Robust Benchmark Design*

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Abstract

Recent scandals over the manipulation of LIBOR, foreign exchange benchmarks, and other financial benchmarks have spurred policy discussions over their appropriate design. We characterize the optimal fixing of a benchmark as an estimator of a market value or reference rate. The fixing data are the reports or transactions of agents whose profits depend on the fixing, and who may therefore have incentives to manipulate it. If the benchmark administrator cannot detect or deter the strategic splitting of trades, we show that the best linear unbiased fixing is the commonly used volume-weighted average price (VWAP).

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1 Introduction

This paper solves a version of the problem faced by a financial benchmark administrator. The benchmark administrator constructs a "fixing," meaning an estimator of a market value or reference rate that is based on transactions or other submission data. The data are often generated by agents whose profits depend on the realization of the fixing. Agents may therefore misreport or trade at distorted prices in order to manipulate the fixing. We characterize optimal transactions weights for benchmark fixings, assuming that the benchmark administrator cannot use transfers. If the benchmark administrator is also unable to detect or deter the strategic splitting of trades, we show the best linear unbiased fixing is the commonly used volume-weighted average price (VWAP).

The London Interbank Offered Rate (LIBOR) is arguably the single most important benchmark used in financial markets. Literally millions of different financial contracts, including interest rate swaps, futures, options, variable rate bank loans, and mortgages, have payments that are contractually linked to LIBOR. The aggregate outstanding amount of LIBOR-linked contracts has been estimated by the Alternative Reference Rate Committee (2018) at $200 trillion. LIBOR and related reference rates such as EURIBOR and TIBOR also serve an important price discovery function,[1] as benchmarks for evaluating investment performance and as indicators of current conditions in credit and interest-rate markets. Similar concerns have been raised over the manipulation of foreign exchange and commodity benchmarks.[2] Given the important role of benchmarks in financial markets, reports that they have been systematically manipulated have triggered regulatory reforms. Among other jurisdictions, the European Union (2016) introduced legislation[3] in support of robust benchmarks, which came into force on January 1, 2018.

LIBOR is an estimate of the interest rate at which large banks can borrow short-term wholesale funds on an unsecured basis in the interbank market. Each day, in each major currency and for each of a range of key maturities, LIBOR is currently reported as a trimmed average of the rates reported by a panel of banks to the benchmark administrator.[4] Investigations have revealed purposeful misreporting of these rates. Two rather different incentives for manipulation have been identified. The first, dramatically exacerbated by the financial crisis

[1] The transparency role of benchmarks is explained in Duffie, Dworczak and Zhu (2017).

[2] See Financial Stability Board (2014).

[3] Financial Conduct Authority (2016) explains how the EU regulation "aims to ensure benchmarks are robust and reliable, and to minimise conflicts of interest in benchmark-setting processes."

[4] For details, see, for example, Hou and Skeie (2013). The reports of each individually named bank are revealed to the market. See also Financial Conduct Authority, 2012; BIS, 2013; Market Participants Group on Reference Rate Reform, 2014. In order to weaken the incentive to under report funding costs it has been suggested that the bank-level reports be made public with a three-month lag.

of 2007-2009, was to improve market perceptions of a submitting bank's creditworthiness, by understating the rate at which the bank could borrow. The second incentive was to profit from LIBOR-linked positions held by the bank. For example, in a typical email uncovered by investigators, a trader at a reporting bank wrote to the LIBOR rate submitter: "For Monday we are very long 3m cash here in NY and would like setting to be as low as possible...thanks."[5] This second form of manipulation, revealed by investigators to have been active over many years, is the main subject of this paper.

Manipulation has been reported across a range of financial market benchmarks. By February 2017, the Commodity Futures Trading Commission, alone, had fined dealers[6] \$5.29 billion for manipulation of LIBOR, Euribor, foreign exchange benchmarks and the swap rate benchmark known as ISDAFIX. Benchmark manipulation has also been a recent concern for the equity volatility benchmark known as VIX,[7] and in the markets for various commodities,[8] precious metals such as gold,[9] and manufactured goods such as pharmaceuticals.[10]

The Financial Stability Board is leading an ongoing global process to overhaul key reference rate and foreign currency benchmarks with a view to improving their robustness to manipulation. A key principle of International Organization of Securities Commissions (2013) is that fixings of key benchmarks should be "anchored" in actual market transactions or executable quotations.

This paper has a theoretical focus. Under restrictive conditions, we focus on the optimal design of a transactions-based weighting scheme. In order to illustrate the problem that we study, we ask the reader to imagine the following abstract situation. An econometrician is choosing an efficient estimator of an unknown parameter. Data are generated by strategic agents whose utilities depend on the realized outcome of the estimator. Thus, the chosen estimator influences the data generating process. This game-theoretic component must be

[5] December 14, 2006, Trader in New York to Submitter; source: Malloch and Mamorsky (2013). Another example: "We have another big fixing tom[orrow] and with the market move I was hoping we could set the 1M and 3M Libors as high as possible."

[6] See "CFTC Orders The Royal Bank of Scotland to Pay \$85 Million Penalty for Attempted Manipulation of U.S. Dollar ISDAFIX Benchmark Swap Rates."

[7] See Griffin and Shams (2017).

[8] For cases of oil, natural gas, and propane benchmark manipulation, see "Federal Court Orders \$13 Million Fine in CFTC Crude Oil Manipulation Action against Parnon Energy Inc., Arcadia Petroleum Ltd., and Arcadia Energy (Suisse) SA, and Crude Oil Traders James Dyer and Nicholas Wildgoose," CFTC, August 4, 2014; "CFTC Files and Settles Charges against Total Gas & Power North America, Inc. and Therese Tran for Attempted Manipulation of Natural Gas Monthly Index Settlement Prices," CFTC, December 7, 2015; and "CFTC Finds Statoil ASA Attempted to Manipulate the Argus Far East Index, a Propane Benchmark, to Benefit Statoil's NYMEX-cleared Swaps Position," CFTC, November 14, 2017.

[9] See Vaughn (2014) and "How a Barclays' options trader manipulated the gold price fix," Reuters, May 23, 2014.

[10] See Gencarelli (2002).

considered in the design of the estimator.

Our model features a benchmark administrator who acts as a mechanism designer. The agents that might manipulate the benchmark could be banks, broker-dealers, asset-management firms, or individual traders within any of these types of firms. The mechanism designer observes the transactions generated by the anonymous agents. The data generated by each transaction consist only of the price and size (the notional amount) of the transaction. Whether or not manipulated, the transactions prices are noisy signals of the fundamental value. For non-manipulated transactions, noise arises from market microstructure effects, as explained by Aït-Sahalia and Yu (2009), and also from asynchronous reporting. For example, the WM/Reuters benchmarks for major foreign exchange rates are fixed each day based on transactions that occur within 5 minutes of 4:00pm London time. In an over-the-counter market, moreover, each pair of transacting counterparties is generally unaware of the prices at which other pairs of counterparties are negotiating trades at around the same time.

In our model, the benchmark administrator is restricted to a fixing that is linear with respect to transactions prices, with weighting coefficients that can depend on the size of the transaction. A common method of benchmark fixing is the "volume weighted average price" (VWAP), for which the weight on a given transaction price is proportional to the size of the transaction. The VWAP benchmark is approximated, with a large number of transactions, within the family of fixing designs that our modeled benchmark administrator can consider.

Agents have private information about their exposures to the benchmark, and observe private signals of the fundamental value of the benchmark asset. If an agent decides to trade according to the signal received, there is no manipulation. However, the agent can choose to manipulate by generating a transaction with an artificially inflated or reduced price in order to gain from the associated distortion of the benchmark. Manipulation is assumed to be costly for agents. For example, in order to cause an upward distortion in the benchmark, a trader would need to buy the underlying asset at a price above its fair market value. In order to manipulate the price downward, the agent would need to sell the asset at a price below its true value. Either way, by trading at a distorted price, the agent suffers a loss. On the other hand, the agent has pre-existing contracts (for example swaps) that can be settled at market values linked to the benchmark. On a large pre-existing swap position, the agent may be able to distort the benchmark enough to generate a profit that exceeds the cost of creating the distortion.

This suggests the benefit of avoiding benchmarks whose underlying asset market is thinly traded relative to the market for financial instruments that are contractually linked to the benchmark. In the case of LIBOR, unfortunately, the volume of transactions in the underlying market for interbank loans that determines LIBOR is tiny by comparison with the volume of swap contracts that are contractually settled on LIBOR. As emphasized by Duffie and Stein

(2015), this situation dramatically magnifies the incentive to manipulate LIBOR.

Our model implies that manipulation is unavoidable when the potential benefits from manipulation, measured by the monetary gain from changing the fixing by one unit, are large relative to *(i)* the cost of manipulation, *(ii)* the average size of the transaction, and *(iii)* the number of transactions in the market for the benchmark asset.

Our main findings are the following. First, even if a benchmark can be found that induces only honest (unmanipulated) transactions, this is not necessarily optimal from the viewpoint of the efficiency of the estimator. This is because the transaction weights required for statistical efficiency can be quite different from those minimizing the incentive to manipulate. Typically, a statistically optimal benchmark fixing allows for a nonzero probability of manipulation. Second, a robust benchmark must put nearly zero weight on small transactions. This is intuitive, and stems from the fact that it is cheap for agents to make small manipulated transactions. For instance, Scheck and Gross (2013) describe a strategy said to be used by oil traders to manipulate the daily oil price benchmark published by Platts: "Offer to sell a small amount at a loss to drive down published oil prices, then snap up shiploads at the lower price." Third, although the optimal transaction weight is always non-decreasing in the size of a transaction, the optimal benchmark assigns nearly equal weight to all large transactions. This follows from the fact that the optimal weighting function is concave in size, with a slope that goes goes to zero as trade sizes become large. In many cases, the optimal weight is actually constant above some threshold transaction size. This avoids overweighting transactions made by agents with particularly strong incentives to manipulate. Fourth, our main result characterizes the exact shape of the optimal weighting function, as a solution to a certain second-order differential equation. In the examples that we study, this optimal shape is well approximated by a weighting function that is linear in size up to a threshold, and constant afterwards.

In our baseline model, we assume that each trader's transactions are aggregated into a single composite transaction before it enters into the fixing, so that there is effectively one transaction per agent. In Section 5, we relax this and allow traders to split their total desired transaction into smaller trades. If the benchmark administrator cannot detect or deter order splitting, we show the optimal fixing is the volume-weighted average price (VWAP). VWAP fixings are popular, for example, for the settlement of futures contracts on the Chicago Mercantile Exchange.[11]

In addition to choice of the benchmark asset and the fixing design, regulators can im-

[11] See Quick Facts on Settlements at CME Group, CME Group, October, 2014. For the NYMEX crude oil futures contract, "If a trade(s) occurs on Globex between 14:28:00 and 14:30:00 ET, the active month settles to the volume-weighted average price (VWAP), rounded to the nearest tradable tick."

plement a range of governance and compliance safeguards, raising the cost of manipulation, consistent with the suggestions of Financial Conduct Authority (2012) and the International Organization of Securities Commissions (2013). Our setting allows for an extra cost for trading at a price away from the fair value, associated with the risk of detection of manipulation by the authorities, and resulting penalties or loss of reputation.

