.5} \sin. \phi^2 \cos. \phi - \tfrac{2.4}{3.5} \cos. \phi$$
$$\int \partial \phi \sin. \phi^6 = -\tfrac{1}{6} \sin. \phi^5 \cos. \phi - \tfrac{1.5}{4.6} \sin \phi^3 \cos. \phi - \tfrac{1.3.5}{2.4.6} \sin. \phi \cos. \phi + \tfrac{1.3.5}{2.4.6} \phi$$

etc.

Corollarium 1.

243. Quoties n est numerus impar, integrale per solum sinum et cosinum exhibetur, at si n est numerus par, integrale insuper ipsum augulum involvit, ideoque est functio transcendens.

Corollarium 2.

244. Casibus ergo quibus n est numerus impar, id imprimis notari convenit; etiamsi angulus seu arcus ϕ in infinitum crescat, integrale tamen nunquam ultra certum limitem excrescere posse, cum tamen si n sit numerus par, etiam in infinitum excrescat.

Scholion.

245. Simili modo formula $\partial \phi \cos. \phi^n$ tractatur, quae in hos factores resoluta $\cos. \phi^{n-1} . \partial \phi \cos. \phi$, praebet,

$$\int\partial\Phi\cos.\Phi^n = \cos.\Phi^{n-1}\sin.\Phi + (n-1)\int\partial\Phi\cos.\Phi^{n-2}\sin.\Phi^2$$
$$= \cos.\Phi^{n-1}\sin.\Phi + (n-1)\int\partial\Phi\cos.\Phi^{n-2} - (n-1)\int\partial\Phi\cos.\Phi^n:$$

unde cum postrema formula propositae sit similis, colligitur

$$\int\partial\Phi\cos.\Phi^n = \tfrac{1}{n}\sin.\Phi\cos.\Phi^{n-1} + \tfrac{n-1}{n}\int\partial\Phi\cos.\Phi^{n-2}.$$

Quare cum casibus $n = 0$, et $n = 1$ integratio sit in promptu, ad altiores potestates patet progressio;

$$\int\partial\Phi\cos.\Phi^0 = \Phi$$
$$\int\partial\Phi\cos.\Phi = \sin.\Phi$$
$$\int\partial\Phi\cos.\Phi^2 = \tfrac{1}{2}\sin.\Phi\cos.\Phi + \tfrac{1}{2}\Phi$$
$$\int\partial\Phi\cos.\Phi^3 = \tfrac{1}{3}\sin.\Phi\cos.\Phi^2 + \tfrac{2}{3}\sin.\Phi$$
$$\int\partial\Phi\cos.\Phi^4 = \tfrac{1}{4}\sin.\Phi\cos.\Phi^3 + \tfrac{1.3}{2.4}\sin.\Phi\cos.\Phi + \tfrac{1.3}{2.4}\Phi$$
$$\int\partial\Phi\cos.\Phi^5 = \tfrac{1}{5}\sin.\Phi\cos.\Phi^4 + \tfrac{1.4}{3.5}\sin.\Phi\cos.\Phi^2 + \tfrac{2.4}{3.5}\sin.\Phi$$
$$\int\partial\Phi\cos.\Phi^6 = \tfrac{1}{6}\sin.\Phi\cos.\Phi^5 + \tfrac{1.5}{4.6}\sin.\Phi\cos.\Phi^3 + \tfrac{1.3.5}{2.4.6}\sin.\Phi\cos.\Phi + \tfrac{1.3.5}{2.4.6}\Phi$$

etc.

Problema 26.

246. Formulae $\partial\Phi\sin.\Phi^m\cos.\Phi^n$ integrale invenire.

Solutio.

Quo hoc facilius praestetur, consideremus factum $\sin\Phi^\mu\cos.\Phi^\nu$, quod differentiatum fit $\mu\partial\Phi\sin.\Phi^{\mu-1}\cos.\Phi^{\nu+1} - \nu\partial\Phi\sin.\Phi^{\mu+1}\cos.\Phi^{\nu-1}$. Jam prout vel in parte priori $\cos.\Phi^2 = 1 - \sin.\Phi^2$, vel in posteriori $\sin\Phi^2 = 1 - \cos.\Phi^2$ statuitur, oritur

$$\text{vel } \partial.\sin.\Phi^\mu\cos.\Phi^\nu = \mu\partial\Phi\sin.\Phi^{\mu-1}\cos.\Phi^{\nu-1} - (\mu+\nu)\partial\Phi\sin.\Phi^{\mu+1}\cos.\Phi^{\nu-1},$$
$$\text{vel } \partial.\sin.\Phi^\mu\cos.\Phi^\nu = -\nu\partial\Phi\sin.\Phi^{\mu-1}\cos.\Phi^{\nu-1} + (\mu+\nu)\partial\Phi\sin.\Phi^{\mu-1}\cos.\Phi^{\nu+1}.$$

Hinc igitur duplicem reductionem adipiscimur:

I. $\int \partial\phi \sin.\phi^{\mu+1} \cos.\phi^{\nu-1} = -\frac{1}{\mu+\nu}\sin.\phi^{\mu}\cos.\phi^{\nu}$
$+\frac{\mu}{\mu+\nu}\int\partial\phi \sin.\phi^{\mu-1}\cos.\phi^{\nu-1}$

II. $\int \partial\phi \sin.\phi^{\mu-1} \cos.\phi^{\nu+1} = \frac{1}{\mu+\nu}\sin.\phi^{\mu}\cos.\phi^{\nu}$
$+\frac{\nu}{\mu+\nu}\int\partial\phi \sin.\phi^{\mu-1}\cos.\phi^{\nu-1}$

Quare formula proposita $\int\partial\phi \sin.\phi^{m}\cos.\phi^{n}$ successive continuo ad simpliciores potestates tam ipsius $\sin.\phi$ quam ipsius $\cos.\phi$ reducitur, donec alter vel penitus abeat, vel simpliciter adsit, quo casu integratio per se patet, cum sit

$\int\partial\phi \sin.\phi^{m}\cos.\phi = +\frac{1}{m+1}\sin.\phi^{m+1}$ et
$\int\partial\phi \sin.\phi \cos.\phi^{n} = -\frac{1}{n+1}\cos.\phi^{n+1}$.

Exemplum.

247. *Formulae* $\partial\phi \sin.\phi^{8}\cos.\phi^{7}$ *integrale invenire.*

Per priorem reductionem ob $\mu = 7$ et $\nu = 8$, impetramus

$\int\partial\phi \sin.\phi^{8}\cos.\phi^{7} = -\frac{1}{15}\sin.\phi^{7}\cos.\phi^{8} + \frac{7}{15}\int\partial\phi \sin.\phi^{6}\cos.\phi^{7}$;

istam per posteriorem reductionem tractemus:

$\int\partial\phi \sin.\phi^{6}\cos.\phi^{7} = \frac{1}{13}\sin.\phi^{7}\cos.\phi^{6} + \frac{6}{13}\int\partial\phi \sin.\phi^{6}\cos.\phi^{5}$,

hoc modo ulterius progrediamur:

$\int\partial\phi \sin.\phi^{6}\cos.\phi^{5} = -\frac{1}{11}\sin.\phi^{5}\cos.\phi^{6} + \frac{5}{11}\int\partial\phi \sin.\phi^{4}\cos.\phi^{5}$
$\int\partial\phi \sin.\phi^{4}\cos.\phi^{5} = \frac{1}{9}\sin.\phi^{5}\cos.\phi^{4} + \frac{4}{9}\int\partial\phi \sin.\phi^{4}\cos.\phi^{3}$
$\int\partial\phi \sin.\phi^{4}\cos.\phi^{3} = -\frac{1}{7}\sin.\phi^{3}\cos.\phi^{4} + \frac{3}{7}\int\partial\phi \sin.\phi^{2}\cos.\phi^{3}$
$\int\partial\phi \sin.\phi^{2}\cos.\phi^{3} = \frac{1}{5}\sin.\phi^{3}\cos.\phi^{2} + \frac{2}{5}\int\partial\phi \sin.\phi^{2}\cos.\phi$
$\int\partial\phi \sin.\phi^{2}\cos.\phi = -\frac{1}{3}\sin.\phi\cos.\phi^{2} + \frac{1}{3}\int\partial\phi\cos.\phi (+\frac{1}{3}\sin.\phi)$.

