ON THE DIFFERENCE BETWEEN CONSECUTIVE PRIMES

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1. LET p_n denote the nth prime, and $\pi(x)$ the number of primes p not exceeding x. The existence of an absolute constant $\theta < 1$ such that

$$\pi(x+x^{\theta})-\pi(x) \sim \frac{x^{\theta}}{\log x} \tag{1}$$

when $x \to \infty$, and therefore

$$p_{n+1}-p_n = O(p_n^{\theta}) \tag{2}$$

when $n \to \infty$, was first proved by Hoheisel.† His proof was based on two propositions concerning the zeros of the Riemann zeta-function $\zeta(s) = \zeta(\sigma+ti)$:

(i) Littlewood's theorem‡ that $\zeta(s)$ has no zeros in a domain of the type

$$\sigma > 1-A\frac{\log\log t}{\log t}, \qquad t > t_0,$$

where $A > 0$, $t_0 > 3$;

(ii) Carlson's theorem, or rather a refinement of it,§ namely that

$$N(\sigma, T) = O(T^{4\sigma(1-\sigma)}\log^6 T) \tag{3}$$

uniformly for $\frac{1}{2} < \frac{1}{2}+\delta \leqslant \sigma \leqslant 1$ as $T \to \infty$, where $N(\sigma, T)$ is the number of zeros $\rho = \beta+\gamma i$ of $\zeta(s)$ with $\beta \geqslant \sigma$, $0 < \gamma \leqslant T$.

Hoheisel proved (1) and (2) with $\theta = \frac{32999}{33000}$, and Heilbronn|| reduced this to $\frac{249}{250}$ by increasing the numerical value of A in (i). It was pointed out by Hoheisel in his original paper that the value $\theta = \frac{3}{4}+\epsilon$ (where ϵ is an arbitrarily small positive number) would follow from his analysis if A could be replaced by an $A(t)$ tending to infinity with t. This advance has now been made by Tchudakoff with the aid of Vinogradoff's results on the estimation of trigonometrical sums.††

The aim of this paper is to reduce the index θ still further by a reconsideration of the exponent of T in (3). The main result (Theorem 4) is that, if

$$\zeta(\tfrac{1}{2}+ti) = O(t^c) \tag{4}$$

† Hoheisel (8). See the list of references at the end.

‡ Landau (12), ii, Satz 397; Titchmarsh (15), Theorem 13.

§ Hoheisel (8), § 2. || Heilbronn (7). †† Tchudakoff (1), (2).

as $t \to \infty$, where c is a positive (absolute) constant, then (1) and (2) are true with

$$\theta = \frac{1+4c}{2+4c}+\epsilon.$$

Thus even the classical value $c = \frac{1}{4}+\epsilon$ reduces θ from $\frac{3}{4}+\epsilon$ to $\frac{2}{3}+\epsilon$. The Hardy-Littlewood value† $c = \frac{1}{6}+\epsilon$ gives $\theta = \frac{5}{8}+\epsilon$, while the best published value‡ $c = \frac{229}{1392}$ gives $\theta = \frac{577}{925}+\epsilon$. Moreover, *if the Lindelöf hypothesis is true* (that is to say, if (4) holds with an arbitrarily small c), then (1) and (2) are true with $\theta = \frac{1}{2}+\epsilon$. This may be compared with Cramér's theorem§ that, *if the Riemann hypothesis is true*, then

$$p_{n+1}-p_n = O(p_n^{\frac{1}{2}}\log p_n).$$

The standard proofs of theorems of the type of (3) are based on the estimation of a certain integral (the integral in Theorem 2 below). In Heilbronn's proof of (1) and (2) this integral is introduced directly into the arithmetical problem without explicit mention of $N(\sigma, T)$, and his method has the advantage of being applicable to other problems. In spite of this we shall follow Hoheisel's line of argument, partly because $N(\sigma, T)$ is of interest in itself and partly because a new difficulty appears in Heilbronn's method with the reduction of the index θ below $\frac{3}{4}$.

The inequality for $N(\sigma, T)$ which we actually prove (in Theorem 3) for application to (1) and (2) is of interest only in the neighbourhood of $\sigma = 1$. In Theorem 5 we indicate the proof of a result which, though less precise in this particular region and less useful for the arithmetical application, is perhaps more interesting in itself in that it supersedes existing theorems over the whole range $\frac{1}{2} < \sigma < 1$. Taken together the two theorems give

$$N(\sigma, T) = O(T^{\lambda(\sigma)(1-\sigma)}\log^5 T)$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$ as $T \to \infty$, where

$$\lambda(\sigma) = \min(1+2\sigma,\ 2+4c),$$

† Landau (**12**), ii, Satz 414; Titchmarsh (**15**), Theorem 15.

