Advances in Group Theory and Applications

Intorno alla generazione dei gruppi di operazioni

Giovanni Frattini

(*Originally communicated by G. Battaglini on Rend. Accad. Lincei 4, 1885*)

«Given a group G of finitely many substitutions or, generally, operations, we will say, naturally, that a system[1] (g) of substitutions of G is a system of independent generating substitutions, when every substitution of G is either in (g) or can be obtained as the product of substitutions contained in (g), and any substitution of (g) cannot be obtained as a product of substitutions chosen among the remaining ones of the system. Lacking the second condition, we will say the system to be simply a generating system.[2] If we imagine now forming and having in sight all possible independent generating systems $(g)', (g)'' \ldots$, some substitution of G (at least the unit) shall be missing.

«If, for instance, the group G were the group of powers of the substitution: $S(a, b, c, d)$, its even powers would not be part of any independent generating system of G, since, as it can be easily seen, there should be an odd power of S contributing to the generation of G, by which the even ones would be generated.[3]

«And so: The substitutions of any group can be divided in two classes: the class of those which can efficaciously contribute to the generation of the group, as they can be part of a generating system without being themselves generated by the remaining ones, and the class of those which cannot contribute to the generation above.

«The subject of this Note is to stress that:

[1] Here and hereafter with *system* is meant *subset*.

[2] The author is defining what is currently said a (*minimal*) *system of generators* for the finite group G.

[3] G is taken as a cyclic group of order 4 and it is shown that its subgroup of order 2 cannot lie inside any minimal system of generators of G.

1. «The substitutions of the group, which cannot efficaciously contribute to its generation, make an exceptional subgroup (the subgroup Φ).[4]

2. «The group Φ coincides with the group of those substitutions which are modules with respect to the generating systems of the fundamental group,[5] in other words, they are such that every generating system can be transformed into a new generating system when the substitutions of the former are taken into account abstracting from factors equal to those modules.[6]

3. «The group Φ coincides with the group which is common to all maximal subgroups (¹) of the fundamental group.

4. «In order that the fundamental group can be generated by a given subgroup combined with some of the others, it is a necessary and sufficient condition that the former is not totally formed with substitutions of Φ.

«And finally:

5. «The group Φ is a Capelli's Ω_0-group (²).

«The fact that the substitutions g of G, which cannot efficaciously contribute to the generation of G itself, form a subgroup, can be shown in the following way: Let us suppose that $g^{(\alpha)}$, $g^{(\beta)}$ cannot be part of any independent generating system. The product $g^{(\alpha)}.g^{(\beta)}$ shall not be part of it,

(¹) We will say that a subgroup[7] of G is maximal when there will be in G no other subgroup containing it.

(²) I name Capelli's Ω_0-groups those groups which, being of order $p^{\alpha}.q^{\beta}.r^{\gamma}\ldots$, do contain nothing but one group of order: $p^{\alpha}, q^{\beta}, r^{\gamma}, \ldots$, respectively,[8] as, in his Memoir: *Sopra la composizione di gruppi di sostituzioni* (R. Accademia dei Lincei, v. XIX),[9] Capelli has proved many properties concerning these groups, and among the others the following two properties: The composition factors of the Ω_0-groups are prime numbers: Every subgroup of an Ω_0-group is itself an Ω_0-group. Combining this second property with our fourth proposition,[10] it will easily follow that: If it is not possible for the fundamental group to be generated by a given subgroup combined with any of the others, the first subgroup will belong to the species of Capelli's Ω_0-groups.

4 The subset Φ is nowadays known as the Frattini subgroup of G. In this context, *exceptional* means *normal*, what an ironic twist.

5 The fundamental group is G.

6 Frattini is now characterizing Φ as the subgroup of non-generator elements of G.

7 In this work, the word "subgroup" stands for "proper subgroup".

8 Ω_0-groups are nothing but finite nilpotent groups.