Ex his colligitur formulae propositae integrale

$$\int \partial \phi \sin. \phi^8 \cos. \phi^7$$
$$= -\frac{1}{15} \sin. \phi^7 \cos. \phi^8 + \frac{1 \cdot 7}{15 \cdot 13} \sin. \phi^7 \cos. \phi^6 - \frac{1 \cdot 7 \cdot 6}{15 \cdot 13 \cdot 11} \sin. \phi^5 \cos. \phi^6$$
$$+ \frac{1 \cdot 7 \cdot 6 \cdot 5}{15 \cdot 13 \cdot 11 \cdot 9} \sin. \phi^5 \cos. \phi^4 - \frac{1 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{15 \cdot 13 \cdot 11 \cdot 9 \cdot 7} \sin. \phi^3 \cos. \phi^4$$
$$+ \frac{1 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{15 \cdot 13 \cdot 11 \cdot 9 \cdot 7 \cdot 5} \sin. \phi^3 \cos. \phi^2 - \frac{1 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{15 \cdot 13 \cdot 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3} \sin. \phi \cos. \phi^2$$
$$+ \frac{1 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{15 \cdot 13 \cdot 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3} \sin. \phi.$$

Scholion.

248. Quando autem hujusmodi casus occurunt, semper praestat productum $\sin. \phi^m \cos. \phi^n$ in sinus vel cosinus angulorum multiplorum resolvere, quo facto singulae partes facillime integrantur. Caeterum hic brevitatis gratia angulum simpliciter littera ϕ indicavi, nihiloque res foret generalior, si per $\alpha\phi + \beta$ exprimeretur, quemadmodum etiam ante haec expressio Ang. sin. x aeque late patet, ac si loco x functio quaecunque scriberetur. Contemplemur ergo ejusmodi formulas, in quibus sinus cosinusve denominatorem occupant, ubi quidem simplicissimae sunt

$$\text{I. } \frac{\partial \phi}{\sin. \phi}; \text{ II. } \frac{\partial \phi}{\cos. \phi}; \text{ III. } \frac{\partial \phi \cos. \phi}{\sin. \phi}; \text{ IV. } \frac{\partial \phi \sin. \phi}{\cos. \phi};$$

quarum integralia imprimis nosse oportet. Pro prima adhibeantur hae transformationes

$$\frac{\partial \phi}{\sin. \phi} = \frac{\partial \phi \sin. \phi}{\sin. \phi^2} = \frac{\partial \phi \sin. \phi}{1 - \cos. \phi^2} = \frac{-\partial x}{1 - xx} \text{ (posito } \cos. \phi = x\text{)},$$

unde fit

$$\int \frac{\partial \phi}{\sin. \phi} = -\frac{1}{2} l \frac{1 + x}{1 - x} = -\frac{1}{2} l \frac{1 + \cos. \phi}{1 - \cos. \phi}.$$

Pro secunda

$$\frac{\partial \phi}{\cos. \phi} = \frac{\partial \phi \cos. \phi}{\cos. \phi^2} = \frac{\partial \phi \cos. \phi}{1 - \sin. \phi^2} = \frac{\partial x}{1 - xx} \text{ (posito } \sin. \phi = x\text{)}$$

ergo

$$\int \frac{\partial \phi}{\cos. \phi} = \frac{1}{2} l \frac{1 + x}{1 - x} = \frac{1}{2} l \frac{1 + \sin. \phi}{1 - \sin. \phi}.$$

Tertiae et quartae integratio manifesto logarithmis conficitur: quare haec integralia probe notasse juvabit

I. $\int \frac{\partial \phi}{\sin. \phi} = -\frac{1}{2} l \frac{1+\cos. \phi}{1-\cos. \phi} = l \frac{\sqrt{(1-\cos. \phi)}}{\sqrt{(1+\cos. \phi)}} = l \operatorname{tang.} \frac{1}{2} \phi$,

II. $\int \frac{\partial \phi}{\cos. \phi} = \frac{1}{2} l \frac{1+\sin. \phi}{1-\sin. \phi} = l \frac{\sqrt{(1+\sin. \phi)}}{\sqrt{(1-\sin. \phi)}} = l \operatorname{tang.} (45^\circ + \frac{1}{2}\phi)$,

III. $\int \frac{\partial \phi \cos. \phi}{\sin. \phi} = l \sin. \phi = \int \frac{\partial \phi}{\operatorname{tang.} \phi} = \int \partial \phi \cot. \phi$

IV. $\int \frac{\partial \phi \sin. \phi}{\cos. \phi} = -l \cos. \phi = \int \partial \phi \operatorname{tang.} \phi$

hincque sequitur III. + IV.

$$\int \frac{\partial \phi}{\sin. \phi \cos. \phi} = l \frac{\sin. \phi}{\cos. \phi} = l . \operatorname{tang.} \phi.$$

Problema 27.

249. Formularum $\frac{\partial \phi \sin. \phi^m}{\cos. \phi^n}$ et $\frac{\partial \phi \cos. \phi^m}{\sin. \phi^n}$ integralia investigare.

Solutio.

Primo statim perspicitur, alteram formulam in alteram transmutari, posito $\phi = 90^\circ - \psi$, quia tum fit $\sin. \phi = \cos. \psi$ et $\cos. \phi = \sin. \psi$, dummodo notetur fore $\partial \phi = -\partial \psi$. Quare sufficit priorem tantum tractasse. Reductio autem prior §. 246. data, sumto $\mu + 1 = m$ et $\nu - 1 = -n$, praebet

$$\int \frac{\partial \phi \sin. \phi^m}{\cos. \phi^n} = -\frac{1}{m-n} \cdot \frac{\sin. \phi^{m-1}}{\cos. \phi^{n-1}} + \frac{m-1}{m-n} \int \frac{\partial \phi \sin. \phi^{m-2}}{\cos. \phi^n}$$

quo pacto in numeratore exponens ipsius $\sin. \phi$ continuo binario deprimitur, ita ut tandem perveniatur vel ad $\int \frac{\partial \phi}{\cos. \phi^n}$ vel ad $\int \frac{\partial \phi \sin. \phi}{\cos \phi^n} = \frac{1}{(n-1) \cos. \phi^{n-1}}$: ideoque sola formula $\int \frac{\partial \phi}{\cos. \phi^n}$ tractanda supersit. Altera autem reductio ibidem tradita (246.) sumto $\mu - 1 = m$ et $\nu - 1 = -n$, dat