‡ Phillips (**13**). A better value $c = \frac{19}{116}$ $(= \frac{228}{1392})$ has been obtained by Prof. Titchmarsh in an unpublished manuscript. This gives $\theta = \frac{48}{77}+\epsilon$.

§ Cramér (**3**); see also (**4**). Another proof of this result may be constructed by performing the operation $\Delta_h^{(2)}$, where $\Delta_h^{(2)} f(x) = f(x+2h)-2f(x+h)+f(x)$, on the explicit formula for $\psi_1(x)$ [Theorem 28 of my tract (**10**)], using the inequality

$$\left|\frac{\Delta_h^{(2)} x^{\rho+1}}{\rho(\rho+1)}\right| < \min\left(h^2 x^{\Theta-1}, \frac{4(2x)^{\Theta+1}}{\gamma^2}\right) \quad (1 < h < x)$$

(where Θ is the upper bound of $\Re\rho$), and taking $h = Cx^{\Theta}\log x$ with a sufficiently large absolute constant C.

c being a number for which (4) is true. The index $(1+2\sigma)(1-\sigma)$ of Theorem 5 is an advance on Titchmarsh's index†

$$1-(2\sigma-1)/(3-2\sigma) = 4(1-\sigma)/(3-2\sigma),$$

since
$$\frac{4}{3-2\sigma}-(1+2\sigma) = \frac{(2\sigma-1)^2}{3-2\sigma} > 0 \quad (\tfrac{1}{2} < \sigma \leqslant 1).$$

The principal weapon is a convexity theorem for integrals.‡

2. Theorem 1. *Suppose* (i) *that* $\zeta(s)$ *has no zeros in the domain*

$$\sigma > 1-A\frac{\log\log t}{\log t} \qquad (A > 0;\, t > t_0 > 3),$$

and (ii) *that*
$$N(\sigma, T) = O(T^{b(1-\sigma)}\log^B T) \tag{5}$$
uniformly for $\tfrac{1}{2} \leqslant \sigma \leqslant 1$ *as* $T\to\infty$, *where* $b > 0$, $B \geqslant 0$. *Then* (1) *and* (2) *are true for any fixed* θ *satisfying*

$$1-\frac{1}{b+A^{-1}B} < \theta < 1. \tag{6}$$

This is essentially the content of Hoheisel's main theorem. For completeness we reproduce the proof (simplified in detail).

By a known formula§ we have

$$\psi(x) = x-\sum_{|\gamma|<T}\frac{x^\rho}{\rho}+O\Big(\frac{x}{T}\log^2 x\Big)$$

uniformly for $3 \leqslant T \leqslant x$ as $x\to\infty$, where $\psi(x) = \sum_{p^m\leqslant x}\log p$, and $\rho = \beta+\gamma i$ is a typical complex zero of $\zeta(s)$. Hence

$$\psi(x+h)-\psi(x) = h-\sum_{|\gamma|<T}\frac{(x+h)^\rho-x^\rho}{\rho}+O\Big(\frac{x}{T}\log^2 x\Big)$$

where O's are uniform for $3 \leqslant T \leqslant x$, $0 < h \leqslant x$, as $x\to\infty$. Since

$$\left|\frac{(x+h)^\rho-x^\rho}{\rho}\right| = \left|\int\limits_x^{x+h} u^{\rho-1}\,du\right| \leqslant \int\limits_x^{x+h} u^{\beta-1}\,du \leqslant hx^{\beta-1},$$

this implies that

$$\frac{\psi(x+h)-\psi(x)}{h} = 1+O\Big(\sum_{|\gamma|\leqslant T}x^{\beta-1}\Big)+O\Big(\frac{x}{Th}\log^2 x\Big). \tag{7}$$

† Titchmarsh (14). ‡ Hardy, Ingham, and Pólya (5), Theorem 7.
§ Landau (11), Satz 1.

Now we have†

$$\sum_{|\gamma|\leqslant T}(x^{\beta-1}-x^{-1}) = \sum_{|\gamma|\leqslant T}\int_0^\beta x^{\sigma-1}\log x\,d\sigma = \int_0^1 \sum_{\substack{|\gamma|\leqslant T\\ \beta>\sigma}} x^{\sigma-1}\log x\,d\sigma,$$

or

$$\sum_{|\gamma|\leqslant T} x^{\beta-1} = 2x^{-1}N(0,T)+2\int_0^1 N(\sigma,T)x^{\sigma-1}\log x\,d\sigma. \tag{8}$$