For our theoretical analysis, we assume that the mechanism designer cannot use transfers. In particular, fines or litigation damages, forms of negative transfer, may affect the cost of manipulation exogenously but cannot be actively controlled by a benchmark administrator. Building on our framework, Coulter, Shapiro and Zimmerman (2017) address the optimal design of fines in a "revealed preference mechanism" that directly elicits private information from the agents. Because Coulter *et al.* (2017) do not study the problem of designing an optimal fixing, our approaches are complementary.

Our work falls into a growing literature on mechanism design without transfers. This body of research, however, typically focuses on allocation problems.[12] The techniques we use are reminiscent of those used to study direct revelation mechanisms and, to some degree, principal-agent models. There are, however, essential differences. Because of the restriction on the class of mechanisms (linear estimators), we cannot rely on the Revelation Principle. The objective function is not typical. Our mechanism designer is minimizing the mean squared error of the estimator (benchmark). Agents face a cost of misreporting their type which is proportional to the deviation from the true type.[13] Overall, we are forced to develop new techniques that draw on tools from optimal control theory.

We do not analyze estimators that assign different weights to transactions based on the transactions prices themselves (that is, nonlinear estimators). This extension is an obvious next step. For example, some benchmarks such as LIBOR dampen or eliminate the influence of prices that are outliers. Eisl, Jankowitsch and Subrahmanyam (2014) and Youle (2014) argue that the median estimator can significantly reduce the incentive to manipulate. However, the net effect on the statistical efficiency of a median-based fixing as an estimator of the underlying market value is unknown in a setting such as ours with strategic data generation.

The remainder of the paper is organized as follows. Section 2 introduces the primitives of the model and the solution concept. Section 3 offers some preliminary analysis in preparation for a treatment of the main problem in Section 4. Section 5 treats fixings that are robust to order splitting. In Section 6, we introduce two specific models of manipulation that micro-found our reduced-form baseline framework. Section 7 concludes and discusses some

[12] See for example Ben-Porath, Dekel and Lipman (2014) and Mylovanov and Zapechelnyuk (2017).

[13] Lacker and Weinberg (1989) analyze a model of an exchange economy where an agent may falsify public information at a cost; Kartik (2009) studies a cheap talk game in which the Sender pays a cost for deviating from the truth; Kephart and Conitzer (2016) formulate a Revelation Principle for a class of models in which the agent faces a reporting cost.

extensions and future research directions. Most proofs are relegated to the appendix.

2 The baseline model

A mechanism designer (benchmark administrator) will estimate an uncertain variable Y, which can be viewed as the "true" market value of an asset. To this end, she designs a benchmark fixing, which is an estimator $\hat{Y}$ that can depend on the transaction data $\left\{(\hat{X}_i,\, \hat{s}_i)\right\}_{i=1}^{n}$ generated by a fixed set $\{1,\, 2, \ldots,\, n\}$ of agents. Here, $\hat{X}_i$ is the price and $\hat{s}_i$ is the quantity of the transaction of agent i. The size $\hat{s}_i$ of each transaction is restricted to $[0, \bar{s}]$, a technical simplification that could be motivated as a risk limit imposed by a market regulator or by an agent's available capital. The price $\hat{X}_i$ is a noisy or manipulated signal of Y, in a sense to be defined. Agents are strategic: they have preferences, to be explained, over their respective transactions and over the benchmark $\hat{Y}$. The sensitivity of a given agent's utility to $\hat{Y}$ is known only to that agent. The agents do not collude.

We describe in detail the problem of the benchmark administrator and the agents. Further interpretation of our assumptions is postponed to the end of the section.

2.1 The problem of the benchmark administrator

The benchmark administrator minimizes the mean squared error $\mathbb{E}\left[(Y - \hat{Y})^2\right]$ of the benchmark fixing $\hat{Y}$, which is restricted to a linear estimator of the form

$$\hat{Y} = \sum_{i=1}^{n} f(\hat{s}_i)\hat{X}_i,$$

where $f : [0,\, \bar{s}] \to \mathbb{R}^+$ is a transaction weighting function to be chosen. In particular, the weight placed on a given transaction depends only on its size, and not on its price or on the identities of the agents. We do not require that the weights sum to one, but we do require the estimator to be unbiased. We will provide distributional conditions under which unbiasedness is equivalent to the condition that the weights sum to one in expectation, that is,

$$\mathbb{E}\left[\sum_{i=1}^{n} f(\hat{s}_i)\right] = 1.$$

Later, we will discuss the restriction to fixing weights that are based only on transactions sizes, as opposed to weights that could depend jointly on both the sizes and prices of transactions.

We impose a mild regularity condition that is needed to ensure the existence of a solution to the administrator's problem. Let $\mathcal{C}^{K,M}$ be the set of upper semi-continuous $f : [0,\, \bar{s}] \to \mathbb{R}^+$

with the property that there exist at most K points $0 = s_1 < s_2 < ... < s_{K-1} < s_K = \bar{s}$ such that f is Lipshitz continuous with Lipshitz constant M in each $(s_i,\, s_{i+1})$. The constants K and M are assumed to be finite but large.[14] This regularity allows the weighting function f to have finitely many jump discontinuities and points of non-differentiability.

We summarize the problem of the benchmark administrator as

$$\inf_{f\,\in\,\mathcal{C}^{K,M}} \; \mathbb{E}\left[\left(Y - \sum_{i=1}^{n} f(\hat{s}_i)\hat{X}_i\right)^2\right] \quad \text{subject to} \quad \mathbb{E}\left[\sum_{i=1}^{n} f(\hat{s}_i)\right] = 1. \tag{$\mathcal{P}$}$$

2.2 The problem of the agents

We now explain how the transaction data $\left\{(\hat{X}_i,\, \hat{s}_i)\right\}_{i=1}^{n}$ are generated by strategic agents. We assume that an agent can conduct a manipulated transaction at some reduced-form net benefit, without explicitly modeling the market in which the transaction takes place. In Section 6, we propose two alternative stylized models of market trading that endogenize the reduced-form costs and benefits of a manipulator.

Agent i privately observes her type R_i, which is interpreted as the agent's profit exposure to the benchmark. Specifically, the agent's payoff includes a profit component $R_i\hat{Y}$. This type R_i can be negative, corresponding to cases when the agent holds a short position in the asset whose value is positively correlated with the benchmark.

Having observed R_i, the agent chooses a pair $(\hat{z}_i,\, \hat{s}_i) \in \{-z_i,\, 0,\, z_i\} \times [0,\, \bar{s}]$, where $\hat{z}_i$ is a price distortion and $\hat{s}_i$ is a trade size. The absolute magnitude $z_i > 0$ of the price distortion is assumed to be exogenous, and can be a random variable, as discussed in Section 6. If agent i chooses not to manipulate, in that $\hat{z}_i = 0$, the benchmark administrator observes $(\hat{X}_i,\, \hat{s}_i) = (X_i,\, s_i)$, where $(X_i,\, s_i)$ can be thought of the transaction (price and quantity) of agent i that would be naturally preferred in the absence of manipulation incentives. The transaction $(X_i,\, s_i)$ is determined by hedging or speculative motives that we do not model. That is, we take the distribution of $(X_i,\, s_i)$ as given. This is further discussed in the next subsection. On the other hand, if agent i chooses to manipulate, in that $\hat{z}_i \neq 0$, then the benchmark administrator observes the transaction $(\hat{X}_i,\, \hat{s}_i) = (X_i + \hat{z}_i,\, \hat{s}_i)$.

Intuitively, the agent can trade some quantity $\hat{s}_i$ at a distorted price to manipulate the benchmark fixing. This substitution, however, induces a cost $\gamma\hat{s}_i|X_i - \hat{X}_i|$ to the agent that is proportional to the size of the transaction and to the deviation of the price from the market level X_i, where $\gamma > 0$ is a fixed parameter.

We assume that the agent does not observe X_i at the time of making the report to the

[14]Formally, all our results hold for large enough M and K.

benchmark administrator. That is, the agent cannot predict the market price when deciding whether to manipulate or not. This assumption is motivated by tractability but is also realistic in some settings, an example of which is discussed in Section 6.

Without loss of generality, we normalize to zero the payoff to the agent associated with the truthful reporting choice $(\hat{X}_i,\, \hat{s}_i) = (X_i,\, s_i)$. Given the additivity of the benchmark across transactions, each agent can ignore the contribution of any of the other transactions to the distortion-related profit $R_i\hat{Y}$. We can thus summarize the problem of agent i as

$$\max_{\hat{z}_i \in \{-z_i,\, 0,\, z_i\},\, \hat{s}_i \in [0,\, \bar{s}]} \left[R_i \mathbb{E}\left[\Delta_{\hat{z}_i} \hat{Y}\right] - \gamma \hat{s}_i |\hat{z}_i| \right] \mathbf{1}_{\{\hat{z}_i \neq 0\}}, \tag{$\mathcal{A}$}$$

where $\mathbb{E}\left[\Delta_{\hat{z}_i}\hat{Y}\right]$ is the expected change in the benchmark fixing relative to choosing $\hat{z}_i = 0$. (Thus, $\Delta_0 \hat{Y} = 0$.) We will later provide an explicit calculation of this expected change. Existence of solutions to the agent's problem is guaranteed by the assumption that the weighting function is upper semi-continuous. We assume that the agent chooses not to manipulate when she is indifferent. If, conditional on manipulation, there are multiple optimal $\hat{s}_i$, then we assume that the agent chooses the largest of these transaction sizes. (This tie breaker does not affect our subsequent results.)

2.3 The distribution of transactions data

The unmanipulated transactions $\{(X_i,\, s_i)\}_{i=1}^n$ are generated as follows. First, Y is drawn from some probability distribution with mean normalized to zero, and with some finite variance σ_Y^2. Then, a pair $(\epsilon_i,\, s_i)$ is drawn for every agent, *i.i.d.* across agents and independently of Y. We assume that $\mathbb{E}\left(\epsilon_i \,|\, s_i\right) = 0$, and that $\text{var}\left(\epsilon_i |\, s_i\right) = \sigma_\epsilon^2$ for some $\sigma_\epsilon^2 > 0$. The size s_i has a cumulative distribution function (cdf) G with a continuous density g that is strictly positive on $[0,\, \bar{s}]$. The unmanipulated price is $X_i = Y + \epsilon_i$, which is therefore a noisy and unbiased signal of Y, with variance $\sigma_U^2 \equiv \sigma_Y^2 + \sigma_\epsilon^2$. The subscript U is a mnemonic for "unmanipulated."