$$\int \frac{\partial \phi \sin. \phi^m}{\cos. \phi^{n-2}} = \frac{1}{m-n+2} \cdot \frac{\sin. \phi^{m+1}}{\cos. \phi^{n-1}} - \frac{n-1}{m-n+2} \int \frac{\partial \phi \sin. \phi^m}{\cos. \phi^n}:$$

unde colligitur

$$\int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi^n} = \frac{1}{n-1} \cdot \frac{\sin. \Phi^{m+1}}{\cos. \Phi^{n-1}} - \frac{m-n+2}{n-1} \int \frac{\partial \Phi \sin. \Phi^m}{\cos \Phi^{n-2}},$$

cujus reductionis ope exponens ipsius cos. Φ in denominatore continuo binario deprimitur, ita ut tandem vel ad $\int \partial \Phi \sin. \Phi^m$ vel ad $\int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi}$ perveniatur. Illius integratio jam supra est monstrata, hujus vero forma, si $m > 1$, per priorem reductionem tandem vel ad $\int \frac{\partial \Phi}{\cos. \Phi}$ vel ad $\int \frac{\partial \Phi \sin. \Phi}{\cos. \Phi}$ revocatur, illius autem integrale est $l \text{ tang.} (45^\circ + \frac{1}{2}\Phi)$, hujus vero $- l \cos. \Phi$.

Corollarium 1.

250. Prior reductio non habet locum, quoties est $m = n$; hoc scilicet casu formula $\int \frac{\partial \Phi \sin. \Phi^n}{\cos. \Phi^n}$ non reduci potest ad formulam $\int \frac{\partial \Phi \sin. \Phi^{n-2}}{\cos. \Phi^n}$. Altera autem reductione semper uti licet, etsi enim casus $n = 1$ inde excluditur, ejus tamen integratio per priorem effici potest.

Corollarium 2.

251. Ratio autem illius exclusionis in hoc est posita, quod formula $\int \frac{\partial \Phi \sin. \Phi^{n-2}}{\cos. \Phi^n}$ est absolute integrabilis, habens integrale $= \frac{1}{n-1} \cdot \frac{\sin. \Phi^{n-1}}{\cos. \Phi^{n-1}}$. Erit ergo pro his casibus:

$$\int \frac{\partial \Phi}{\cos. \Phi^2} = \frac{\sin. \Phi}{\cos. \Phi} = \text{tang. } \Phi;$$

$$\int \frac{\partial \Phi \sin. \Phi}{\cos. \Phi^3} = \frac{1}{2} \cdot \frac{\sin. \Phi^2}{\cos. \Phi^2} = \frac{1}{2} \text{ tang. } \Phi^2;$$

$$\int \frac{\partial \Phi \sin. \Phi^2}{\cos. \Phi^4} = \tfrac{1}{3} \cdot \frac{\sin. \Phi^3}{\cos. \Phi^3} = \tfrac{1}{3} \text{tang}. \Phi^3;$$

$$\int \frac{\partial \Phi \sin. \Phi^3}{\cos. \Phi^5} = \tfrac{1}{4} \cdot \frac{\sin. \Phi^4}{\cos. \Phi^4} = \tfrac{1}{4} \text{tang}. \Phi^4.$$

etc.

Exemplum 1.

252. *Formulae* $\frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi}$ *integrale assignare.*

Prior reductio dat:

$$\int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi} = \frac{-1}{m-1} \sin. \Phi^{m-1} + \int \frac{\partial \Phi \sin. \Phi^{m-2}}{\cos. \Phi}.$$

Hinc a casibus per se notis incipiendo, habebimus:

$$\int \frac{\partial \Phi}{\cos. \Phi} = l \text{ tang}. (45^\circ + \tfrac{1}{2} \Phi)$$

$$\int \frac{\partial \Phi \sin. \Phi}{\cos. \Phi} = - l \cos. \Phi = l \sec. \Phi$$

$$\int \frac{\partial \Phi \sin. \Phi^2}{\cos. \Phi} = - \sin. \Phi + \int \frac{\partial \Phi}{\cos. \Phi}$$

$$\int \frac{\partial \Phi \sin. \Phi^3}{\cos. \Phi} = - \tfrac{1}{2} \sin. \Phi^2 + l \sec. \Phi$$

$$\int \frac{\partial \Phi \sin. \Phi^4}{\cos. \Phi} = - \tfrac{1}{3} \sin. \Phi^3 - \sin. \Phi + \int \frac{\partial \Phi}{\cos. \Phi}$$

$$\int \frac{\partial \Phi \sin. \Phi^5}{\cos. \Phi} = - \tfrac{1}{4} \sin. \Phi^4 - \tfrac{1}{2} \sin. \Phi^2 + l \sec. \Phi$$

$$\int \frac{\partial \Phi \sin. \Phi^6}{\cos. \Phi} = - \tfrac{1}{5} \sin. \Phi^5 - \tfrac{1}{3} \sin. \Phi^3 - \sin. \Phi + \int \frac{\partial \Phi}{\cos. \Phi}$$

$$\int \frac{\partial \Phi \sin. \Phi^7}{\cos. \Phi} = - \tfrac{1}{6} \sin. \Phi^6 - \tfrac{1}{4} \sin. \Phi^4 - \tfrac{1}{2} \sin. \Phi^2 + l \sec. \Phi$$

etc.

Scholion.

253. Pro reliquis casibus denominatoris totum negotium conficietur his reductionibus:

$$\int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi^2} = \frac{\sin. \Phi^{m+1}}{\cos. \Phi} - m \int \partial \Phi \sin. \Phi^m$$

$$\int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi^3} = \tfrac{1}{2} \cdot \frac{\sin. \Phi^{m+1}}{\cos. \Phi^2} - \frac{m-1}{2} \int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi}$$

$$\int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi^4} = \tfrac{1}{3} \cdot \frac{\sin. \Phi^{m+1}}{\cos. \Phi^3} - \frac{m-2}{3} \int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi^2}$$

$$\int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi^5} = \tfrac{1}{4} \cdot \frac{\sin. \Phi^{m+1}}{\cos. \Phi^4} - \frac{m-3}{4} \int \frac{\partial \Phi \sin. \Phi^m}{\cos. \Phi^3}$$

etc.

Exemplum 2.