Since $\zeta(s)$ has only a finite number of zeros $\rho = \beta+\gamma i$ with $\frac{1}{2} \leqslant \beta < 1$, $|\gamma| \leqslant t_0$ and none with $\beta \geqslant 1$, it follows from hypothesis (i) that we can find a $T_0 > 3$ so that $N(\sigma, T) = 0$ for $T \geqslant T_0$, $\sigma > 1-\eta(T)$, where $\eta(T) = A(\log\log T)/(\log T)$. Further, since $N(\frac{1}{2}, T) \neq o(T)$, the case $\sigma = \frac{1}{2}$ of hypothesis (ii) shows that $b \geqslant 2$. Also $N(\sigma, T) \leqslant 2N(\frac{1}{2}, T)$ for $\sigma \leqslant \frac{1}{2}$, so that (5) holds uniformly for $0 \leqslant \sigma \leqslant 1$. Hence by (8), since $N(0, T) = O(T\log T)$,

$$\sum_{|\gamma|\leqslant T} x^{\beta-1} = O(x^{-1}T\log T)+O\left(\int_0^{1-\eta(T)}\left(\frac{T^b}{x}\right)^{1-\sigma}\log^B T\log x\,d\sigma\right)$$

uniformly for $T_0 \leqslant T \leqslant x$ as $x \to \infty$.

Take $T = x^\alpha$, where α is a constant satisfying $0 < \alpha < b^{-1}$ ($\leqslant \frac{1}{2}$). Then

$$\begin{aligned}\sum_{|\gamma|\leqslant T} x^{\beta-1} &= O(x^{\alpha-1}\log x)+O(x^{(\alpha b-1)\eta(x^\alpha)}\log^B x)\\ &= O(x^{\alpha-1}\log x)+O(e^{(b-\alpha^{-1})A\log(\alpha\log x)}\log^B x) = O((\log x)^{-\delta}),\end{aligned} \tag{9}$$

where $\delta = (\alpha^{-1}-b)A-B$. Choose α so that $\alpha^{-1} > b+A^{-1}B$ ($\geqslant b$). Then $\delta > 0$, so that by (9) and (7) (with $T = x^\alpha$)

$$\psi(x+h)-\psi(x) \sim h$$

when $x \to \infty$, if $h = x^\theta$ and θ is a constant satisfying

$$1 > \theta > 1-\alpha \ (> \tfrac{1}{2}).$$

This implies (1), and therefore (2), since (when $h = x^\theta$)

$$\begin{aligned}\psi(x+h)-\psi(x) &= \sum_{x<p\leqslant x+h}\log p+O\left(\sum_{p^2\leqslant x+h}\log p\left[\frac{\log(x+h)}{\log p}\right]\right)\\ &= \sum_{x<p\leqslant x+h}\{\log x+O(1)\}+O\left(\sum_{p^2\leqslant 2x}\log 2x\right)\\ &= \{\pi(x+h)-\pi(x)\}\{\log x+O(1)\}+O(x^{\frac{1}{2}}\log x).\end{aligned}$$

† Or, using the Stieltjes integral,

$$\sum_{|\gamma|\leqslant T} x^{\beta-1} = -2\int_0^{1+0} x^{\sigma-1}d_\sigma N(\sigma,T) = 2x^{-1}N(0,T)+2\int_0^1 N(\sigma,T)d_\sigma(x^{\sigma-1}).$$

The conditions on α and θ imply (6), and for any given θ satisfying (6) an appropriate α can be found.

3. THEOREM 2. *Let*

$$f_X(s) = \zeta(s) \sum_{n<X} \mu(n) n^{-s} - 1 = \zeta(s) M_X(s) - 1,$$

where $\mu(n)$ *is Möbius's function. Then, if* c *is an absolute constant for which* (4) *is true,*

$$\int_1^T |f_X(\sigma+ti)|^2\, dt < C \frac{T^{4c(1-\sigma)}}{X^{2\sigma-1}} (T+X)\log^4(T+X) \tag{10}$$

for $\frac{1}{2} \leqslant \sigma \leqslant 1$, $T > 1$, $X > 1$, *where* C *is a positive absolute constant.*

We may suppose that $X \geqslant 2$, since $f_X(s) = f_2(s)$ for $1 < X < 2$. We subject T in the first instance only to the restriction $T \geqslant 0$. The absolute constant c of (4) is necessarily positive, and we may suppose that $c < \frac{1}{2}$, since (4) is certainly true for some $c < \frac{1}{2}$. The symbols C_1, C_2,... denote positive absolute constants.