9 Communicated on March 2nd, 1884.

10 *In order that the fundamental group can be generated by a given subgroup combined with some of the others, it is a necessary and sufficient condition that the first is not totally formed by substitutions of* Φ. In other words, a subgroup K of a group G is contained in $\Phi(G)$ if and only if, for each $K' \leqslant G$ such that $\langle K, K' \rangle = G$, it follows that $K' = G$.

neither. For, otherwise, if we substitute the product $g^{(\alpha)}.g^{(\beta)}$ with its factors in the independent generating system $(g)^{(\omega)}$, the modified system would be a new generating system, still.[11] And however we reduce this new system to one made only by independent generators suppressing some unnecessary substitutions, the substitutions $g^{(\alpha)}$, $g^{(\beta)}$ should necessarily vanish. But one would hence obtain a generating system made of substitutions of $(g)^{(\omega)}$ without the product $g^{(\alpha)}.g^{(\beta)}$, which would thus be superfluous in $(g)^{(\omega)}$.

«By transforming[12] now the substitutions of a system[13] (g) by any substitution $g^{(\mu)}$ of G, we will surely obtain a new system (g). In fact, if the substitutions of the first system generate G, the substitutions of the transformed system will generate the group G transformed by $g^{(\mu)}$, namely G itself. And none of the transformed substitutions will be unnecessary in the second of the two systems. In fact, if this were the case, superfluous would also be the corresponding substitution in the first of the two systems the second one is transformed into, through the inverse of $g^{(\mu)}$. Thus it follows that if a g belongs to any or no system (g), all the transformations of g will be subject to the same condition. The group of the substitutions which cannot efficaciously contribute to the generation of G will thence contain all the transformations of any substitution of its by substitutions of G, and it will hence be exceptional in G.

«And now, let K be a subgroup of G which is not entirely contained in Φ, and let $(g)^{(\omega)}$ one of those systems (g) showing substitutions in common with K. By abolishing the substitutions in $(g)^{(\omega)}$ which are in common with K, the remaining ones will generate a group K' smaller than G, for if they would generate the whole G, the abolished substitutions would have been superfluous in $(g)^{(\omega)}$. Having said this, the group K and the group K' will evidently generate the whole G, since G was generated by the substitutions of $(g)^{(\omega)}$. There exists, thence, a group K' of G, which generates G together with K.

«However this would not happen if K were entirely contained in Φ. For if the group K together with a group K' smaller than G would generate the latter, some substitutions of K could efficaciously contribute to the generation of G, and this is contrary to the nature of the substitutions of Φ which compose K.[14]

[11] This paragraph deals with proving the first part of proposition 1, namely that the set Φ is a subgroup: the proof is made by contradiction, assuming that $g^{(\alpha)}.g^{(\beta)}$ belongs to a minimal generating system $(g)^{(\omega)}$.

[12] The action of transforming is that of conjugating, indeed.

[13] Hereafter with the sole word *system* the author refers to an *independent generating system*.

[14] These last two paragraphs prove proposition 4. In fact, it is proved here that a subgroup K of a

«There is the following theorem: A subgroup Γ, exceptional in G, can always efficaciously contribute to the generation of G, when there are in Γ at least two different groups among those having order the biggest power of some of the prime factors of the order of Γ.[15]

«Before proving this theorem, I notice that it is essentially due to Capelli, who proves (1) that under the aforesaid assumption there are subgroups of G contributing with their substitutions to all the periods[16] of Γ.[17] In order to prove the equivalence of the two propositions, we will indeed point out that, if Γ is exceptional in G, every subgroup L of G permutes with Γ (2), so that putting together the periods of Γ having substitutions in common with a period of L, a new distribution in periods of the substitutions of G will take place and, precisely, the distribution pertaining the group generated by Γ and L[18] as I have proved in my Memoir: *Intorno ad alcune proposizioni della teoria delle sostituzioni* (3). This evidently reveals that: a necessary and sufficient condition so that there are subgroups L of G contributing with their substitutions to all the periods of Γ is that Γ generates G together with some subgroup of G and smaller than G,[19] namely that Γ efficaciously contributes to the generation of G with some system of substitutions extraneous to Γ.

(1) loc. cit.

(2) For all α and β, it takes place a relation of the form: $l_\alpha.\gamma_\beta = \gamma'_\beta.l'_\alpha$.

(3) Mem. of the R. Accademia dei Lincei, v. XVIII, 1883–84.