254. *Formulae* $\frac{\partial \Phi}{\cos. \Phi^n}$ *integrale assignare.*

Altera reductio ob $m = 0$ fit

$$\int \frac{\partial \Phi}{\cos. \Phi^n} = \frac{1}{n-1} \cdot \frac{\sin. \Phi}{\cos. \Phi^{n-1}} + \frac{n-2}{n-1} \int \frac{\partial \Phi}{\cos. \Phi^{n-2}};$$

quia jam casus simplicissimi

$$\int \partial \Phi = \Phi \text{ et } \int \frac{\partial \Phi}{\cos. \Phi} = l \operatorname{tang.} (45^\circ + \tfrac{1}{2} \Phi)$$

sunt cogniti, ad eos sequentes omnes revocabuntur:

$$\int \frac{\partial \Phi}{\cos. \Phi^2} = \frac{\sin. \Phi}{\cos. \Phi}$$

$$\int \frac{\partial \Phi}{\cos. \Phi^3} = \tfrac{1}{2} \cdot \frac{\sin. \Phi}{\cos. \Phi^2} + \tfrac{1}{2} \int \frac{\partial \Phi}{\cos. \Phi}$$

$$\int \frac{\partial \Phi}{\cos. \Phi^4} = \tfrac{1}{3} \cdot \frac{\sin. \Phi}{\cos. \Phi^3} + \tfrac{2}{3} \cdot \frac{\sin. \Phi}{\cos. \Phi}$$

$$\int \frac{\partial \Phi}{\cos. \Phi^5} = \tfrac{1}{4} \cdot \frac{\sin. \Phi}{\cos. \Phi^4} + \frac{1.3}{2.4} \cdot \frac{\sin. \Phi}{\cos. \Phi^2} + \frac{1.3}{2.4} \int \frac{\partial \Phi}{\cos. \Phi}$$

$$\int \frac{\partial \Phi}{\cos. \Phi^6} = \tfrac{1}{5} \cdot \frac{\sin. \Phi}{\cos. \Phi^5} + \frac{1.4}{3.5} \cdot \frac{\sin. \Phi}{\cos. \Phi^3} + \frac{2.4}{3.5} \cdot \frac{\sin. \Phi}{\cos. \Phi}$$

etc.

Corollarium 1.

255. Simili modo habebimus has integrationes:

$$\int \frac{\partial \Phi}{\sin. \Phi} = l \operatorname{tang.} \tfrac{1}{2} \Phi; \quad \int \frac{\partial \Phi}{\sin. \Phi^2} = -\frac{\cos. \Phi}{\sin. \Phi};$$

$$\int \frac{\partial \Phi}{\sin. \Phi^3} = -\tfrac{1}{2} \cdot \frac{\cos. \Phi}{\sin. \Phi^2} + \tfrac{1}{2} \int \frac{\partial \Phi}{\sin. \Phi}$$

$$\int \frac{\partial \Phi}{\sin. \Phi^4} = -\tfrac{1}{3} \cdot \frac{\cos. \Phi}{\sin. \Phi^3} - \tfrac{2}{3} \cdot \frac{\cos. \Phi}{\sin. \Phi}$$

$$\int \frac{\partial \Phi}{\sin. \Phi^5} = -\tfrac{1}{4} \cdot \frac{\cos. \Phi}{\sin. \Phi^4} - \frac{1.3}{2.4} \cdot \frac{\cos. \Phi}{\sin. \Phi^2} + \frac{1.3}{2.4} \int \frac{\partial \Phi}{\sin. \Phi}$$

etc.

Corollarium 2.

256. Deinde est

$$\int \frac{\partial \Phi \sin. \Phi}{\cos. \Phi^n} = \frac{1}{n-1} \cdot \frac{1}{\cos. \Phi^{n-1}}; \text{ et}$$

$$\int \frac{\partial \Phi \cos. \Phi}{\sin. \Phi^n} = \frac{-1}{n-1} \cdot \frac{1}{\sin. \Phi^{n-1}}.$$

Porro

$$\int \frac{\partial \Phi \sin. \Phi^2}{\cos. \Phi^n} = \int \frac{\partial \Phi}{\cos. \Phi^n} - \int \frac{\partial \Phi}{\cos. \Phi^{n-2}};$$

$$\int \frac{\partial \Phi \cos. \Phi^2}{\sin. \Phi^n} = \int \frac{\partial \Phi}{\sin. \Phi^n} - \int \frac{\partial \Phi}{\sin. \Phi^{n-2}}$$

et $$\int \frac{\partial\phi \sin.\phi^3}{\cos.\phi^n} = \int \frac{\partial\phi \sin.\phi}{\cos.\phi^n} - \int \frac{\partial\phi \sin.\phi}{\cos.\phi^{n-2}};$$

$$\int \frac{\partial\phi \cos.\phi^3}{\sin.\phi^n} = \int \frac{\partial\phi \cos.\phi}{\sin.\phi^n} - \int \frac{\partial\phi \cos.\phi}{\sin.\phi^{n-2}};$$

quibus reductionibus continuo ulterius progredi licet.

Problema 28.

257. Formulae $\frac{\partial\phi}{\sin.\phi^m \cos.\phi^n}$ integrale investigare.

Solutio.

Reductiones supra adhibitas huc accommodare licet, sumendo in praecedente problemate m negative: ita erit

$$\int \frac{\partial\phi}{\sin.\phi^m \cos.\phi^n} = + \frac{1}{m+n} \cdot \frac{1}{\sin.\phi^{m+1} \cos.\phi^{n-1}} + \frac{m+1}{m+n} \int \frac{\partial\phi}{\sin.\phi^{m+2} \cos.\phi^n},$$

unde loco m scribendo $m-2$, per conversionem fit

$$\int \frac{\partial\phi}{\sin.\phi^m \cos.\phi^n} = - \frac{1}{m-1} \cdot \frac{1}{\sin.\phi^{m-1} \cos.\phi^{n-1}} + \frac{m+n-2}{m-1} \int \frac{\partial\phi}{\sin.\phi^{m-2} \cos.\phi^n}$$

Altera huic similis est

$$\int \frac{\partial\phi}{\sin.\phi^m \cos.\phi^n} = \frac{1}{n-1} \cdot \frac{1}{\sin.\phi^{m-1} \cos.\phi^{n-1}} + \frac{m+n-2}{n-1} \int \frac{\partial\phi}{\sin.\phi^m \cos.\phi^{n-2}}$$

Cum jam in hoc genere formae simplicissimae sint:

$$\int \frac{\partial \Phi}{\sin. \Phi} = l. \text{ tang. } \tfrac{1}{2} \Phi; \quad \int \frac{\partial \Phi}{\cos. \Phi} = l. \text{ tang. } (45^\circ + \tfrac{1}{2} \Phi);$$

$$\int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi} = l. \text{tang.} \Phi; \quad \int \frac{\partial \Phi}{\sin. \Phi^2} = -\cot. \Phi; \quad \int \frac{\partial \Phi}{\cos. \Phi^2} = \text{tang. } \Phi;$$

hinc magis compositas eliciemus:

$$\int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^2} = \frac{1}{\cos. \Phi} + \int \frac{\partial \Phi}{\sin. \Phi};$$

$$\int \frac{\partial \Phi}{\sin. \Phi^2 \cos. \Phi} = -\frac{1}{\sin. \Phi} + \int \frac{\partial \Phi}{\cos. \Phi};$$

$$\int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^4} = \tfrac{1}{3} \cdot \frac{1}{\cos. \Phi^3} + \int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^2};$$

$$\int \frac{\partial \Phi}{\sin. \Phi^4 \cos. \Phi} = -\tfrac{1}{3} \cdot \frac{1}{\sin. \Phi^3} + \int \frac{\partial \Phi}{\sin. \Phi^2 \cos. \Phi};$$

$$\int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^6} = \tfrac{1}{5} \cdot \frac{1}{\cos. \Phi^5} + \int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^4};$$

$$\int \frac{\partial \Phi}{\sin. \Phi^6 \cos. \Phi} = -\tfrac{1}{5} \cdot \frac{1}{\sin. \Phi^5} + \int \frac{\partial \Phi}{\sin. \Phi^4 \cos. \Phi};$$

$$\int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^3} = \tfrac{1}{2} \cdot \frac{1}{\cos. \Phi^2} + \int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi};$$

$$\int \frac{\partial \Phi}{\sin. \Phi^3 \cos. \Phi} = -\tfrac{1}{2} \cdot \frac{1}{\sin. \Phi^2} + \int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi};$$

$$\int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^5} = \tfrac{1}{4} \cdot \frac{1}{\cos. \Phi^4} + \int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^3};$$

$$\int \frac{\partial \Phi}{\sin. \Phi^5 \cos. \Phi} = -\tfrac{1}{4} \cdot \frac{1}{\sin. \Phi^4} + \int \frac{\partial \Phi}{\sin. \Phi^3 \cos. \Phi};$$

$$\int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^7} = \tfrac{1}{6} \cdot \frac{1}{\cos. \Phi^6} + \int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi^5};$$

$$\int \frac{\partial \Phi}{\sin. \Phi^7 \cos. \Phi} = -\tfrac{1}{6} \cdot \frac{1}{\sin. \Phi^6} + \int \frac{\partial \Phi}{\sin. \Phi^5 \cos. \Phi};$$

etc.