The exposure types $R_1, \ldots, R_n$ are *i.i.d.* and independent of all other primitive random variables in the model. The cdf $\tilde{H}$ of R_i has support contained by some interval $[-\bar{R},\, \bar{R}]$. We allow the case of $\bar{R} = \infty$. We assume that the probability distribution of R_i is symmetric around zero, and that bigger incentives to manipulate are relatively less likely to occur than smaller incentives. That is, $\tilde{H}$ has a finite variance and a density $\tilde{h}$ that is symmetric around zero and strictly decreasing on $(0,\, \bar{R})$. Examples include the normal and Laplace ("double exponential") distributions. Given the symmetry of $\tilde{H}$, we can define a cdf H on $[0,\, \bar{R}]$ such that

$$\tilde{H}(R) = \begin{cases} \frac{1}{2} - \frac{1}{2}H(-R) & \text{if } R < 0, \\ \frac{1}{2} + \frac{1}{2}H(R) & \text{if } R \geq 0. \end{cases}.$$

That is, H is the distribution of R_i conditional on $R_i \geq 0$. We let h denote the density of H, and we assume that h is twice continuously differentiable.

The agents' respective price-distortion magnitudes $z_1, \ldots, z_n$ are *i.i.d.*, and are independent of all other primitive model variables. The variance σ_z^2 of z_i is finite and strictly positive. We let $\sigma_M^2 \equiv \sigma_U^2 + \sigma_z^2 > \sigma_U^2$ denote the variance of the reported price $X_i + \hat{z}_i$ conditional on the event $\{\hat{z}_i \neq 0\}$ of a manipulation. The subscript M is thus a mnemonic for "manipulated."

2.4 Comments on assumptions

For tractability, we have restricted attention to estimators that are linear with respect to price, with weights depending only on the sizes of the respective transactions. The common volume-weighted-average-price (VWAP) form of benchmark has *relative* size weights

$$\frac{\hat{s}_i}{\sum_{j=1}^{n} \hat{s}_j}.$$

For the case of a large number n of underlying transactions, the VWAP is therefore approximately of the form that we study. Our assumption that the weight $f(\hat{s}_i)$ on transaction i does not depend on the sizes of other transactions could be justified by the desire of the administrator to avoid strategic interactions between agents' decisions. This is similar in spirit to motivations for strategy-proofness in mechanism-design problems. An implication of this assumption is that the benchmark that we study is robust to some forms of collusion. For example, agents cannot benefit by sharing information about their exposure types with each other, nor from attempting to coordinate their decisions.

We have assumed that variance of the price noise ϵ_i associated with an unmanipulated transaction has a variance that does not depend on the size of the transaction. It would be more realistic to allow the price precision to be increasing with the size of the transaction, as implicitly supported by volume-weighted-average-price (VWAP) schemes often used to report representative prices in financial markets.[15] Let $\kappa(s_i) = \text{var}\,(X_i \mid s_i)^{-1}$ denote the precision of the unmanipulated price X_i conditional on the transaction size s_i. Focusing on the case of a constant $\kappa(s)$ allows us to greatly simplify our arguments and sharpen the results. However, as shown in a preliminary version of this paper (Duffie and Dworczak, 2014), our qualitative conclusions remain valid provided that $\kappa(s)$ is a non-decreasing and concave function, and that σ_z^2 is large enough that a manipulated transaction has more price noise than an unmanipulated transaction, regardless of its size.

The problem faced by each agent is stylized. We aim to capture some of a manipulator's key incentives. The agent's type $(X_i,\, s_i)$ can be interpreted as the transaction that the agent

[15] See, for example, Berkowitz *et al.* (1988).

would make, given current market conditions, to fulfill her usual "legitimate" business needs. For example, such a trade could be the result of a natural speculative, market making, or hedging motive. The assumption that each agent can make only one transaction is relaxed in Section 5. Formally, this assumption is justified if all of the transactions of a given agent are first aggregated and only then provided as a single input to the estimator. This is the method currently used in the fixing of LIBOR by ICE Benchmark Administrator. Section 5 considers the problem of a benchmark administrator when such an aggregation is infeasible or undesirable.

For simplicity, we have also assumed that the size of a price manipulation is bounded by z_i. Alternatively, we could assume that there there is an increasing cost $\psi(|z|)$ of manipulation, based for instance on an increasing probability of detection. Formally, in our setting, $\psi(|z|) = c\mathbf{1}_{\{z \notin [-z_i, z_i]\}}$, for some large $c > 0$. The results depend mainly, in this regard, on the assumption that the manipulation levels chosen by agents are high enough that manipulated transactions are less precise signals of price than unmanipulated transactions. This property would hold across many plausible alternative model specifications.

The cost of manipulation reflects the losses that the agent incurs when trading away from market prices in order to manipulate the fixing. We take a partial-equilibrium approach, relegating an endogenous model of trading and payoffs to Section 6. Our particular functional form for the cost of manipulation, chosen in large part for its tractability, can be further justified by an alternative interpretation of the nature of manipulations. Namely, imagine that agents can submit "shill trades," in the form of fictitious transactions at distorted prices, with reimbursements,"kickbacks," arranged through side payments. Then $\hat{s}_i|X_i - \hat{X}_i|$ is precisely the kickback cost of manipulation.[16] Assuming that the cost of manipulation is linear, rather than strictly convex, with respect to size and price distortion is a conservative approach in that it allows for relatively higher profits associated with larger manipulations.

Finally, R_i can be thought of as the position that the agent holds in contracts whose settlement price is tied to the administered benchmark fixing. For example, many manipulators of LIBOR were motivated by the fact that they held interest rate derivatives whose settlement payments are contractually based on the fixing of LIBOR. For positions such as options whose market values are nonlinear with respect to a benchmark, one can view R_i as the so-called "delta" (first-order) sensitivity of the position value to the benchmark. The assumption that R_i is symmetric around zero is, in effect, a belief by the benchmark administrator that upward and downward manipulative incentives are similar, other than with respect to their signs.

[16]This assumption is that the cost is linear in size. We can view γ as a per-dollar cost of using an illegal transfer channel, for example resulting from the possibility of detection and punishment.

3 Using the fixing to deter manipulations

In this section we provide some basic properties of an optimal benchmark, and present solutions to some preliminary cases that provide intuition as well as elements on which to build when solving the general case.

3.1 Solution without manipulation

For comparison purposes, we first solve the problem assuming that agents do not manipulate. The law of iterated expectation implies that

$$\mathbb{E}\,[\hat{Y}\,|\,Y] \;=\; \left[\mathbb{E}\sum_{i=1}^{n} f(s_i)\right] Y. \tag{3.1}$$

Thus, $\hat{Y}$ is unbiased if and only if $\mathbb{E}\sum_{i=1}^{n} f(s_i) = 1$. It follows that

$$\mathbb{E}\left[(Y-\hat{Y})^2\right] = -\frac{\sigma_Y^2}{n} + \sum_{i=1}^{n} \mathbb{E}\left[f^2(s_i)\sigma_U^2\right].$$

Using the symmetry assumption, we can formulate the problem of the benchmark administrator as

$$\inf_{f \in \mathcal{C}^{\mathcal{K},\mathcal{M}}} \; \sigma_U^2 \int_0^{\bar{s}} f^2(s) g(s)\, ds \quad \text{subject to} \quad \int_0^{\bar{s}} f(s) g(s)\, ds = \frac{1}{n}.$$

Proposition 1 *Absent manipulation, the weighting function that solves problem $\mathcal{P}$ is given by $f^\star(s) = 1/n$.*

The proof is skipped. This problem can be viewed as a simple case of ordinary-least-squares estimation. The benchmark administrator's optimal weights are proportional to the precision of each price observation. Because the precisions are assumed to be identical and in particular invariant to the sizes of transactions, the optimal weights are equal. There is an obvious generalized-least-squares extension to the case of a general covariance structure for the observation "noises" $\epsilon_1, \ldots, \epsilon_n$.

3.2 Incentives to manipulate

We now turn to the manipulation problem $\mathcal{A}$ facing an agent. By symmetry, we may concentrate on the event of a positive manipulation incentive, $R_i \geq 0$. Using the assumptions of

Subsections 2.2 and 2.3, we can express the problem $\mathcal{A}$ of agent i as

$$\max_{\hat{z}_i \in \{0,\, z_i\},\, \hat{s}_i \in [0,\, \bar{s}]} \; [R_i f(\hat{s}_i) - \gamma \hat{s}_i]\, \hat{z}_i \mathbf{1}_{\{\hat{z}_i \neq 0\}}. \tag{3.2}$$

An agent with type R_i manipulates if and only if there is some $s \in [0,\, \bar{s}]$ such that $R_i f(s) > \gamma s$, that is, if there is a size for the manipulated trade at which the impact on the benchmark fixing is high enough to cover the associated manipulation cost.

If an agent with type R_i chooses to manipulate, then all agents with types higher than R_i also manipulate. Similarly, if an agent with type R_i chooses not to manipulate, all agents with types below R_i also choose not to manipulate. It follows that with any weighting function f we may associate a unique threshold R_f defined by

$$R_f = \sup\{R \leq \bar{R} : Rf(s) \leq \gamma s, \quad s \in [0,\, \bar{s}]\}.$$

That is, given f, the types above R_f manipulate and the types below R_f do not. This easy observation leads to the following result.

Proposition 2 *The benchmark administrator can ensure that there are no manipulations if and only if $\bar{R} \leq n\gamma \mathbb{E}[s_1]$. If the benchmark administrator is further constrained to implement non-manipulation, then the optimal weighting function $f^\star$ is given by $f^\star(s) = \gamma \bar{R}^{-1} s$ on $[0,\, s_0]$ and $f^\star(s) = f^\star(s_0)$ on $[s_0,\, \bar{s}]$, where s_0 is chosen to satisfy the constraint*

$$\int_0^{\bar{s}} f^\star(s) g(s)\, ds = \frac{1}{n}.$$

Proof: We sketch the proof. The remaining details are easy. By the above characterization, it is possible to implement no-manipulation (truthful reporting) if and only if, for every $s \in [0,\, \bar{s}]$, we have $\bar{R} f(s) \leq \gamma s$. Because the administrator is constrained by $\int_0^{\bar{s}} f(s) g(s)\, ds = 1/n$, it is necessary that

$$\frac{1}{n} \leq \frac{\gamma}{\bar{R}} \int_0^{\bar{s}} s g(s)\, ds.$$

This condition is also sufficient. If this condition holds, we can obtain the optimal weighting function by applying standard techniques from optimal control theory. □

The result states that implementing truthful reporting may sometimes be possible. However, the condition $\bar{R} \leq n\gamma \mathbb{E}[s_1]$ is likely to be violated in practice, especially when the underlying asset market for the benchmark is thinly traded relative to the market for instruments that determine the incentives to manipulate, as is the case for LIBOR. A thinly traded underlying market corresponds to the case in which $\bar{R}$ is large relative to $\mathbb{E}[s_1]$, the expected size of a typical transaction in the benchmark market. This condition is even less likely to be satisfied when manipulation is relatively cheap (γ is small) or when there are few transactions (n is small). The latter case is indeed a practical concern because banks are increasingly reluctant to support benchmarks in the face of potential regulatory penalties and the risk of private litigation, as documented by Brundsen, 2014 for the case of EURIBOR. The head of the U.K. Financial Conduct Authority, Andrew Bailey, has similarly announced[17] that LIBOR may be discontinued because the number of reporting banks may become too small once the agreement of the banks to continue reporting expires at the end of 2021.