For $\sigma > 1$ we have

$$f_X(s) = \sum_{n \geqslant X} a_X(n) n^{-s},$$

where

$$a_X(n) = \sum_{\substack{d|n \\ d<X}} \mu(d), \tag{11}$$

so that $a_X(1) = 1$, $a_X(n) = 0$ for $1 < n < X$, and $|a_X(n)| \leqslant d(n)$ for all n. Hence, if $0 < \delta < 1$,

$$\int_0^T |f_X(1+\delta+ti)|^2\, dt = \sum_{m,n \geqslant X} \frac{a_X(m)a_X(n)}{(mn)^{1+\delta}} \int_0^T \left(\frac{m}{n}\right)^{ti} dt = \sum_{m=n} + 2\Re \sum_{m<n}$$

$$\leqslant T \sum_{n \geqslant X} \frac{d^2(n)}{n^{2+2\delta}} + 4 \sum_{n>m \geqslant X} \frac{d(m)d(n)}{(mn)^{1+\delta}\log(n/m)}. \tag{12}$$

These sums are easily estimated by means of the known inequalities†

$$\sum_{n \leqslant x} d^2(n) < C_1 x \log^3 x \quad (x \geqslant 2), \tag{13}$$

$$\sum_{m<n \leqslant x} \frac{d(m)d(n)}{(mn)^{\frac{1}{2}}\log(n/m)} < C_2 x \log^3 x \quad (x > 1). \tag{14}$$

† For (14) see Ingham (9), 296, Lemmas B2 and B3. (13) is included in an asymptotic formula stated by S. Ramanujan and proved analytically by B. M. Wilson, but an elementary proof of (13) is suggested by Heilbronn (7) (cf. 413, Hilfssatz 20). It may be remarked in passing that the result and proof of Lemma B2 of my paper (9) remain valid even in the case $k = 0$ if we replace $\sum_{d|k}$ by $\sum_{\substack{d|k \\ d \leqslant \sqrt{x}}}$; this provides another elementary proof of (13).

We deduce, in fact, from (13) that, for $0 < \xi < 3$,

$$\sum_{n>X} \frac{d^2(n)}{n^{1+\xi}} = \sum_{n>X} d^2(n) \int_n^\infty \frac{1+\xi}{x^{2+\xi}}\,dx = \int_X^\infty \frac{1+\xi}{x^{2+\xi}} \sum_{X<n\leqslant x} d^2(n)\,dx$$

$$< \int_X^\infty \frac{(1+\xi)C_1 \log^3 x}{x^{1+\xi}}\,dx < \frac{C_3}{\xi X^\xi}\left(\frac{1}{\xi}+\log X\right)^3 \qquad (15)$$

by the substitution $x = Xy^{1/\xi}$; and from (14) (since $1 < \log\lambda+\lambda^{-1} < \log\lambda+\lambda^{-\frac{1}{2}}$ for $\lambda > 1$) that

$$\sum_{n>m\geqslant X} \frac{d(m)d(n)}{(mn)^{1+\xi}\log(n/m)} < \sum_{n>m\geqslant X} \frac{d(m)d(n)}{(mn)^{1+\xi}} + \sum_{n>m\geqslant X} \frac{d(m)d(n)}{m^\xi n^{1+\xi}(mn)^{\frac{1}{2}}\log(n/m)}$$

$$< \left(\sum_{n=1}^\infty \frac{d(n)}{n^{1+\xi}}\right)^2 + \sum_{n>m\geqslant 1} \frac{d(m)d(n)}{(mn)^{\frac{1}{2}}\log(n/m)} \int_n^\infty \frac{1+\xi}{x^{2+\xi}}\,dx$$

$$= \zeta^4(1+\xi) + \int_1^\infty \frac{1+\xi}{x^{2+\xi}} \sum_{m<n\leqslant x} \frac{d(m)d(n)}{(mn)^{\frac{1}{2}}\log(n/m)}\,dx$$

$$< \zeta^4(1+\xi) + \int_1^\infty \frac{(1+\xi)C_2\log^3 x}{x^{1+\xi}}\,dx < \frac{C_4}{\xi^4}. \qquad (16)$$

Hence by (12) (since $(\log X)^3/X^{2\delta} < \delta^{-3}$)

$$\int_0^T |f_X(1+\delta+ti)|^2\,dt < C_5\left(\frac{T}{X}+1\right)\delta^{-4}. \qquad (17)$$

For $\sigma = \frac{1}{2}$ we use the inequalities

$$|f_X|^2 \leqslant 2(|\zeta|^2|M_X|^2+1),$$

$$\int_0^T |M_X(\tfrac{1}{2}+ti)|^2\,dt \leqslant T\sum_{n<X} \frac{\mu^2(n)}{n} + 4\sum_{m<n<X} \frac{|\mu(m)\mu(n)|}{(mn)^{\frac{1}{2}}\log(n/m)}$$

$$\leqslant T\sum_{n<X}\frac{1}{n} + 4\sum_{m<n<X}\left(\frac{1}{(mn)^{\frac{1}{2}}}+\frac{1}{n-m}\right) < C_6(T+X)\log X$$

(since $1/\log\lambda < \lambda/(\lambda-1) < 1+\lambda^{\frac{1}{2}}/(\lambda-1)$ for $\lambda > 1$), and deduce by (4) that

$$\int_0^T |f_X(\tfrac{1}{2}+ti)|^2\,dt < C_7 T^{2c}(T+X)\log X, \qquad (18)$$

the inequality holding down to $T = 0$ since the integral is at most $C_8 X.T \leqslant C_8 T^{2c}X$ for $0 \leqslant T \leqslant 1$.