$$\int \frac{\partial \Phi}{\sin.\Phi \cos.\Phi^2} = \frac{1}{\sin.\Phi\cos.\Phi} + 2\int \frac{\partial \Phi}{\sin.\Phi^2} = -\frac{1}{\sin \Phi\cos.\Phi} + 2\int \frac{\partial \Phi}{\cos.\Phi^2}$$

$$\int \frac{\partial \Phi}{\sin.\Phi^2 \cos.\Phi^4} = \tfrac{1}{3} \cdot \frac{1}{\sin.\Phi\cos.\Phi^3} + \tfrac{4}{3}\int \frac{\partial \Phi}{\sin.\Phi^2\cos.\Phi^2}$$

$$\int \frac{\partial \Phi}{\sin.\Phi^4 \cos.\Phi^2} = -\tfrac{1}{3} \cdot \frac{1}{\sin.\Phi^3\cos.\Phi} + \tfrac{4}{3}\int \frac{\partial \Phi}{\sin.\Phi^2\cos.\Phi^2}.$$

Sicque formulae quantumvis compositae ad simpliciores, quarum integratio est in promtu, reducuntur.

Corollarium 1.

258. Ambo exponentes ipsius sin. Φ et cos. Φ simul binario minui possunt: erit enim per priorem reductionem

$$\int \frac{\partial \Phi}{\sin.\Phi^\mu \cos.\Phi^\nu} = -\frac{1}{\mu-1} \cdot \frac{1}{\sin.\Phi^{\mu-1} \cos.\Phi^{\nu-1}} + \frac{\mu+\nu-2}{\mu-1}\int \frac{\partial \Phi}{\sin.\Phi^{\mu-2} \cos.\Phi^\nu}$$

nunc haec formula per posteriorem ob $m = \mu - 2$ et $n = \nu$ dat

$$\int \frac{\partial \Phi}{\sin.\Phi^{\mu-2} \cos.\Phi^\nu} = \frac{1}{\nu-1} \cdot \frac{1}{\sin.\Phi^{\mu-3} \cos.\Phi^{\nu-1}} + \frac{\mu+\nu-4}{\nu-1}\int \frac{\partial \Phi}{\sin.\Phi^{\mu-2} \cos.\Phi^{\nu-2}}$$

unde concluditur

$$\int \frac{\partial \Phi}{\sin.\Phi^\mu \cos.\Phi^\nu} = -\frac{1}{\mu-1} \cdot \frac{1}{\sin.\Phi^{\mu-1} \cos.\Phi^{\nu-1}} + \frac{\mu+\nu-2}{(\mu-1)(\nu-1)} \cdot \frac{1}{\sin.\Phi^{\mu-3} \cos.\Phi^{\nu-1}} + \frac{(\mu+\nu-2)(\mu+\nu-4)}{(\mu-1)(\nu-1)}\int \frac{\partial \Phi}{\sin.\Phi^{\mu-2} \cos.\Phi^{\nu-2}}$$

Corollarium 2.

259. Prioribus membris ad communem denominatorem reductis obtinebitur

$$\int \frac{\partial \Phi}{\sin. \Phi^{\mu} \cos. \Phi^{\nu}} = \frac{(\mu - 1) \sin. \Phi^2 - (\nu - 1) \cos. \Phi^2}{(\mu - 1)(\nu - 1) \sin. \Phi^{\mu-1} \cos. \Phi^{\nu-1}}$$
$$+ \frac{(\mu + \nu - 2)(\mu + \nu - 4)}{(\mu - 1)(\nu - 1)} \int \frac{\partial \Phi}{\sin. \Phi^{\mu-2} \cos. \Phi^{\nu-2}}$$

qua reductione semper ad calculum contrahendum uti licet, nisi vel $\mu = 1$ vel $\nu = 1$.

Scholion.

260. Hujusmodi formulae $\frac{\partial \Phi}{\sin. \Phi^m \cos. \Phi^n}$ etiam hoc modo maxime obvio ad simpliciores reduci possunt; dum numerator per $\sin. \Phi^2 + \cos. \Phi^2 = 1$ multiplicatur, unde fit

$$\int \frac{\partial \Phi}{\sin. \Phi^m \cos. \Phi^n} = \int \frac{\partial \Phi}{\sin. \Phi^{m-2} \cos. \Phi^n} + \int \frac{\partial \Phi}{\sin. \Phi^m \cos. \Phi^{n-2}}$$

quae eousque continuari potest, donec in denominatore unica tantum potestas relinquatur. Ita erit

$$\int \frac{\partial \Phi}{\sin. \Phi \cos. \Phi} = \int \frac{\partial \Phi \sin. \Phi}{\cos. \Phi} + \int \frac{\partial \Phi \cos. \Phi}{\sin. \Phi} = l \frac{\sin. \Phi}{\cos. \Phi}$$
$$\int \frac{\partial \Phi}{\sin. \Phi^2 \cos. \Phi^2} = \int \frac{\partial \Phi}{\sin. \Phi^2} + \int \frac{\partial \Phi}{\cos. \Phi^2} = \frac{\sin. \Phi}{\cos. \Phi} - \frac{\cos. \Phi}{\sin. \Phi}.$$

Quodsi proposita sit haec formula $\int \frac{\partial \Phi}{\sin. \Phi^n \cos. \Phi^n}$, in subsidium vocari potest, esse $\sin. \Phi \cos. \Phi = \frac{1}{2} \sin. 2\Phi$; unde habetur $\int \frac{2^n \partial \Phi}{\sin. 2\Phi^n} = 2^{n-1} \int \frac{\partial \omega}{\sin. \omega^n}$, posito $\omega = 2\Phi$, quae formula per superiora praecepta resolvitur. His igitur adminiculis observatis circa formulam $\partial \Phi \sin. \Phi^m \cos. \Phi^n$, si quidem m et n fuerint numeri integri sive positivi sive negativi, nihil amplius desideratur: sin autem fuerint numeri fracti, nihil admodum praecipiendum occurrit, quandoquidem casus, quibus integratio succedit, quasi sponte se produnt. Quemadmodum autem integralia, quae exhiberi nequeunt, per series exprimi conveniat, in capite sequente accuratius exponamus.

mus. Nunc vero formulas fractas consideremus, quarum denominator est $a+b\cos.\phi$ ejusque potestas, tales enim formulae in Theoria Astronomiae frequentissime occurrunt.