---

group G is contained in $\Phi(G)$ if and only if for each $K' \leqslant G$ such that $\langle K, K' \rangle = G$, it follows that $K' = G$; notice also that proposition 2 can follow from the above result taking K as a cyclic subgroup of G.

15 A more contemporary reading of this theorem is the following: Let G be a finite group and N a normal subgroup of G. If N contains two distinct Sylow p-subgroups for a given prime p, then $N \nleqslant \Phi(G)$ (cf. note 14).

16 The sentence *contributing to the periods of a given subgroup Γ of a group* G stands for *having non-trivial intersection with all the cosets of Γ in* G.

17 The proof of this result (pp. 263–264 of the quoted paper of Capelli) uses the argument which is now known as the Frattini argument. Moreover, the proof shows that the subgroups contributing to all periods of Γ are the normalizers of the distinct Sylow p-subgroups of Γ.

18 In other words, let $g_1\Gamma, \ldots, g_n\Gamma$ and $h_1L, \ldots, h_mL$ be the cosets in G of Γ and L, respectively. Moreover, let

$$M'_{h_i} = \{g_j\Gamma | g_j\Gamma \cap h_iL \neq \emptyset\}$$

and put

$$M_{h_i} = \bigcup_{H \in M'_{h_i}} H.$$

It is stated here that $M_{h_i} = h_iL\Gamma$.

19 Let us state this is a more modern fashion. *Let* G *be a group and* Γ *be a proper subgroup of* G *containing two distinct Sylow* p*-subgroups for a given prime* p*. There exists a proper subgroup* L *of* G *having non-empty intersection with each coset of* Γ *in* G *if and only if there exists a proper subgroup* K *such that* $\Gamma K = G$. The necessary condition is trivial, since $\Gamma L = G$. As for the sufficient condition, since $G = \Gamma K = M_1$ (see note 18), it follows that K must have non-trivial intersection with all cosets of Γ in G.

«That being said, let us come to the proof of the stated theorem. Let P be one of the subgroups of order p^{α} (the biggest α) contained in Γ, and S be a substitution of G. Let us say P' the group of order p^{α} (contained in Γ) which P is transformed into by S.

«We know there are substitutions in Γ which transform P into P'. Let γ be one of these. The substitution $S\gamma^{-1} = \sigma$ will belong to the group of substitutions of G which transform P into itself, and we will have: $S = \sigma\gamma$.

«Being S any substitution of G, we conclude that the group Γ and that of the substitutions transforming P into itself generate G. Now the group Γ efficaciously contribute to this generation, provided that the substitutions of G transforming P into itself do not form the whole G. But in this case P' would coincide with P and there would not be in Γ two distinct subgroups of order p^{α}.[20]

«And now, since the group Φ, which is exceptional in G, cannot, owing to its definition, efficaciously contribute to the generation of G, there will not be in Φ two distinct groups of orders p^{α}, $q^{\beta}, \ldots$ respectively. The group Φ will hence be an Ω_0-group.[21]

«The group Φ is then made by those substitutions which are modules with respect to the generation of G. In fact, let: $P\varphi_1\, Q\varphi_2\, R\varphi_3, \ldots$ be a substitution of a generating system regarded as a product in which the factors φ_1, φ_2, φ_3, $\ldots$ stand for substitutions of Φ. Since it is: $P\varphi_1 = \varphi_1' P$, $PQ\varphi_2 = \varphi_2' PQ, \ldots$ for Φ is exceptional in G, we will have: $P\varphi_1\, Q\varphi_2\, R\varphi_3 \ldots = (\varphi_1'\varphi_2'\varphi_3' \ldots)(PQR\ldots) = \varphi PQR\ldots$

«It will so happen that, whereas by substituting in the generating system the φ and the product $PQR\ldots$ in place of the considered substitution, we will evidently obtain a new generating system, abolishing the φ, which is superfluous, it will be possible to place $PQR\ldots$ instead of the original substitution, just as it would have happened setting: $\varphi_1 = \varphi_2 = \varphi_3 \ldots = 1$ in it. Conversely, if the substitution $g^{(h)}$ is a module, it will necessarily be in Φ. In fact, $g^{(h)}$ could not be a part of any independent generating system, since otherwise one could set there $g^{(h)} = 1$ and abolish $g^{(h)}$ as superfluous.[22]