The following example shows that it need not be optimal for the benchmark administrator to induce truthful reporting with certainty, even in the case when it is possible.

Example 1 *Suppose that* $\gamma = 1$, $n = 10$, $\bar{R} = 5$, $\sigma_Y^2 = \sigma_\epsilon^2 = \sigma_z^2 = 1$, *$g$ is the uniform density on* $[0, 1]$, *and $\tilde{h}$ is the uniform density*[18] *on* $[-5, 5]$. *Then, f is feasible and implements truthful reporting if and only if* $f(s) = s/5$. *The value of the benchmark administrator's objective function is* $1/6$. *Consider an alternative weighting function f_α that is linear up to a threshold and then flat, in that*

$$f_\alpha(s) = \alpha \max\{s,\, s_0^\alpha\},$$

where $\alpha \geq 1/5$ *and s_0^α is chosen such that the constraint in problem $\mathcal{P}$ holds. If* $|R_i| > 1/\alpha$ *then agent i manipulates, choosing* $\hat{s}_i = s_0^\alpha$. *The value of the administrator's objective function is in this case strictly below* $1/6$ *for all α between* $1/5$ *and* $2/5$. *Thus, it is optimal to allow manipulation. At the optimal choice of 1/4 for $\alpha^\star$, the objective function is approximately equal to* 0.16, *and the unconditional probability of manipulation is* $1/5$.

A consequence of Proposition 2 is that the benchmark administrator should not restrict attention to weighting functions that fully deter manipulation. The optimal weighting function instead influences the *degree* to which agents manipulate. For clarity of exposition, we henceforth assume that $\bar{R} = \infty$, so that implementing truthful reporting is not possible.[19]

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[17] On November 24, 2017, a press release of the Financial Conduct Authority stated: "Andrew Bailey, FCA Chief Executive, set out in a speech earlier this year that, whilst significant improvements have been made to LIBOR since April 2013, the absence of active underlying markets means that the future sustainability of LIBOR cannot be guaranteed. The support of the panels for LIBOR is needed until the end of 2021, by when a transition can be made to alternative rates. The FCA has been working with the panel banks to finalise an agreement for the banks to remain on the panels they currently submit to until the end of 2021."

[18] The uniform density is not strictly decreasing, as assumed in Section 2, but this property is not needed for this example.

[19] This is practically without loss of generality because we can specify the cdf H of R_i to place arbitrarily small probability mass above some finite $\bar{R}$.

3.3 Administrator's problem under manipulation

We now derive a concise mathematical formulation of the problem $\mathcal{P}$ faced by the benchmark administrator under the assumptions of Subsections 2.2 and 2.3.

First, we use our symmetry assumptions to simplify the problem. From the viewpoint of the benchmark administrator, the events $\hat{z}_i = z_i$ and $\hat{z}_i = -z_i$ are equally likely, even after conditioning on $\hat{s}_i$. Therefore, equation (3.1) still holds if we replace s_i by $\hat{s}_i$. That is, forcing the estimator $\hat{Y}$ to be unbiased is equivalent to the requirement that $\mathbb{E}\left[\sum_{i=1}^{n} f(\hat{s}_i)\right] = 1$. We denote by $\Psi_f(\,\cdot\,)$ the cdf of the transaction size $\hat{s}_i$, conditional on its manipulation. That is,

$$\Psi_f(s) = \mathbb{P}_{R_i \sim H}(\operatorname{argmax}_{\hat{s}} R_i f(\hat{s}) - \gamma \hat{s} \leq s |\, R_i > R_f). \tag{3.3}$$

By the law of iterated expectations and because of arguments presented in Subsection 3.2,

$$\mathbb{E}\left[(Y - \hat{Y})^2\right] = \sum_{i=1}^{n} \int_0^{\bar{s}} f^2(\hat{s}_i) \left[\sigma_U^2 H(R_f) g(\hat{s}_i)\, d\hat{s}_i + \sigma_M^2 (1 - H(R_f))\, d\Psi_f(\hat{s}_i)\right] - \frac{\sigma_Y^2}{n}.$$

The displayed equation states that if $|R_i| \leq R_f$ (which happens with probability $H(R_f)$), then the transaction of agent i is unmanipulated, $\hat{s}_i = s_i$, $\hat{X}_i$ has variance σ_U^2, and $\hat{s}_i$ has probability density g. On the other hand, if the transaction is manipulated, which happens with probability $(1 - H(R_f))$, then $\hat{X}_i$ has variance σ_M^2 from the viewpoint of the benchmark administrator.

Similarly, we can express the constraint in problem $\mathcal{P}$ as

$$1 = \mathbb{E}\left[\sum_{i=1}^{n} f(\hat{s}_i)\right] = \sum_{i=1}^{n} \int_0^{\bar{s}} f(\hat{s}_i) \left[H(R_f) g(\hat{s}_i)\, d\hat{s}_i + (1 - H(R_f))\, d\Psi_f(\hat{s}_i)\right].$$

To characterize the optimal benchmark, we use an approach familiar from principal-agent models. We address the best way, given some target manipulation threshold R, for the administrator to implement an outcome in which an agent with $|R_i| \leq R$ chooses not to manipulate. As we saw before, this requires that the benchmark weight function satisfies the additional constraint $f(s) \leq (\gamma/R)s$. Solving this auxiliary problem is a key step towards solving the original problem $\mathcal{P}$. This auxiliary problem is illuminating in its own right. For example, the benchmark administrator may have exogenous preferences for deterring manipulation, which could be modeled by setting a high manipulation threshold R. Formally, using the assumption that agents are symmetric, we can formulate the auxiliary problem as

$$\inf_{f \in \mathcal{C}^{K,M}} \int_0^{\bar{s}} f^2(s) \left[\sigma_U^2 H(R) g(s)\, ds + \sigma_M^2 (1 - H(R))\, d\Psi_f(s)\right] \tag{$\mathcal{P}(R)$}$$

subject to[20]

$$f(s) \leq \frac{\gamma}{R} s, \quad s \in [0, \bar{s}], \tag{3.4}$$

$$\int_0^{\bar{s}} f(s) \left[H(R) g(s) \, ds + (1 - H(R)) \, d\Psi_f(s) \right] = \frac{1}{n}. \tag{3.5}$$

If the target manipulation threshold R is too high, then no function f inducing that threshold (that is, satisfying constraint (3.4)) will satisfy constraint (3.5). Among all weighting functions f satisfying (3.4), $f(s) = (\gamma/R)s$ maximizes the left hand side of (3.5). In particular, under this transaction weighting all manipulators choose the maximal transaction size $\bar{s}$. Therefore, if we define

$$\hat{R} = \max \left\{ R \geq 0 : \frac{\gamma H(R)}{R} \mathbb{E}(s_1) + \frac{\gamma(1 - H(R))}{R} \bar{s} \geq \frac{1}{n} \right\},$$

then the condition $R \leq \hat{R}$ is both necessary and sufficient for the set of feasible f to be non-empty for the problem $\mathcal{P}(R)$.

4 The optimal benchmark

In this section we present the solution to the problem faced by the benchmark administrator. Theorem 1a lists the main properties of the optimal benchmark. Theorem 1b describes the exact shape of the optimal fixing under a technical assumption. When this technical assumption fails, the optimal fixing can still be described as a solution to a parameterized differential equation. The full description of this solution is relegated to Appendix A. Following the statement of the main result, we discuss the intuition and sketch its proof. The remaining details can be found in Appendix B.

Theorem 1a *For any $R \in (0, \hat{R})$, there exists a unique solution $f^\star$ to problem $\mathcal{P}(R)$. Moreover, $f^\star$ is non-decreasing, concave, continuously differentiable, and satisfies $(f^\star)'(\bar{s}) = 0$. There is some $s_0 > 0$ such that $f^\star(s)$ coincides with $(\gamma/R)s$ whenever $s \leq s_0$.*

The following ordinary differential equation (ODE), which will play a key role in determining the shape of the optimal fixing, is indexed by two parameters: s_0 and s_1, with $0 < s_0 < s_1 < \bar{s}$. Consider

$$f''(s) = -\frac{\left[f(s_1) - f(s)\right] H(R) g(s) + \frac{\sigma_M^2}{\sigma_U^2} \gamma h\left(\frac{\gamma}{f'(s)}\right)}{\left[\frac{\sigma_M^2}{\sigma_U^2} f(s) - f(s_1)\right] \left(-h'\left(\frac{\gamma}{f'(s)}\right)\right) \frac{\gamma^2}{(f'(s))^3}}, \tag{4.1}$$

[20] We abuse notation slightly by treating $\Psi_f(s)$ as being defined by (3.3) with R_f replaced by R.

with boundary conditions

$$f(s_0) = (\gamma/R)s_0, \qquad f'(s_0) = (\gamma/R).$$

Theorem 1b *Suppose that there exist $s_0 < s_1 < \bar{s}$ such that $f^\star$, defined by*

$$f^\star(s) = \begin{cases} \frac{\gamma}{R}s, & s \in [0,\, s_0] \\ \text{solves (4.1)}, & s \in (s_0,\, s_1) \\ f^\star(s_1), & s \in [s_1,\, \bar{s}], \end{cases}$$

is continuously differentiable and satisfies (3.5). Then, for any $R \in (0,\, \hat{R})$, $f^\star$ is the unique solution to the optimal fixing problem $\mathcal{P}(R)$.

When no s_0 and s_1 satisfying the condition of Theorem 1b can be found, the solution to the optimal fixing problem $\mathcal{P}(R)$ satisfies a generalization of (4.1) on the interval $[s_0,\, \bar{s}]$ that is provided in Appendix A.