Problema 29.

261. Formulae differentialis $\frac{\partial\phi}{a+b\cos.\phi}$ integrale investigare.

Solutio.

Haec investigatio commodius institui nequit, quam ut formula proposita ad formam ordinariam reducatur, ponendo $\cos\phi = \frac{1-xx}{1+xx}$, ut rationaliter fiat $\sin.\phi = \frac{2x}{1+xx}$, hincque $\partial\phi\cos.\phi = \frac{2\partial x(1-xx)}{(1+xx)^2}$, sicque $\partial\phi = \frac{2\partial x}{1+xx}$. Quia igitur $a+b\cos.\phi = \frac{a+b+(a-b)xx}{1+xx}$, erit formula nostra $\frac{\partial\phi}{a+b\cos.\phi} = \frac{2\partial x}{a+b+(a-b)xx}$, quae prout fuerit $a>b$ vel $a<b$, vel angulum vel logarithmum praebet.

Casu $a>b$ reperitur

$$\int\frac{\partial\phi}{a+b\cos.\phi} = \frac{2}{\sqrt{(aa-bb)}}\text{Arc. tang.}\frac{(a-b)x}{\sqrt{(aa-bb)}}:$$

casu $a<b$ vero est

$$\int\frac{\partial\phi}{a+b\cos.\phi} = \frac{1}{\sqrt{(bb-aa)}}\, l\frac{\sqrt{(bb-aa)}+x(b-a)}{\sqrt{(bb-aa)}-x(b-a)}.$$

Nunc vero est

$$x = \sqrt{\frac{1-\cos.\phi}{1+\cos.\phi}} = \text{tang.}\tfrac{1}{2}\phi = \frac{\sin.\phi}{1+\cos.\phi};$$

qua restitutione facta, cum sit

$$2\text{ Ang. tang.}\frac{(a-b)x}{\sqrt{(aa-bb)}} = \text{Ang. tang.}\frac{2x\sqrt{(aa-bb)}}{a+b-(a-b)xx}$$
$$= \text{Ang. tang.}\frac{2\sin.\phi\sqrt{(aa-bb)}}{(a+b)(1+\cos.\phi)-(a-b)(1-\cos.\phi)}$$
$$= \text{Ang. tang.}\frac{\sin.\phi\sqrt{(aa-bb)}}{a\cos.\phi+b}.$$

Quocirca pro casu $a>b$ adipiscimur:

$$\int\frac{\partial\phi}{a+b\cos.\phi} = \frac{1}{\sqrt{(aa-bb)}}\text{Ang. tang.}\frac{\sin.\phi\sqrt{(aa-bb)}}{a\cos.\phi+b}, \text{ seu}$$
$$\int\frac{\partial\phi}{a+b\cos.\phi} = \frac{1}{\sqrt{(aa-bb)}}\text{Ang. sin.}\frac{\sin.\phi\sqrt{(aa-bb)}}{a+b\cos.\phi}, \text{ sive}$$
$$\int\frac{\partial\phi}{a+b\cos.\phi} = \frac{1}{\sqrt{(aa-bb)}}\text{Ang. cos.}\frac{a\cos.\phi+b}{a+b\cos.\phi}.$$

Pro casu autem $a < b$:

$$\int \frac{\partial \Phi}{a + b \cos.\Phi} = \frac{1}{\sqrt{(bb - aa)}} l \frac{\sqrt{(b+a)(1+\cos.\Phi)} + \sqrt{(b-a)(1-\cos.\Phi)}}{\sqrt{(b+a)(1+\cos.\Phi)} - \sqrt{(b-a)(1-\cos.\Phi)}},$$

seu

$$\int \frac{\partial \Phi}{a + b \cos.\Phi} = \frac{1}{\sqrt{(bb - aa)}} l \frac{a \cos.\Phi + b + \sin.\Phi \cdot \sqrt{(bb - aa)}}{a + b \cos.\Phi}.$$

At casu $b = a$, integrale est $= \frac{x}{a} = \frac{1}{a}$ tang. $\frac{1}{2}\Phi$, unde fit

$$\int \frac{\partial \Phi}{1 + \cos.\Phi} = \text{tang.} \tfrac{1}{2}\Phi = \frac{\sin.\Phi}{1 + \cos.\Phi},$$

quae integralia evanescunt facto $\Phi = 0$.

Corollarium 1.

262. Formulae autem $\frac{\partial \Phi \sin.\Phi}{a + b \cos.\Phi} = \frac{-\partial.\cos.\Phi}{a + b \cos.\Phi}$ integrale est $\frac{1}{b} l \frac{a + b}{a + b \cos.\Phi}$, ita sumtum, ut evanescat posito $\Phi = 0$; sicque habebimus:

$$\int \frac{\partial \Phi \sin.\Phi}{a + b \cos.\Phi} = \frac{1}{b} l \frac{a + b}{a + b \cos.\Phi}.$$

Corollarium 2.

263. Formula autem $\frac{\partial \Phi \cos.\Phi}{a + b \cos.\Phi}$ transformatur in $\frac{\partial \Phi}{b} - \frac{a \partial \Phi}{b(a + b \cos.\Phi)}$, unde integrale per solutionem problematis exhiberi potest:

$$\int \frac{\partial \Phi \cos.\Phi}{a + b \cos.\Phi} = \frac{\Phi}{b} - \frac{a}{b} \int \frac{\partial \Phi}{a + b \cos.\Phi}.$$

Scholion 1.

264. Integratione hac inventa, etiam hujus formulae $\frac{\partial \Phi}{(a + b \cos.\Phi)^n}$ integrale inveniri potest, existente n numero integro; quod fingendo integralis forma commodissime praestari videtur: ponatur

$$\int \frac{\partial \Phi}{(a + b \cos.\Phi)^2} = \frac{A \sin.\Phi}{a + b \cos.\Phi} + m \int \frac{\partial \Phi}{a + b \cos.\Phi};$$

ac reperitur

$A = \frac{-b}{aa-bb}$, et $m = \frac{a}{aa-bb}$. Porro fingatur

$$\int \frac{\partial \Phi}{(a+b\cos.\Phi)^3} = \frac{(A+B\cos.\Phi)\sin.\Phi}{(a+b\cos.\Phi)^2} + m\int \frac{\partial \Phi}{(a+b\cos.\Phi)^2};$$

reperiturque

$$A = \frac{-b}{aa-bb};\ B = \frac{-bb}{2a(aa-bb)};\ m = \frac{2aa+bb}{2a(aa-bb)};$$

similique modo investigatio ad majores potestates continuari potest, labore quidem non parum taedioso. Sequenti autem modo negotium facillime expediri videtur.