«So, if G were the cyclic group of powers of the substitution S of order m, the powers: S^{p_1}, $S^{p_2} \ldots$ with p_1, $p_2 \ldots$ prime to m would be generating systems of the group, or rather the only systems made by one generator. So if S^a belongs to the group Φ, the substitutions S^{a+p_1}, $S^{a+p_2}, \ldots$ will reproduce the above series of powers. The series of the numbers prime to m and smaller than m will hence go back to itself (mod. m) by adding a to

[20] This concludes the proof of the stated theorem.
[21] Proposition 5 is hence proved.
[22] This proves proposition 2.

all its elements. Conversely, if this happens to be the case, S^a will belong to Φ. In fact, if $S^u, \dots S^a$ could be an independent generating system, the product $S^{u'u} \dots S^{a'a}$ with suitable values of $u', \dots a'$ would reduce to S with exponent $u'u + .. + a'a$ prime to m, and $u'u + \dots$ would hence be prime to m. The $S^u \dots$ would therefore suffice to generate the group.[23]

«That being so, let: $m = p^\alpha . q^\beta . r^\gamma \dots$ The series of numbers prime to m and smaller than m will evidently go back to itself with a multiple of the product $p.q.r \dots$ and will not go back to itself in any other case. If, indeed, a is not divisible by some of the prime factors of m, e.g. by p, the series $\omega + a$, $\omega + 2a \dots$ in which ω represents a number prime to m, will contain some term which is divisible by p, being a prime to p, and so the series of the numbers prime to m and smaller than m will not go back to itself by adding a to all of its elements. We conclude hence that in the simple case we are considering the group Φ is the group made by the powers of the substitution $S^{p.q.r\cdots}$. On the generating systems of the cyclic group one can therefore operate with the hypothetical equivalence: $S^{p.q.r\cdots} = 1$.

«Finally, let H' be a maximal subgroup of G. If Φ did not exist in H', the group generated by H' and Φ would coincide with G since H' is maximal, and Φ could contribute to the generation of G, which is unacceptable. But if any substitution $g^{(\alpha)}$ extraneous to Φ could be in common to all groups H,[24] since $g^{(\alpha)}$ combined with some subgroup K would generate G, it would generate it also combined with a subgroup $H^{(i)}$[25] which would be either K itself if K is maximal in G, or else would contain K as a subgroup. However this is impossible since we have supposed that $H^{(i)}$ contains $g^{(\alpha)}$.[26] We will therefore highlight the theorem: The group in common to all maximal subgroups of any group, is exceptional in the group and it is an Ω_0-group. In a similar fashion, one can prove that the group Φ is in common to all exceptional maximal subgroups of G. But it can also be a subgroup of the whole group in common to these.

«We will mention finally that, since an Ω_0-group of order $p^\alpha . q^\beta . r^\gamma \dots$ includes in itself subgroups of any smaller order $p^{\alpha'} . q^{\beta'} . r^{\gamma'} \dots$, which are generated by subgroups having orders $p^{\alpha'}, q^{\beta'}, r^{\gamma'} \dots$ respectively, being the single substitutions of each generating group permutable with the single ones of all the others, it is easy to prove that the subgroup Φ of an Ω_0-group coincides with the one generated by the groups respectively in

[23] Let $G = \langle g \rangle$ be a (non-trivial) cyclic group of order m. This paragraph proves the following auxiliary lemma: $g^a \in \Phi$ if and only if for each p prime to m, it follows that $p + a$ is still prime to m. This equivalence will be soon used to determine $\Phi(G)$ when G is cyclic.

[24] Here H is an arbitrary maximal subgroup of G.

[25] Also in this case Frattini speaks of a maximal subgroup.

[26] This proves proposition 3.

common to all the groups of orders $p^{\alpha-1}$, $q^{\beta-1}$,... which are contained in Ω_0 ».

Translated by

M. Brescia, M. Trombetti
Dipartimento di Matematica e Applicazioni "Renato Caccioppoli"
Università degli Studi di Napoli Federico II
Via Cintia, Napoli (Italy)
e-mail: mattia.brescia@unina.it; marco.trombetti@unina.it