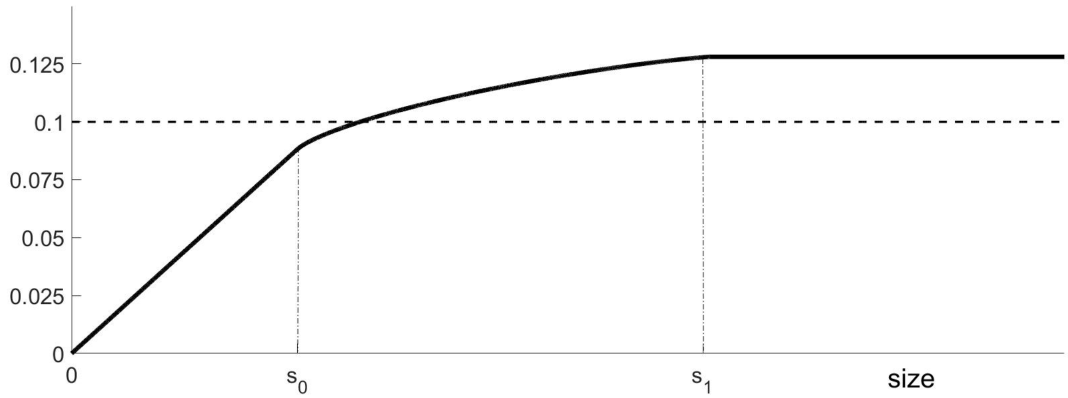
Intuitively, in Theorem 1b, for any given s_0, the point s_1 is chosen so that $f'(s_1) = 0$. (This is called the "shooting method.") This construction guarantees that $f^\star$ is continuously differentiable. Then s_0 can be chosen to satisfy (3.5). However, especially when $\bar{s}$ is relatively small, suitable choices for s_0 and s_1 might not exist. In such a case, the optimal $f^\star$ asymptotes to a constant function without being constant on any interval; $f^\star$ satisfies a generalized version of (4.1) given in Appendix A, which depends on an additional parameter chosen to satisfy the boundary condition $(f^\star)'(\bar{s}) = 0$.

Because there is no explicit solution to the differential equation (4.1), a closed-form solution for the optimal fixing-weight function $f^\star$ is not available. However, Theorem 1a provides a number of economic predictions about the form of the optimal benchmark.

One robust finding is that the optimal weighting function becomes flat as the transaction size increases, as captured by the property $(f^\star)'(\bar{s}) = 0$. The optimal $f^\star$ is typically flat after some threshold transaction size $s_1 < \bar{s}$, as predicted by Theorem 1b and as illustrated in Example 2, to follow. Intuitively, assigning too much weight to very large transactions is suboptimal because it induces agents with high manipulation incentives to choose large transaction sizes, resulting in overweighting such large transactions in the estimator.

Another general feature of the optimal benchmark is that $f^\star(s)$ coincides with $(\gamma/R)s$ for a sufficiently small transaction size s. In particular, $f^\star$ attaches small weight to small transactions. The shape of the optimal fixing for small transactions is pinned down by the binding constraint that an agent with the cutoff type R prefers to avoid manipulation. This is intuitive. If a benchmark fixing places small weight on small transactions, then unmanipulated transactions are underweighted compared to the weight that they would receive in a statistically efficient estimator. Therefore, it is optimal to place the maximal weight on small transactions that is consistent with deterring manipulation by types above R.

Fig. 4.1: Optimal weighting function for Example 2 (The dotted line depicts the optimal solution in the absence of manipulations)



Finally, Theorem 1a indicates that the optimal benchmark provides an incentive for "smoothing out" manipulations, preventing them from "bunching" around a given transaction size. This is perhaps somewhat surprising. The manipulated transactions have the same precisions as signals of Y, and yet it is optimal to attach different weights to them. In particular this shows that the functions considered in Example 1 are not optimal. As added intuition, we note that local behavior of f' has only second-order effects on the incremental variance term $\sigma_U^2 f^2(s)H(R)\,dG(s)$ associated with unmanipulated transactions. In contrast, the incremental variance term $\sigma_M^2 f^2(s)(1-H(R))\,d\Psi_f(s)$ for manipulated transactions is sensitive to the local behavior of f'. This follows from from the influence of f' on the distribution Ψ_f, in that relatively small changes in the slope of f can lead to large changes in the optimal transaction volume chosen by a manipulator. Under our assumptions, this variance term is convex in f'. Thus, minimizing the variance term requires minimizing the variation of f' (subject to meeting other criteria). As a result, f' changes continuously rather than exhibiting discrete jumps.

When the fixing function $f^\star$ is that given by Theorem 1b, all manipulators choose a transaction size in $[s_0, s_1]$, and the distribution of sizes has full support in that interval. However, as shown in Figure 4.2, the distribution of manipulated transaction sizes is typically concentrated around s_0. Under these conditions, the optimal benchmark can be well-approximated by a simple fixing that is linear up to a threshold, and constant afterwards. We comment further on this point in Example 4.

To illustrate the above discussion we consider the following numerical example.

Example 2 *We take the parameters of Example 1, with the exception that* $h(x) = \exp(-x/2)/2$. *The given density* h *implies that, on average, the exposure to the benchmark asset is equal to* 2. *We set the manipulation threshold to be twice the mean,* $R = 4$. *The type threshold* $R = 4$ *corresponds to a probability of manipulation of around* 14%. *The optimal weighting function is depicted in Figure 4.1. This function is smooth* (C^1)*, but its first derivative changes rapidly close to* $s_0 \approx 0.40$. *All of the manipulated transactions are in the interval* $[s_0, s_1]$. *As can be seen in Figure 4.2, manipulations are in fact highly concentrated around* s_0.

While the qualitative properties of the optimal weighting function are intuitive, the particular form of the ODE (4.1) is less clear. To gain intuition, we can rewrite (4.1) as

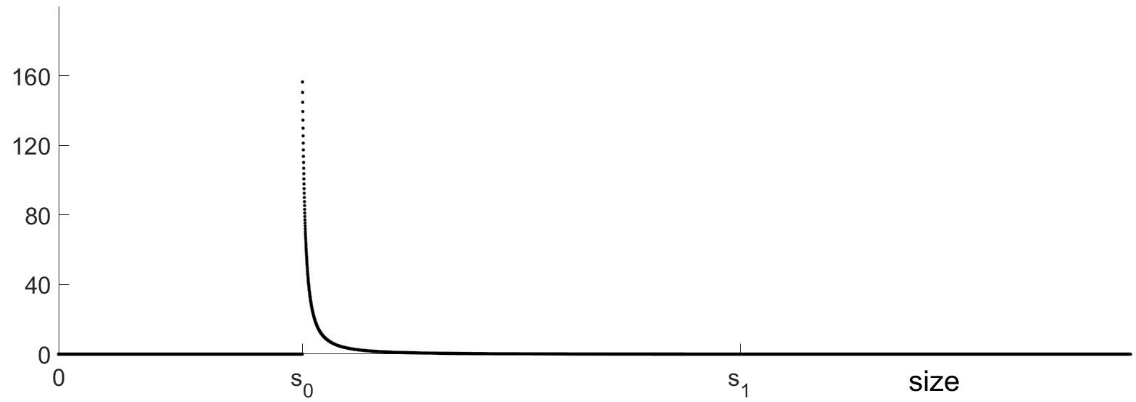
$$\underbrace{\left[f(s)\sigma_M^2 - f(s_1)\sigma_U^2\right] dH\left(\frac{\gamma}{f'(s)}\right)}_{I_M} + \underbrace{\left[f(s) - f(s_1)\right]\sigma_U^2 H(R) g(s)}_{I_U} = \frac{d}{ds}\underbrace{\left[\left(f(s)\sigma_M^2 - f(s_1)\sigma_U^2\right) h\left(\frac{\gamma}{f'(s)}\right)\frac{\gamma}{f'(s)}\right]}_{I_A}.$$

In the above formula, one may think of $f(s_1)$ as the optimal constant weight that would be assigned to unmanipulated transactions for the efficient estimator (fixing) that would be chosen in the absence of manipulation incentives. The term I_U is zero, that is $f(s) = f(s_1)$, when the weight is chosen optimally from the point of view of unmanipulated transactions. This term is proportional to the density of sizes corresponding to unmanipulated transactions. On the other hand, the term I_M is zero, that is $f(s)\sigma_M^2 = f(s_1)\sigma_U^2$, when the weight is chosen optimally from the point of view of manipulated transactions. This term is proportional to the density of sizes that arises from manipulated transactions. In both of these cases, individually, the term I_A is also zero because $h(\gamma/f'(s))(\gamma/f'(s)) = 0$ when f is constant.[21] Ideally the benchmark administrator would like to set both of the terms I_M and I_U to zero, but this is impossible when $\sigma_M^2 > \sigma_U^2$. Thus, the administrator faces a trade-off. She either puts insufficient weight on unmanipulated transactions, which are relatively precise signals of the fundamental value, or she puts too much weight on manipulated transactions, which are relatively noisy signals of the fundamental value Y.

In balancing these two effects, the administrator takes into account the term I_A. By assumption, types in $[R, \infty)$ manipulate. By controlling f', the administrator controls the sizes of transactions chosen by types R_i in $[R, \infty)$. Because the optimal fixing is concave and differentiable, the optimal size of a manipulation is pinned down by the first-order condition for the manipulator's problem (3.2). Thus, an agent with type $R_i = \gamma/f'(s)$ chooses size

[21] While this seems to present some issues associated with division by zero, the result follows from integrability of h, and is formally stated in Lemma 7 of Appendix B.

Fig. 4.2: Density of manipulations



s. The term $\gamma/f'(s)$ starts at R when $s = 0$, and ends at ∞ when $s = \bar{s}$. It follows that $dH(\gamma/f'(s))$ describes the density of manipulated transactions. The term I_A accounts for the fact that when the benchmark administrator chooses $f(s)$ at s, she considers the effect of the speed with which the slope changes on the distribution of the remaining mass of manipulated transactions.

To complete the characterization of the optimal fixing function $f^\star$, we observe that for the case $R = 0$ (at which every type manipulates), the optimal solution is $f^\star(s) = 1/n$. This is analogous to Proposition 1, replacing σ_U^2 with σ_M^2. For the case $R = \hat{R}$, there is only one feasible fixing function, that with $f(s) = (\gamma/R)s$, which is thus trivially optimal.