From (17) and (18) we shall deduce an inequality valid for $\frac{1}{2} \leqslant \sigma \leqslant 1+\delta$ by means of a convexity theorem. Write

$$I_\sigma(T) = \int_0^T |f_X(\sigma+ti)|^2\,dt, \qquad J_\sigma = \int_{-\infty}^{\infty} |\phi(\sigma+ti)|^2\,dt,$$

where $$\phi(s) = \phi_{X,\tau}(s) = \frac{s-1}{s\cos(s/2\tau)} f_X(s) \quad (\tau > 3/\pi).$$

In the strip $\frac{1}{2} \leqslant \sigma \leqslant 1+\delta$, $\phi(s)$ is regular and satisfies

$$|\phi(s)|^2 \leqslant C_9\, e^{-|t|/\tau} |f_X(s)|^2 \tag{19}$$

and is therefore certainly bounded (for fixed X and τ). Further, for $\frac{1}{2} \leqslant \sigma \leqslant 1+\delta$, $\sigma \neq 1$,

$$J_\sigma \leqslant 2\int_0^\infty C_9\, e^{-t/\tau} |f_X(\sigma+ti)|^2\,dt = 2C_9 \int_0^\infty e^{-u} I_\sigma(\tau u)\,du,$$

by partial integration and the substitution $t = \tau u$; whence, by (17) and (18),

$$J_{1+\delta} < C_{10} \int_0^\infty e^{-u}\left(\frac{\tau u}{X}+1\right)\delta^{-4}\,du < C_{11}\left(\frac{\tau}{X}+1\right)\delta^{-4},$$

$$J_{\frac{1}{2}} < C_{12} \int_0^\infty e^{-u}(\tau u)^{2c}(\tau u+X)\log X\,du < C_{13}\tau^{2c}(\tau+X)\log X.$$

It follows† that, for $\frac{1}{2} \leqslant \sigma \leqslant 1+\delta$,

$$J_\sigma < \left\{C_{11}\left(\frac{\tau}{X}+1\right)\delta^{-4}\right\}^{\frac{\sigma-\frac{1}{2}}{\frac{1}{2}+\delta}} \{C_{13}\tau^{2c}(\tau+X)\log X\}^{\frac{1+\delta-\sigma}{\frac{1}{2}+\delta}}. \tag{20}$$

Now $\quad |\phi(s)|^2 \geqslant C_{14}\, e^{-t/\tau} |f_X(s)|^2 \quad (\frac{1}{2} \leqslant \sigma \leqslant 1+\delta,\ t \geqslant 1).$

Hence (20) implies that, for $T > 1$, $\frac{1}{2} \leqslant \sigma \leqslant 1$,

$$C_{14}\, e^{-T/\tau} \int_1^T |f_X(\sigma+ti)|^2\,dt$$
$$< X^{\frac{1-2\sigma}{1+2\delta}} \tau^{\frac{4c(1+\delta-\sigma)}{1+2\delta}} (\tau+X)\max(C_{11}\delta^{-4}, C_{13}\log X)$$

(on simplification of the right-hand side). Taking $\tau = C_{15} T$, $\delta = C_{16}/\log(T+X)$, we deduce the theorem, since

$$X^{\frac{1-2\sigma}{1+2\delta}} \leqslant X^{-(1-2\delta)(2\sigma-1)} \leqslant X^{-(2\sigma-1)+2\delta} < e^{2C_{16}} X^{-(2\sigma-1)},$$

$$T^{\frac{4c(1+\delta-\sigma)}{1+2\delta}} \leqslant T^{4c(1+\delta-\sigma)} \leqslant T^{4c(1-\sigma)+2\delta} < e^{2C_{16}} T^{4c(1-\sigma)}.$$

† Hardy, Ingham, and Pólya (5), Theorem 7.

THEOREM 3. *If* (4) *is true, then*

$$N(\sigma, T) = O(T^{2(1+2c)(1-\sigma)}\log^5 T)$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$ *when* $T \to \infty$.