Consideretur scilicet formula generalior $\frac{\partial \Phi\,(f+g\cos.\Phi)}{(a+b\cos.\Phi)^{n+1}}$, ac ponatur

$$\int \frac{\partial \Phi(f+g\cos.\Phi)}{(a+b\cos.\Phi)^{n+1}} = \frac{A\sin.\Phi}{(a+b\cos.\Phi)^n} + \int \frac{\partial \Phi\,(B+C\cos.\Phi)}{(a+b\cos.\Phi)^n},$$

sumtisque differentialibus, ista prodibit aequatio:

$$f+g\cos.\Phi = A\cos.\Phi\,(a+b\cos.\Phi) + nAb\sin.\Phi^2$$
$$+ (B+C\cos.\Phi)(a+b\cos.\Phi);$$

quae ob $\sin.\Phi^2 = 1 - \cos.\Phi^2$ hanc formam induit

$$\left.\begin{array}{l} -f \quad - \quad g\cos.\Phi + Ab\cos.\Phi^2 \\ +nAb + Aa\cos.\Phi - nAb\cos.\Phi^2 \\ +Ba \quad + Bb\cos.\Phi + Cb\cos.\Phi^2 \\ \qquad\quad + Ca\cos.\Phi \end{array}\right\} = 0;$$

unde singulis membris nihilo aequatis, elicitur:

$$A = \frac{ag-bf}{n(aa-bb)};\ B = \frac{af-bg}{aa-bb}\ \text{et}\ C = \frac{(n-1)(ag-bf)}{n(aa-bb)}.$$

Ita ut haec obtineatur reductio

$$\int \frac{\partial \Phi\,(f+g\cos.\Phi)}{(a+b\cos.\Phi)^{n+1}} = \frac{(ag-bf)\sin.\Phi}{n(aa-bb)(a+b\cos.\Phi)^n}$$
$$+ \frac{1}{n(aa-bb)}\int \frac{\partial \Phi\,[n(af-bg)+(n-1)(ag-bf)\cos.\Phi]}{(a+b\cos.\Phi)^n}$$

cujus ope tandem ad formulam $\int \frac{\partial \Phi\,(h+k\cos.\Phi)}{a+b\cos.\Phi}$ pervenitur, cujus integrale $= \frac{k}{b}\Phi + \frac{bh-ak}{b}\int \frac{\partial \Phi}{a+b\cos.\Phi}$ ex superioribus constat. Perspicuum autem est semper fore $k = 0$.

Scholion 2.

265. Occurrunt etiam ejusmodi formulae, in quas insuper quantitas exponentialis $e^{\alpha\phi}$, angulum ipsum ϕ in exponente gerens, ingreditur, quas quomodo tractari oporteat, ostendendum videtur, cum hinc methodus reductionum supra exposita maxime illustretur. Hic enim per illam reductionem ad formulam propositae similem pervenitur, unde ipsum integrale colligi poterit. In hunc finem notetur esse $\int e^{\alpha\phi}\,\partial\phi = \frac{1}{\alpha}e^{\alpha\phi}$.

Problema 30.

266. Formulae differentialis $\partial y = e^{\alpha\phi}\,\partial\phi\,\sin.\phi^n$ integrale investigare.

Solutio.

Sumto $e^{\alpha\phi}\,\partial\phi$ pro factore differentiali, erit

$$y = \frac{1}{\alpha}e^{\alpha\phi}\sin.\phi^n - \frac{n}{\alpha}\int e^{\alpha\phi}\,\partial\phi\,\sin.\phi^{n-1}\cos.\phi:$$

simili modo reperitur

$$\int e^{\alpha\phi}\,\partial\phi\,\sin.\phi^{n-1}\cos.\phi = \frac{1}{\alpha}e^{\alpha\phi}\sin.\phi^{n-1}\cos.\phi$$
$$-\frac{1}{\alpha}\int e^{\alpha\phi}\,\partial\phi\,[(n-1)\sin.\phi^{n-2}\cos.\phi^2 - \sin.\phi^n],$$

quae postrema formula, ob $\cos.\phi^2 = 1 - \sin.\phi^2$, reducitur ad has

$$(n-1)\int e^{\alpha\phi}\,\partial\phi\,\sin.\phi^{n-2} - n\int e^{\alpha\phi}\,\partial\phi\,\sin.\phi^n:$$

unde habebitur

$$\int e^{\alpha\phi}\,\partial\phi\,\sin.\phi^n = \frac{1}{\alpha}e^{\alpha\phi}\sin.\phi^n - \frac{n}{\alpha\alpha}e^{\alpha\phi}\sin.\phi^{n-1}\cos.\phi$$
$$+\frac{n(n-1)}{\alpha\alpha}\int e^{\alpha\phi}\,\partial\phi\,\sin.\phi^{n-2} - \frac{nn}{\alpha\alpha}\int e^{\alpha\phi}\,\partial\phi\,\sin.\phi^n.$$

Quare hanc postremam formulam cum prima conjungendo, elicitur:

$$\int e^{\alpha\phi}\,\partial\phi\,\sin.\phi^n = \frac{e^{\alpha\phi}\sin.\phi^{n-1}(\alpha\sin.\phi - n\cos.\phi)}{\alpha\alpha + nn}$$
$$+\frac{n(n-1)}{\alpha\alpha + nn}\int e^{\alpha\phi}\,\partial\phi\,\sin.\phi^{n-2}.$$

Duobus ergo casibus integrale absolute datur, scilicet $n = 0$ et $n = 1$, eritque

$$\int e^{\alpha\Phi}\partial\Phi = \tfrac{1}{\alpha}e^{\alpha\Phi} - \tfrac{1}{\alpha}, \text{ et}$$

$$\int e^{\alpha\Phi}\partial\Phi \sin.\Phi = \frac{e^{\alpha\Phi}(\alpha \sin.\Phi - \cos.\Phi)}{\alpha\alpha + 1} + \frac{1}{\alpha\alpha + 1}$$

atque ad hos sequentes omnes, ubi n est numerus integer unitate major, reducuntur.

Corollarium 1.

267. Ita si $n = 2$, acquirimus hanc integrationem

$$\int e^{\alpha\Phi}\partial\Phi \sin.\Phi^2 = \frac{e^{\alpha\Phi}\sin.\Phi(\alpha \sin.\Phi - 2\cos.\Phi)}{\alpha\alpha + 4}$$
$$+ \frac{1.2}{\alpha(\alpha\alpha + 4)}e^{\alpha\Phi} - \frac{1.2}{\alpha(\alpha\alpha + 4)}:$$

at si sit $n = 3$, istam

$$\int e^{\alpha\Phi}\partial\Phi \sin\Phi^3 = \frac{e^{\alpha\Phi}\sin.\Phi^2(\alpha \sin.\Phi - 3\cos.\Phi)}{\alpha\alpha + 9}$$
$$+ \frac{2.3e^{\alpha\Phi}(\alpha\sin.\Phi - \cos.\Phi)}{(\alpha\alpha + 1)(\alpha\alpha + 9)} + \frac{2.3}{(\alpha\alpha + 1)(\alpha\alpha + 9)}$$

integralibus ita sumtis, ut evanescant, posito $\Phi = 0$.

Corollarium 2.