4.1 Choosing the optimal manipulation threshold R

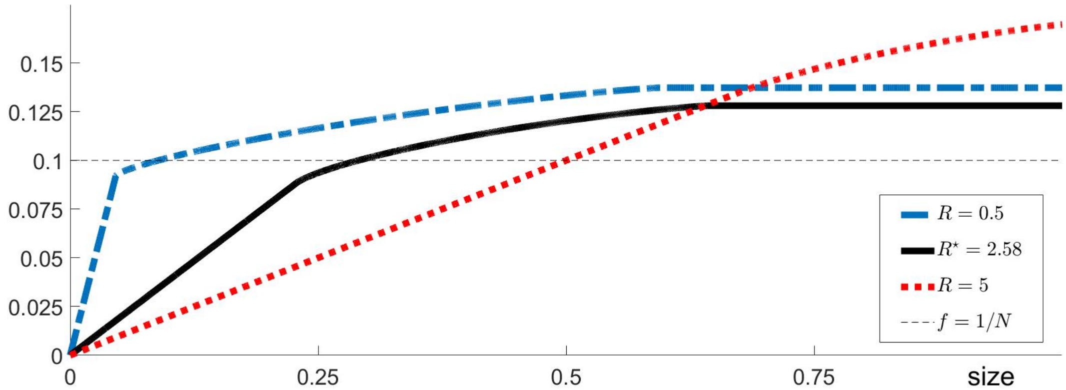
Having characterized the solution to problem $\mathcal{P}(R)$ for a fixed manipulation threshold R, one can solve the original problem $\mathcal{P}$ by choosing an optimal threshold $R^\star$. This involves computing the optimal weighting function $f^\star$ for every $R \in [0, \hat{R}]$, evaluating the objective function, and finding the maximum over all R, achieved at some $R^\star$. This optimum is attained, by Berge's Theorem. While analytic solutions are infeasible, this step can be done numerically. We show below that the optimal $R^\star$ is interior, and thus deters some manipulation but does not minimize the probability of manipulation among all feasible weighting functions.

Proposition 3 *The optimal manipulation threshold for problem $\mathcal{P}$ is interior: $R^\star \in (0, \hat{R})$.*

A consequence of Proposition 3 is that the predictions of Theorem 1a about the shape of the fixing hold at the optimal $R^\star$.

Example 3 *With the parametric assumptions of Example 2, it turns out that $R^\star \approx 2.58$ achieves the minimum for the benchmark administrator's problem $\mathcal{P}$. Figure 4.3 presents the*

Fig. 4.3: Optimal weighting functions for Example 3



optimal weighting function for $R = 0.5$, $R = 2.58$, *and* $R = 5$. *The ex-ante probabilities of manipulation under these target levels are approximately* 0.78, 0.28, *and* 0.08, *respectively.*[22]

4.2 Derivation of the optimal benchmark

In this section, we sketch the proof of Theorems 1a and 1b. The remaining details are presented in the Appendix.

To solve the problem $\mathcal{P}(R)$, we must first determine $\Psi_f(\cdot)$ for each admissible $f \in \mathcal{C}^{K,M}$. This is complicated by the fact that f need not be well behaved. For example, f is not necessarily differentiable or even concave. However, we can use the structure of the manipulation problem faced by agents to overcome this difficulty. We do this in a series of Lemmas which establish that the optimal benchmark exists, and the weighting function f must be continuous, non-decreasing, and concave.

Lemma 1 *The problem* $\mathcal{P}(R)$ *admits a solution for any* $R \leq \hat{R}$.

Our proof of this lemma is relatively involved because the standard argument (exploiting upper semi-continuity of the objective function on a compact domain) does not apply directly. The weighting functions are allowed to have jump discontinuities, which can lead to discontinuities in the objective function (especially if a small change in the weighting function induces a large change in the behavior of manipulators) and failure of compactness. We deal

[22] Although Figure 4.3 may suggest otherwise, the function corresponding to $R = 5$ has a zero derivative at $s = \bar{s}$. The second derivative gets large close to $s = \bar{s}$, so the first derivative changes rapidly in a small neighborhood of $\bar{s}$. This is the case in which Theorem 1b does not apply and the solution is described by Theorem 1 in Appendix A. Figure 4.3 shows that it is possible for two feasible weighting functions to never cross. If the distribution of sizes $\hat{s}_i$ were fixed, this would clearly be impossible because any two such functions could not have the same expectation with respect to the distribution of $\hat{s}_i$. However, this is possible when the distribution of $\hat{s}_i$ depends on the shape of f.

with these difficulties by exploiting the special structure of the problem and the regularity conditions imposed on feasible f. For unmanipulated transactions, due to the continuous distribution of trade sizes, the properties of f on a measure-zero set (in particular at the finitely many points of discontinuity) are irrelevant. For manipulated transactions, we observe that discontinuities in the choice of the optimal size $\hat{s}_i$ can occur only in cases for which the manipulator is indifferent between several transaction sizes. However, such cases are non-generic with respect to R_i. Because R_i has a continuous distribution, any such cases can be ignored when computing the expected payoff.

A simple corollary of Lemma 1 is that the full problem $\mathcal{P}$ also admits a solution. This follows from the Maximum Theorem (also known as Berge's Theorem) which implies continuity of the value of the problem $\mathcal{P}(R)$ in the threshold type R.

Having established existence, we can derive a series of restrictions on the shape of the optimal weighting function.

Lemma 2 *If f is a solution to problem $\mathcal{P}(R)$, then f is non-decreasing.*

The proof of this lemma is technical and thus relegated to the Appendix, but the intuition behind this result is straightforward and instructive. Suppose that a feasible weighting function f is not non-decreasing. Then we can find an interval $[s_0, s_1] \subset [0, \bar{s}]$ such that no manipulator chooses a transaction size in this interval. Intuitively, manipulators never choose transactions that give them the same influence on the benchmark as some smaller (hence less costly) transaction. Absent manipulation, however, we saw in Proposition 1 that the optimal weight is constant. Thus, we can modify f in such an interval so as to retain feasibility but improve the value of the program $\mathcal{P}$. This rules out the optimality of f.

Lemma 3 *If f is a solution to problem $\mathcal{P}(R)$, then f is continuous.*

By Lemma 2 and the regularity conditions imposed on any weighting function, we can prove Lemma 3 merely by ruling out cases in which f jumps up at some s_0. If there is a jump at s_0, then there are no manipulations in $(s_0 - \epsilon, s_0)$ for small $\epsilon > 0$ because the manipulator can discretely increase the influence on the benchmark by choosing s_0 instead, at a negligibly higher cost. Absent manipulations, the jump in f is suboptimal because the optimal weight for unmanipulated transactions is constant. So, we can improve on a discontinuous f by "smoothing it out" in the neighborhood of s_0.

Lemma 4 *If f is a solution to problem $\mathcal{P}(R)$, then f is concave.*

To prove Lemma 4, we use the fact that there can be no manipulations in intervals over which the weighting function f fails to be concave, that is, where f lies below some affine function. This follows from the linearity of costs. In such cases, we can modify f in such an interval without inducing manipulation, so as to improve the weighting of the non-manipulated transactions.

Given Lemmas 1-4, it is without loss of generality that we consider only weighting functions in the set

$$\mathcal{F} = \{f \in \mathcal{C}^{K,M} : f \text{ is continuous, nondecreasing, and concave}\}.$$

The concavity of f implies that we can use first-order conditions to solve the agent's manipulation problem. However, f is not necessarily differentiable, so we use "superdifferential" calculus.[23] We denote by $\partial f(s)$ the superdifferential of f at the point s. A function $f \in \mathcal{F}$ is superdifferentiable at any point $s \in (0, \bar{s})$ because f is concave, and the existence of a superdifferential at 0 and $\bar{s}$ follows from $R_i f(s) \leq \gamma s$, and the fact that f is non-decreasing. Moreover, $\partial f(s)$ is a non-decreasing correspondence in the strong set order that is singleton-valued for almost all s. A transaction size $\hat{s}_i$ is a global maximum of $R_i f(s) - \gamma s$ if and only if $0 \in \partial(R_i f(\hat{s}_i) - \gamma \hat{s}_i)$, or simply $\gamma/R_i \in \partial f(\hat{s}_i)$. If f is actually differentiable at s, the condition for optimality boils down to the usual first-order condition $R_i f'(s) = \gamma$.

We can now characterize $\Psi_f(\,\cdot\,)$ for any $f \in \mathcal{F}$. For some $s \in [0, \bar{s}]$ and some manipulation threshold R,

$$\begin{aligned}
\Psi_f(s) &= \mathbb{P}\left(\hat{s}_i \leq s \;\middle|\; |R_i| \geq R\right) \\
&= \mathbb{P}\left(\partial f(\hat{s}_i) \geq \partial f(s) \;\middle|\; R_i \geq R\right) \\
&= \mathbb{P}\left(\frac{\gamma}{R_i} \geq f'(s^+) \;\middle|\; R_i \geq R\right) \\
&= \frac{H\left(\frac{\gamma}{f'(s^+)}\right) - H(R)}{1 - H(R)}.
\end{aligned}$$

Here, $f'(s^+)$ denotes the right derivative of f at s and (when applied to sets) the inequality $\geq$ is the strong set order.[24] Because the right derivative of a concave function is a right-continuous and non-increasing function, $\Psi_f(\,\cdot\,)$ is a well defined cdf. Discontinuities in f' correspond to atoms in the distribution of manipulated transaction sizes.

The concavity of f implies that the derivative of f, whenever it exists, lies below γ/R. Indeed, the derivative is non-increasing and the constraint $f(s) \leq (\gamma/R)s$ implies that $f'(0^+) \leq \gamma/R$. Because f is non-decreasing, we also know that $f'(s) \geq 0$. The inequality $f(s) \leq (\gamma/R)s$ implies that $f(0) = 0$. Once these properties are imposed, the constraint

[23]See Rockafellar (1970) for the definitions of the subderivative and subdifferential of a convex function. The superderivative and superdifferential have the analogous definitions for a concave function.

[24]That is, for subsets X and Y of the real line, $X \geq Y$ if for any x in X and y in Y, we have $\max\{x, y\} \in X$ and $\min\{x, y\} \in Y$.

$f(s) \le (\gamma/R)s$ is redundant. We will study a relaxed problem in which we do not impose concavity of f, and instead apply the weaker conditions listed above. We will then verify that the solution to the relaxed problem is concave, validating our approach.

The relaxed problem can be phrased as an optimal control problem in which the control variable is the derivative of f. This approach is valid because our assumptions and previous analysis imply that f is absolutely continuous. So, we have

$$\min_{u:\, u(s) \in [0,\gamma/R]} \quad \int_0^{\bar{s}} f^2(s) \left[\sigma_U^2 H(R)\, g(s)\, ds + \sigma_M^2\, dH\left(\frac{\gamma}{u(s)}\right)\right] \tag{4.2}$$

subject to

$$f(0) = 0, \quad f'(s) = u(s), \quad \int_0^{\bar{s}} f(s) \left[H(R)\, dG(s) + dH\left(\frac{\gamma}{u(s)}\right)\right] = \frac{1}{n}. \tag{4.3}$$

To solve this problem, we apply a theorem that gives sufficient conditions for a control variable and the associated state variable to be optimal. Because the objective function is quadratic in the state variable f and the constraint (4.3) is linear in f, the Hamiltonian is convex in the state variable, implying a unique minimizer.