This may be deduced from Theorem 2 by a familiar argument. We have, in the notation of Theorem 2,

$$1-f_X^2 = \zeta M_X(2-\zeta M_X) = \zeta g = h,$$

where $g(s) = g_X(s)$ and $h(s) = h_X(s)$ are regular except at $s = 1$; and

$$\log|h| \leqslant \log(1+|f_X|^2) \leqslant |f_X|^2. \tag{21}$$

Also, for $\sigma \geqslant 2$,

$$|f_X|^2 \leqslant \Big(\sum_{n>X} \frac{d(n)}{n^2}\Big)^2 < \frac{1}{2X} < \frac{1}{2},$$

if $X > C_{17} > 1$; whence

$$\left.\begin{aligned} \Re h &> \tfrac{1}{2} \qquad\qquad\qquad\qquad\qquad\quad (22)\\ -\log|h| &\leqslant -\log(1-|f_X|^2) \leqslant 2|f_X|^2 < X^{-1} \quad (23)\end{aligned}\right\} \quad (\sigma \geqslant 2; X > C_{17}).$$

Take $X > C_{17}$, $T > 4$, and choose T_1 and T_2 so that $3 < T_1 < 4$, $T < T_2 < T+1$, and $h(s)$ has no zeros on either of the segments $t = T_1$ or $t = T_2$ ($\frac{1}{2} \leqslant \sigma \leqslant 2$). Then, writing

$$N_\zeta(\sigma; T_1, T_2) = N_\zeta(\sigma, T_2) - N_\zeta(\sigma, T_1),$$

where the suffix ζ refers to the function $\zeta(s)$, and extending the notation to $g(s)$ and $h(s)$, we have, applying a theorem of Littlewood† and taking account of (22),

$$\int_\sigma^2 N_h(\sigma; T_1, T_2)\, d\sigma = \int_{T_1}^{T_2} (\log|h(\sigma_0+ti)| - \log|h(2+ti)|)\, dt + \\ + \int_{\sigma_0}^2 (\arg h(\sigma+T_2 i) - \arg h(\sigma+T_1 i))\, d\sigma \tag{24}$$

for $\frac{1}{2} \leqslant \sigma_0 \leqslant 1$, where $\arg h(s)$ is 0 when $s = 2$ and varies continuously along the lines $\sigma = 2$, $t = T_1$, $t = T_2$. By (21), Theorem 2, and (23), the first integral on the right is less than

$$C(T+1)^{4c(1-\sigma_0)} X^{1-2\sigma_0}(T+1+X)\log^4(T+1+X) + TX^{-1}.$$

In the second integral

$$|\arg h(\sigma+T_r i)| \leqslant (m_r+1)\pi \quad (r = 1, 2),$$

where m_r is the number of points of the segment $t = T_r$, $\sigma_0 < \sigma < 2$, at which $\Re h(s) = 0$; for $\arg h(s)$ cannot vary by more than π on any of the m_r+1 pieces into which these points divide the broken line

† Titchmarsh (15), 3.52, 3.53; or (16), 3.8.

$(2, 2+T_r i,\ \sigma_0+T_r i)$, since $\Re h(s) \neq 0$ on the vertical part by (22). But m_r is the number of zeros of the function

$$H_r(s) = \tfrac{1}{2}\{h(s+T_r i)+h(s-T_r i)\}$$

on the segment $t = 0$, $\sigma_0 < \sigma < 2$, and therefore cannot exceed the number of zeros of $H_r(s)$ in the circle $|s-2| \leqslant \frac{3}{2}$. Hence, since $H_r(s)$ is regular for $|s-2| \leqslant \frac{7}{4}$, we have†

$$\left(\frac{7}{6}\right)^{m_r} \leqslant \max_{|s-2|<\frac{7}{4}} \left|\frac{H_r(s)}{H_r(2)}\right| \leqslant \max_{\substack{\sigma>\frac{1}{4} \\ 1\leqslant t\leqslant T+3}} \frac{|h(s)|}{\Re h(2+T_r i)} < (T+X)^{C_{18}} \cdot$$

by (22) and the definition of $h(s)$. The second integral on the right of (24) is therefore less than $C_{19}\log(T+X)$. Collecting these results, we obtain

$$\int\limits_{\sigma_0}^{2} N_h(\sigma;\, T_1, T_2)\, d\sigma < C_{20}\, T^{4c(1-\sigma_0)}(TX^{1-2\sigma_0}+X^{2(1-\sigma_0)})\log^4(T+X),$$

since $TX^{-1} \leqslant TX^{1-2\sigma_0}$ and $\log(T+X) \leqslant X^{2(1-\sigma_0)}\log(T+X)$.