268. Si igitur determinatis hoc modo integralibus, statuatur $\alpha\Phi = -\infty$, ut $e^{\alpha\Phi}$ evanescat, erit in genere

$$\int e^{\alpha\Phi}\partial\Phi \sin.\Phi^n = \tfrac{n(n-1)}{\alpha\alpha + nn}\int e^{\alpha\Phi}\partial\Phi \sin.\Phi^{n-2};$$

hincque integralia pro isto casu $\alpha\Phi = -\infty$ erunt

$$\int e^{\alpha\Phi}\partial\Phi = -\tfrac{1}{\alpha}; \qquad \int e^{\alpha\Phi}\partial\Phi\sin.\Phi = \tfrac{1}{\alpha\alpha+1};$$

$$\int e^{\alpha\Phi}\partial\Phi\sin.\Phi^2 = \tfrac{-1.2}{\alpha(\alpha\alpha+4)}; \qquad \int e^{\alpha\Phi}\partial\Phi\sin.\Phi^3 = \tfrac{1.2.3}{(\alpha\alpha+1)(\alpha\alpha+9)};$$

$$\int e^{\alpha\Phi}\partial\Phi\sin.\Phi^4 = \tfrac{-1.2.3.4}{\alpha(\alpha\alpha+4)(\alpha\alpha+16)}; \qquad \int e^{\alpha\Phi}\partial\Phi\sin.\Phi^5 = \tfrac{1.2.3.4.5}{(\alpha\alpha+1)(\alpha\alpha+9)(\alpha\alpha+25)}$$

Corollarium 3.

269. Quare si proponatur haec series infinita

$$s = 1 + \frac{1.2}{\alpha\alpha+4} + \frac{1.2.3.4}{(\alpha\alpha+4)(\alpha\alpha+16)} + \frac{1.2.3.4.5.6}{(\alpha\alpha+4)(\alpha\alpha+16)(\alpha\alpha+36)} + \text{etc. erit}$$

$$s = -\alpha \int e^{\alpha\Phi} \partial\Phi (1 + \sin.\Phi^2 + \sin.\Phi^4 + \sin.\Phi^6 + \text{etc.})$$

seu $s = -\alpha \int \frac{e^{\alpha\Phi}\partial\Phi}{\cos.\Phi^2}$, posito post integrationem $\alpha\Phi = -\infty$.

Problema 31.

270. Formulae differentialis $e^{\alpha\Phi}\partial\Phi \cos.\Phi^n$ integrale investigare.

Solutio.

Simili modo procedendo ut ante, erit

$$e^{\alpha\Phi}\partial\Phi \cos.\Phi^n = \tfrac{1}{\alpha} e^{\alpha\Phi} \cos.\Phi^n + \tfrac{n}{\alpha}\int e^{\alpha\Phi}\partial\Phi \sin.\Phi \cos.\Phi^{n-1};$$

tum vero

$$\int e^{\alpha\Phi}\partial\Phi \sin.\Phi \cos.\Phi^{n-1} = \tfrac{1}{\alpha} e^{\alpha\Phi} \sin.\Phi \cos.\Phi^{n-1}$$
$$- \tfrac{1}{\alpha}\int e^{\alpha\Phi}\partial\Phi [\cos.\Phi^n - (n-1)\cos.\Phi^{n-2}\sin.\Phi^2],$$

quae postrema formula abit in $-(n-1)\int e^{\alpha\Phi}\partial\Phi \cos.\Phi^{n-2} + n\int e^{\alpha\Phi}\partial\Phi \cos.\Phi^n$, ita ut sit

$$\int e^{\alpha\Phi}\partial\Phi \cos.\Phi^n = \tfrac{1}{\alpha} e^{\alpha\Phi}\cos.\Phi^n + \tfrac{n}{\alpha\alpha} e^{\alpha\Phi}\sin.\Phi\cos.\Phi^{n-1}$$
$$+ \tfrac{n(n-1)}{\alpha\alpha}\int e^{\alpha\Phi}\partial\Phi\cos.\Phi^{n-2} - \tfrac{nn}{\alpha\alpha}\int e^{\alpha\Phi}\partial\Phi\cos.\Phi^n,$$

unde colligimus

$$\int e^{\alpha\Phi}\partial\Phi \cos.\Phi^n = \frac{e^{\alpha\Phi}\cos.\Phi^{n-1}(\alpha\cos.\Phi + n\sin.\Phi)}{\alpha\alpha+nn}$$
$$+ \frac{n(n-1)}{\alpha\alpha+nn}\int e^{\alpha\Phi}\partial\Phi\cos.\Phi^{n-2}.$$

Hinc ergo casus simplicissimi sunt

$$\int e^{\alpha\Phi}\partial\Phi = \tfrac{1}{\alpha}e^{\alpha\Phi} + C;\ \int e^{\alpha\Phi}\partial\Phi\cos.\Phi = \frac{e^{\alpha\Phi}(\alpha\cos.\Phi + \sin.\Phi)}{\alpha\alpha+1} + C,$$

ad quos sequentes omnes, ubi n est numerus integer positivus, reducuntur.

Scholion.

271. Casibus simplicissimis notatis, alia datur via integrale formularum propositarum, quin etiam hujus magis patentis $e^{\alpha\phi}\partial\phi\,\sin.\phi^m\cos.\phi^n$, eruendi. Cum enim productum $\sin.\phi^m\cos.\phi^n$ resolvi possit in aggregatum plurium sinuum vel cosinuum, quorum quisque est hujus formae $M\sin.\lambda\phi$ vel $M\cos.\lambda\phi$, integratio reducitur ad alterutram harum formularum $e^{\alpha\phi}\partial\phi\,\sin.\lambda\phi$ vel $e^{\alpha\phi}\partial\phi\cos.\lambda\phi$. Ponamus ergo $\lambda\phi = \omega$, ut habemus

$$e^{\alpha\phi}\partial\phi\cos.\lambda\phi = \frac{1}{\lambda}e^{\frac{\alpha}{\lambda}\omega}\partial\omega\,\sin.\omega, \text{ et}$$

$$e^{\alpha\phi}\partial\phi\cos.\lambda\phi = \frac{1}{\lambda}e^{\frac{\alpha}{\lambda}\omega}\partial\omega\cos.\omega,$$

quarum integralia per superiora ita dantur:

$$\int e^{\frac{\alpha}{\lambda}\omega}\partial\omega\,\sin.\omega = \frac{\lambda e^{\frac{\alpha}{\lambda}\omega}(\alpha\sin.\omega - \lambda\cos.\omega)}{\alpha\alpha+\lambda\lambda} = \frac{\lambda e^{\alpha\phi}(\alpha\sin.\lambda\phi - \lambda\cos.\lambda\phi)}{\alpha\alpha+\lambda\lambda},$$

$$\int e^{\frac{\alpha}{\lambda}\omega}\partial\omega\cos.\omega = \frac{\lambda e^{\frac{\alpha}{\lambda}\omega}(\alpha\cos.\omega + \lambda\sin.\omega)}{\alpha\alpha+\lambda\lambda} = \frac{\lambda e^{\alpha\phi}(\alpha\cos.\lambda\phi + \lambda\sin.\lambda\phi)}{\alpha\alpha+\lambda\lambda}.$$

Unde tandem colligimus:

$$\int e^{\alpha\phi}\partial\phi\,\sin.\lambda\phi = \frac{e^{\alpha\phi}(\alpha\sin.\lambda\phi - \lambda\cos.\lambda\phi)}{\alpha\alpha+\lambda\lambda}, \text{ et}$$

$$\int e^{\alpha\phi}\partial\phi\cos.\lambda\phi = \frac{e^{\alpha\phi}(\alpha\cos.\lambda\phi + \lambda\sin.\lambda\phi)}{\alpha\alpha+\lambda\lambda},$$

si in genere statim loco $\sin.\phi$ et $\cos.\phi$ scripsissem $\sin.\lambda\phi$ et $\cos.\lambda\phi$, hac reductione non fuisset opus; sed quia hic nihil est difficultatis, brevitati consulendum existimavi.