Lemma 5 *There exists a unique solution to problem* (4.2)-(4.3)*. The solution is non-decreasing, concave, continuously differentiable everywhere, and coincides with* $(\gamma/R)s$ *for small* s*. Moreover, the solution is given by the function* $f^\star$ *described by Theorem 1b whenever such a function exists.*

Because $f^\star$ solves the relaxed problem (4.2)-(4.3) and is feasible for the original problem $\mathcal{P}(R)$, $f^\star$ is also optimal for the original problem. Thus, the proof of Lemma 5 concludes the proof of Theorem 1a-1b.

5 Robustness to order splitting

A practical concern related to the design of benchmarks based on transaction data is that agents intending to trade total quantity s of the asset may split the order into several smaller "chunks" in order to influence the benchmark fixing. So far, we have ruled out this possibility by assuming that each agent conducts exactly one transaction. If we relax this assumption, and in particular assume that the benchmark administrator is not able to aggregate all of the transactions of a single agent, it turns out that the optimal benchmark from Section 4 is susceptible to this type of manipulation. To see this, imagine that agent i, with a positive manipulation incentive R_i, intends to trade the quantity s_1. (See Figure 4.1.) Beyond merely distorting the price of this transaction, the agent can additionally influence the benchmark fixing by submitting two smaller transactions, each with quantity $s_1/2$. Such a manipulation is costless, given our linear cost function, and yields the agent a benefit of

$$R_i(2f^\star(s_1/2) - f^\star(s_1)) > 0.$$

because of the concavity of the weighting function $f^\star$.

By an extension of this argument, if the designer chooses a benchmark fixing f, the effective weighting function that will arise under optimal order-splitting takes the form

$$\bar{f}(s) = \sup\{f(q_1) + \cdots + f(q_k) : q_i \in [0,\, \bar{s}],\ q_1 + \cdots + q_k = s\}.$$

Therefore, if order-splitting is allowed and costless,[25] it is without loss of generality to require that the benchmark administrator chooses a weighting function f that leaves no incentive for this type of order-splitting manipulation. This property is easily seen to be equivalent to the condition that f is superadditive. In particular, for any positive integer k,

$$f(ks) \geq kf(s), \quad s \leq \frac{\bar{s}}{k}. \tag{5.1}$$

Superadditivity is a cumbersome constraint in optimal control problems because it is a global property, ruling out characterizations based on local behavior. Therefore, for tractability, we will assume a slightly stronger mathematical condition by requiring (5.1) to hold for all real $k \geq 1$, and not only for integer k.

Condition 1 *A benchmark weighting function f is robust to order-splitting if*

$$f(ks) \geq kf(s),$$

for all $k \in [1, \infty)$ and all s such that $ks \leq \bar{s}$.

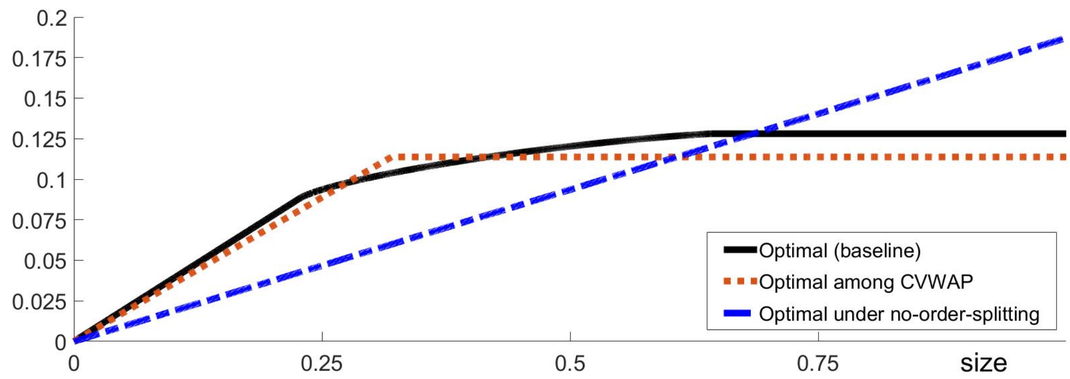
It is clear that the optimal weighting function found in Theorem 1a-1b is not robust to order splitting. In fact, if f is concave but not linear, it cannot satisfy Condition 1.

Theorem 2 *For any $R \leq \hat{R}$, the optimal solution $f^\star$ to problem $\mathcal{P}(R)$, subject to robustness to order splitting, is given by $f^\star(s) = (\gamma/\hat{R})s$. Thus, the optimal manipulation threshold $R^\star$ is equal to $\hat{R}$, and the associated benchmark is the volume-weighted average price (VWAP).*

The proof can be found in Appendix B.7. Theorem 2 states that if the benchmark administrator cannot deter or detect order-splitting, then the optimal benchmark is the volume-weighted average price. The intuition for this result is relatively straightforward. When agents

[25] Costless order splitting amounts to assumption that agents have no price impact in the underlying market. With price impact, submitting smaller orders might actually improve the price received by the agent, which further encourages order-splitting.

Fig. 5.1: Optimal weighting functions for Example 4



engage in strategic order splitting, the optimal weighting function cannot be concave unless it is linear. At the same time, it is not optimal for the weighting function to be strictly convex in any interval, for the reasons explained in the discussion of Lemma 4. Thus, it is optimal to choose a linear weighting function. Under a linear weighting function, all manipulators choose the largest feasible transaction size, which hence receives the highest possible weight. Therefore, the optimal benchmark that is unbiased and robust to order splitting minimizes the probability of manipulation.

Example 4 *We adopt the parameters of Examples 2 and 3.*[26] *The optimal benchmark fixing in the baseline model leads to the threshold* $R^\star \approx 2.58$ *which induces* 28% *of agents to manipulate. The optimal benchmark that deters order splitting is that which minimizes the probability of manipulation subject to unbiasedness. This yields a manipulation incentive threshold* $\hat{R}$ *of about 5.35, leading to manipulation with a probability of about* 7%. *The minimized objective function (mean squared error of the estimator) is* 0.142 *in the baseline case, and* 0.19 *when restricted to Condition 1, robustness to order splitting. This sharp increase in benchmark noise is caused by attaching a higher weight to manipulated transactions and inefficiently small weight to small unmanipulated transactions.*

To put this in context, consider the optimal benchmark fixing in the class of capped-volume weighted average price (CVWAP) fixings, those with a weighting function that in linear in transaction size s *up to some maximal transaction size, after which the weight remains constant. The best such fixing has a mean squared error of* 0.149 *and induces manipulation by an agent in the event that the agent's manipulation incentive* R *exceeds* 2.81, *which has a probability of about* 24%. *These three weighting functions are depicted in Figure 5.1.*

[26]Because we solve the example numerically, all numerical results reported in this and subsequent examples are approximate.

6 Models of manipulation

This section presents two stylized models of trading and manipulation that give rise to the functional forms for costs and incentives assumed in Section 2. Apart from providing a microeconomic foundation for our assumptions, these models give more precise meanings to some model parameters.

6.1 Committed quotes and costly search

We first consider a framework in which manipulation is costly because agents are committed to offering execution at the price quotes they submit to the benchmark administrator. In this framework, as is common in some actual benchmark settings, the submitting agents are dealers whose quotes are used to fix the benchmark. This was the case for the main industry benchmark for interest rate swaps known as ISDAFIX, whose manipulation[27] triggered more than $600 million in fines for several dealers, Deutsche Bank, Goldman Sachs, Royal Bank of Scotland, Citibank, and Barclays, and to a more robust benchmark design, as outlined by Aquilina, Ibikunle, Mollica, Pirrone and Steffen (2018).

Manipulation consists in quoting a price that is an overestimate or underestimate of the true value of the asset to the dealer. If the values for the asset are highly correlated among market participants, then a mispriced quote is likely to be executed by a different investor, yielding a loss to the quoting bank. In an instance of manipulation of ISDAFIX by Deutsche Bank Securities Inc., the CFTC found[28] that "DBSI Swap traders would tell the Swaps Broker their need for a certain swap level at 11:00 a.m. or their need to have the level moved up or down. On at least one occasion, the Swaps Broker expressed the need to know how much 'ammo' certain DBSI traders had to use in order to move the screen at 11:00 a.m." The "ammo" presumably refers to losses that the DBSI would incur from trades at manipulated quotes.

The probability of an execution at a distorted quote depends both on the degree of distortion and also on the transparency of the market. If quotes are public (as would be the case in a centralized limit order book), a significantly distorted quote would be executed with a probability close to one. If the market is more opaque or less active, and especially if quotes are revealed to traders only upon request (as in bilateral over-the-counter markets and on multilateral request-for-quote platforms), then the probability of incurring a loss by offering a distorted quote would be lower.

[27] See "CFTC Orders The Royal Bank of Scotland to Pay $85 Million Penalty for Attempted Manipulation of U.S. Dollar ISDAFIX Benchmark Swap Rates."

[28] See CFTC Orders Deutsche Bank Securities Inc. to Pay $70 Million Penalty for Attempted Manipulation of U.S. Dollar ISDAFIX Benchmark Swap Rates, CFTC, February 1, 2018.

In our model, dealer i chooses $\hat{s}_i \in [0, \bar{s}]$ and $\hat{z}_i \in \{-\bar{z}, 0, \bar{z}\}$, for some constant $\bar{z} > 0$ which we could set to σ_z to match the notation from the baseline model. The variable X_i is interpreted as the actual per-unit value of the asset to dealer i. The dealer commits to trade up to $\hat{s}_i$ units at a price $\hat{X}_i = X_i + \hat{z}_i$, where the pair $(\hat{X}_i, \hat{s}_i)$ is used as a benchmark submission. For simplicity, we set the bid-ask spread to zero, that is, $\hat{X}_i$ is both a bid and an ask. We assume that Y has unbounded support, while ϵ_i has a symmetric distribution on an interval $[-\bar{\epsilon}, \bar{\epsilon}]$, for some $\bar{\epsilon} \leq \bar{z}/2$. This captures the idea that the distortion in prices due to manipulation is larger than the distortion due to idiosyncratic differences in the value of the asset to different traders.

We adopt a stylized search protocol to determine the probability that a committed quote is executed. Before observing its manipulation incentive type R_i, dealer i chooses a search intensity $\lambda_i \in [0, 1]$, paying a cost $c(\lambda_i) = \frac{1}{2}\bar{c}\lambda_i^2$. Here, λ_i is the probability that the dealer will be allowed to trade at the committed quotes of some other (randomly chosen) dealer j. We assume that each dealer is contacted at most once.[29] Upon contacting j, dealer i maximizes the value of its chosen transaction. Because X_i is the unit value of the asset to dealer i, the resulting payoff of dealer i is

$$\max\left\{\max_{s \leq \hat{s}_j}\left(X_i - \hat{X}_j\right)s,\ \max_{s \leq \hat{s}_j}\left(\hat{X}_j - X_i\right)s\right\}.$$

Here, dealer i buys or sells the maximum quantity $\hat{s}_j$ to which dealer j has committed, due to linearity in value. The difference between the value X_i and the quote $\hat{X}_j$ determines the direction of trade.