On the other hand, since $N_h = N_\zeta+N_g \geqslant N_\zeta$,

$$\int\limits_{\sigma_0}^{2} N_h(\sigma;\, T_1, T_2)\, d\sigma \geqslant \int\limits_{\sigma_0}^{\sigma_0+\delta} N_\zeta(\sigma;\, T_1, T_2)\, d\sigma \geqslant \delta N_\zeta(\sigma_0+\delta;\, T_1, T_2),$$

if $0 < \delta < 1$. Writing σ for $\sigma_0+\delta$, we deduce, since

$$N_\zeta(\sigma, T) < N_\zeta(\sigma; T_1, T_2)+C_{21},$$

that

$$N_\zeta(\sigma, T) < C_{22}\,\delta^{-1}T^{4c(1-\sigma+\delta)}(TX^{1-2\sigma+2\delta}+X^{2(1-\sigma+\delta)})\log^4(T+X)$$
$$(\tfrac{1}{2}+\delta \leqslant \sigma \leqslant 1).$$

But (since $T > 4$)

$$N_\zeta(\sigma, T) < C_{23}\, T\log T \leqslant C_{23}\, T^{2(1-\sigma+\delta)}\log T \quad (\tfrac{1}{2} \leqslant \sigma \leqslant \tfrac{1}{2}+\delta).$$

The theorem follows from these inequalities if we take

$$X = T > \max(C_{17}, 4), \qquad \delta = 1/\log T.$$

4. Theorem 4. *If* (4) *is true, then* (1) *and* (2) *are true for any fixed* θ *satisfying*

$$\frac{1+4c}{2+4c} < \theta < 1.$$

† By a well-known corollary of Jensen's formula. For a direct proof of this particular result see, for example, Ingham (**10**), 49, Theorem D.

Tchudakoff has proved† that $\zeta(s)$ has no zeros in a domain of the type

$$1-\frac{1}{(\log t)^{a}},\quad t>t_1,$$

where $a<1$, $t_1>2$. The condition (i) of Theorem 1 is therefore satisfied with an arbitrarily large A and an appropriate t_0.

By Theorem 3 the condition (ii) of Theorem 1 is satisfied with $b=2+4c$ and $B=5$.

Theorem 4 therefore follows from Theorem 1, since

$$\lim_{A\to\infty}\left(1-\frac{1}{b+A^{-1}B}\right)=1-\frac{1}{b}=\frac{1+4c}{2+4c}.$$

5. Theorem 3 is specially designed for application to Theorem 4. The result itself is of no interest over the range

$$\tfrac{1}{2}\leqslant\sigma<(1+4c)/(2+4c)$$

where the exponent of T is greater than 1. We shall now give a result which is non-trivial over the whole range $\frac{1}{2}<\sigma<1$. It is better than Theorem 3 for $\frac{1}{2}<\sigma<\frac{1}{2}+2c$, but worse for $\frac{1}{2}+2c<\sigma<1$. Where the argument is similar to that of § 3 some of the details will be omitted.

THEOREM 5. *We have*

$$N(\sigma,T)=O(T^{(1+2\sigma)(1-\sigma)}\log^5 T)$$

uniformly for $\frac{1}{2}\leqslant\sigma\leqslant 1$ *when* $T\to\infty$.

We need a new estimate of the integral of Theorem 2. We consider first the corresponding integral with $f_X(s)$ replaced by

$$f_{X,Y}(s)=\zeta(s)\sum_{n<X}\mu(n)n^{-s}-\sum_{n<Y}a_X(n)n^{-s}=\zeta(s)M_X(s)-A_{X,Y}(s),$$

say, where $2\leqslant X\leqslant Y$, and the coefficients $a_X(n)$ are defined by (11). Suppose that $T\geqslant 0$.

For $\sigma>1$ we have

$$f_{X,Y}(s)=\sum_{n\geqslant Y}a_X(n)n^{-s},$$

whence, as in § 3,

$$\int_0^T|f_{X,Y}(1+\delta+ti)|^2\,dt<C_5\left(\frac{T}{Y}+1\right)\delta^{-4}\quad(0<\delta<1).\tag{25}$$

† Tchudakoff (1). By using the full force of this result we could replace $x^{\theta}=x^{(1+4c)/(2+4c)+\epsilon}=x^{\theta_0+\epsilon}$ in (1) by $x^{\theta_0}e^{(\log x)^{a'}}$ $(a<a'<1)$, and indeed by something a little better. [Cf. Tchudakoff (2).]

For $\sigma = \frac{1}{2}$ we use the inequalities†

$$\int_0^T |f_{X,Y}|^2\,dt \leqslant \int_0^T 2(|\zeta M_X|^2+|A_{X,Y}|^2)\,dt$$
$$\leqslant 2\left(\int_0^T |\zeta|^4\,dt \int_0^T |M_X^2|^2\,dt\right)^{\frac{1}{2}}+2\int_0^T |A_{X,Y}|^2\,dt;$$

$$\int_0^T |\zeta(\tfrac{1}{2}+ti)|^4\,dt < C_{24}\,T\log^4(T+2), \tag{26}$$

$$\int_0^T |A_{X,Y}(\tfrac{1}{2}+ti)|^2\,dt \leqslant T\sum_{n<Y}\frac{d^2(n)}{n}+4\sum_{m<n<Y}\frac{d(m)d(n)}{(mn)^{\frac{1}{2}}\log(n/m)}$$
$$< C_{25}\,T\log^4 Y+4C_2\,Y\log^3 Y, \tag{27}$$

$$\int_0^T |M_X^2(\tfrac{1}{2}+ti)|^2\,dt \leqslant T\sum_{n<X^2}\frac{d^2(n)}{n}+4\sum_{m<n<X^2}\frac{d(m)d(n)}{(mn)^{\frac{1}{2}}\log(n/m)}$$
$$< C_{25}\,T\log^4 X^2+4C_2 X^2\log^3 X^2, \tag{28}$$

and obtain,

$$\int_0^T |f_{X,Y}(\tfrac{1}{2}+ti)|^2\,dt < C_{26}(T+T^{\frac{1}{2}}X+Y)\log^4(T+Y)$$
$$< C_{27}(T+T^{\frac{1}{2}}X+Y)^{1+2\delta}\delta^{-4}. \tag{29}$$

Using a convexity argument as in § 3 and taking

$$\delta = C_{28}/\log(T+Y) < 2C_{28}/\log(T+T^{\frac{1}{2}}X+Y),$$

we deduce from (25) and (29) that, for $\frac{1}{2} \leqslant \sigma \leqslant 1$, $T > 1$,

$$\int_1^T |f_{X,Y}(\sigma+ti)|^2\,dt < C_{29}\left(\frac{T}{Y}+1\right)^{2\sigma-1}(T+T^{\frac{1}{2}}X+Y)^{2(1-\sigma)}\log^4(T+Y). \tag{30}$$

Now

$$f_X(s) = f_{X,Y}(s)+\sum_{X\leqslant n<Y} a_X(n)n^{-s} = f_{X,Y}(s)+A^*_{X,Y}(s), \tag{31}$$

say, and (for $\frac{1}{2} \leqslant \sigma \leqslant 1$, $T > 1$)‡

$$\int_1^T |A^*_{X,Y}(\sigma+ti)|^2\,dt \leqslant T\sum_{X\leqslant n<Y}\frac{d^2(n)}{n^{2\sigma}}+4\sum_{X\leqslant m<n<Y}\frac{d(m)d(n)}{(mn)^{\sigma}\log(n/m)}$$
$$< C_{30}(TX^{1-2\sigma}+Y^{2(1-\sigma)})\log^4 Y.$$

† (26) is a well-known result of Hardy and Littlewood (6), Theorem D (3.12); see also Titchmarsh (17). (27) and (28) follow from (13) and (14).

‡ The inequalities used here may be deduced from (13) and (14) in much the same way as (15) and (16) were deduced. Alternatively, we can deduce them directly from (15) and (16) by using the inequalities $n^{-2\sigma} \leqslant X^{1-2\sigma}Y^{\xi}n^{-1-\xi}$, $(mn)^{-\sigma} \leqslant Y^{2(1-\sigma+\xi)}(mn)^{-1-\xi}$ ($\xi > 0$), and taking $\xi = 1/\log Y$.

Combining this with (30), we obtain, by (31),

$$\int_1^T |f_X(\sigma+ti)|^2\,dt$$

$$< C_{31}\left\{\left(\frac{T}{Y}+1\right)^{2\sigma-1}(T+T^{\frac{1}{2}}X+Y)^{2(1-\sigma)}+TX^{1-2\sigma}\right\}\log^4(T+Y).$$

By considering separately the ranges $Y \geqslant T$ and $Y \leqslant T$, we see at once that (for given T, X, σ) the right-hand side of this is of lowest order when $Y = T$, and this choice gives

$$\int_1^T |f_X(\sigma+ti)|^2\,dt < C_{32}(T^{2(1-\sigma)}+T^{1-\sigma}X^{2(1-\sigma)}+TX^{1-2\sigma})\log^4 T$$

for $\frac{1}{2} \leqslant \sigma \leqslant 1$, $2 \leqslant X \leqslant T$.

Arguing now as in the deduction of Theorem 3 from Theorem 2, and taking $X = C_{33}T^{\sigma}$, we obtain Theorem 5.

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