6.1.1 Solution

We focus on symmetric Nash equilibria. Dealer i makes two choices, the search intensity λ_i and the manipulation levels $(\hat{z}_i, \hat{s}_i)$. Regarding the first choice, the expected payoff to a dealer conditional on a successful search depends on the probability that other banks choose to manipulate. If p_M denotes the equilibrium probability of manipulation, then that expected payoff is

$$\mathbb{E}\left((1-p_M)|X_i - X_j| + \frac{1}{2}p_M|X_i - \bar{z} - X_j| + \frac{1}{2}p_M|X_i + \bar{z} - X_j|\right)\mathbb{E}(\hat{s}_j)$$
$$= \left[(1-p_M)\mathbb{E}\left(|\epsilon_i - \epsilon_j|\right) + p_M\bar{z}\right]\mathbb{E}(\hat{s}_j) \equiv \phi.$$

[29] Formally, imagine the following iterative procedure. Dealer 1 contacts one of the in dealer $N \setminus \{1\}$ with probability λ_1. If dealer 1 contacts dealer j, then dealer 2 contacts one of the dealers in $N \setminus \{2, j\}$ with probability λ_2, and so on.

The optimal choice of search intensity is thus $\lambda^\star = \min\left\{1,\, \phi\bar{c}^{-1}\right\}$.

As for the choice of manipulation, the dealer can always guarantee a zero payoff by quoting a price equal to the true value X_i, regardless of the size $\hat{s}_i$, by choosing $\hat{z}_i = 0$. On the other hand, choosing $\hat{z}_i \in \{-\bar{z},\, \bar{z}\}$ yields a payoff $-\bar{z}\hat{s}_i$ in the event of being contacted by another dealer. The probability of being contacted is

$$\sum_{k=1}^{n-1} \binom{n-1}{k} (\lambda^\star)^k (1-\lambda^\star)^{n-1-k} \frac{k}{n-1} = \lambda^\star.$$

Taking into account the payoff generated by influencing the benchmark, and normalizing the payoff from not manipulating to zero, we see that the payoff from choosing $(\hat{s}_i,\, \hat{z}_i)$ is equal to

$$(R_i f(\hat{s}_i) - \lambda^\star \hat{s}_i)\, \hat{z}_i$$

which is exactly the expression assumed in Section 2, when taking $\gamma = \lambda^\star$.

6.1.2 Discussion

Based on the simple model of the previous subsection, the parameter γ can be interpreted as the probability of execution of a manipulated quote. If trade takes place on an active limit order book, then it is natural to assume that the cost $\bar{c}$ of search is nearly zero, and hence that $\gamma = \lambda^\star$ is close to 1. That is, manipulation would almost always yield a trading loss. On the other hand, in an opaque over-the-counter markets, $\bar{c}$ may be relatively large, and hence manipulation is less costly – a manipulated quote might not always be executed. As a consequence, holding the benchmark fixed, the probability of manipulation is higher in an opaque market.

Is $\lambda^\star$ is less than one, there is an additional feedback effect between the benchmark fixing and the probability of manipulation. The ex-ante probability p_M of manipulation by any dealer is $1 - H(R_f)$, which is the probability that the dealer's exposure type R exceeds the threshold R_f determined by the weighting function f used in the fixing. If f is changed to reduce manipulation, then R_f goes up and p_M goes down. This, however, implies that the incentive to search is reduced, because the probability of encountering a profitable distorted quote gets smaller. As a consequence, $\lambda^\star$ decreases, and manipulation becomes cheaper. In a sense, the benchmark fixing and the market forces act as substitutes in preventing manipulation when the market is relatively opaque.

This discussion suggests that moving from a centralized to an opaque market may have an ambiguous influence on the shape of the optimal benchmark. On one hand, because a given manipulation of the price is less costly in an opaque market, the fixing that should be chosen in an opaque market would place a relatively smaller weight on small transactions. On the other hand, a fixing that deters manipulation lowers the cost of manipulating through the equilibrium effect on the search intensity of other market participants.

6.2 An auction model

In this subsection, we consider an alternative trading model. When a liquidity shock hits a dealer, it may request quotes from other dealers, as is typical on electronic request-for-quote (RFQ) platforms. We model this as a sealed-bid auction. Absent incentives to manipulate, the dealer will accept the most attractive quote, for example, the lowest ask when it needs to buy the asset. The execution price, along with the corresponding trade volume, is then used to calculate the benchmark fixing. If, however, the dealer wants to inflate the fixing in order to take advantage of a long position in benchmark-linked assets, the dealer has an incentive to trade at the highest ask offered in the auction. This induces a tradeoff between the loss incurred in the auction and the gain associated with distorting the benchmark fixing.

We build a stylized model that aims to capture the main incentives. Dealer i is hit by a liquidity shock δ_i that takes one of the values $\{-\Delta, \Delta\}$ with equal probability, for some $\Delta > 0$. Dealer i then values each unit of the asset at $Y + \delta_i$, for quantities up to s_i. Whenever a dealer is hit by a shock, it requests quotes from two other dealers who have access to an unlimited supply of the asset at the common-value price Y. (The restriction to only two other dealers is not essential for the qualitative results but will yield explicit analytic solutions.) We model the competition between the two quoting dealers as a first-price auction (Bertrand competition). Absent incentives to manipulate, the dealer requesting the quote chooses the more attractive of the quotes, and thus Bertrand forces push the price to Y. However, when the quote-requesting dealer is a manipulator, it chooses the least attractive of the quotes, creating an incentive for dealers to provide quotes further away from the value Y.

6.2.1 Solution

For concreteness, consider the case in which dealer i requests quotes to buy the asset (the opposite case is symmetric). Let p_M be the equilibrium probability that dealer i manipulates by accepting the higher of the quotes, corresponding to the case of a positive exposure R_i. In the unique symmetric equilibrium of the auction, the two dealers that provide quotes randomize their offers according to a continuous distribution function F with support $[Y + \lambda\Delta, Y + \Delta]$, where λ is determined in equilibrium. Following the line of argument in Stahl (1989), this requires each of the two dealers to be indifferent between all per-unit quotes q in the support of F, so that

$$[(1 - p_M)(1 - F(q)) + p_M F(q)]\,(q - Y) = p_M \Delta.$$

Solving, we obtain

$$F(q) = 1 - \frac{p_M}{1 - 2p_M} \frac{Y + \Delta - q}{q - Y}$$

which is a well defined cdf when $p_M < 1/2$. Moreover, we have $\lambda = p_M/(1 - p_M)$. If p_M is small, the quotes are close to Y. When p_M is relatively high (but below 1/2), the quotes are close[30] to $Y + \Delta$. With the above description, we can calculate equilibrium payoffs, and the distribution of transaction data. Let ε_i^k, for $k = 1, 2$, and $i = 1, 2, ..., n$, be the profit margin charged by dealer k in the auction requested by dealer i. That is, $Y + \varepsilon_i^1$ and $Y + \varepsilon_i^2$ are the quotes received by dealer i. Normalizing the payoff from not manipulating to zero, we take the cost of manipulation to be equal to the extra profit margin conceded by dealer i through choosing the less attractive quote for $\hat{s}_i$ units of the asset. This concession is $\hat{s}_i \mathbb{E}\left[\max\{\varepsilon_i^1, \varepsilon_i^2\} - \min\{\varepsilon_i^1, \varepsilon_i^2\}\right]$. Taking into account the benefit from influencing the fixing, the net expected payoff from manipulation is equal to

$$(R_i f(\hat{s}_i) - \hat{s}_i)\, \mathbb{E} z_i,$$

where the random variable z_i is defined by $z_i = \left[\max\{\varepsilon_i^1, \varepsilon_i^2\} - \min\{\varepsilon_i^1, \varepsilon_i^2\}\right]$. This setting can therefore be viewed as a version of our basic model for the case $\gamma = 1$.

6.2.2 Discussion

The model of this section endogenizes the noise structure assumed in Section 2. The noise term ϵ_i reflects the dispersion in bids and asks quoted in the auction requested by dealer i. Manipulated transactions are more noisy than unmanipulated transactions because the worst price is further away from the mean Y than the best price. The noise term ϵ_i is $\pm \min\{\varepsilon_i^1, \varepsilon_i^2\}$, with symmetric probability. Manipulated transactions contain an additional noise term $z_i = \max\{\varepsilon_i^1, \varepsilon_i^2\} - \min\{\varepsilon_i^1, \varepsilon_i^2\}$. Thus, we provided a game-theoretic foundation for our assumption that manipulation reduces the signal-to-noise ratio of a benchmark.

In the framework modeled in this section, there is an additional distortionary channel for manipulation, through its impact on the probability distribution of unmanipulated data. When it is more likely that a counterparty in a transaction is a manipulator, a trader might provide a noisy quote, hoping that it will be accepted when the price distortion happens to be of the sign preferred by the manipulator. As a result, even when the quote requester is not a manipulator, and would take the most attractive quote, the distribution of quotes is more dispersed. As the probability p_M of manipulation rises, the probability distribution F of quotes shifts towards quotes further away from the true value Y. Hence the variance of ϵ_i

[30] We leave out a description of the equilibrium for the case $p_M \geq 1/2$ which is less relevant for our application. In that case, we would observe bids above $Y + \Delta$.

rises, in that $|\epsilon_i|$ is distributed according to the CDF $1-(1-F_\varepsilon(\epsilon))^2$, where

$$F_\varepsilon(\epsilon) = 1 - \frac{p_M}{1-2p_M}\frac{\Delta-\epsilon}{\epsilon},$$

implying that

$$\sigma_\epsilon^2 = 2\Delta^2\left(\frac{p_M}{1-2p_M}\right)^2\left[-\log\left(1+\frac{2p_M-1}{1-p_M}\right)+\frac{2p_M-1}{1-p_M}\right].$$

The noise level σ_ϵ^2 is increasing in Δ and p_M. In particular, $\lim_{p_M\to 1/2}\sigma_\epsilon^2 = \Delta^2$.

In this auction setting, because manipulation adversely impacts the precision of unmanipulated price signals, the slope of the optimal benchmark weighting function f is lowered in order to mitigate the risk of manipulation. The benchmark designer can affect the distribution of ϵ_i by choosing f so that $p_M = 1 - H(R_f)$ is relatively low. As a result, the probability of manipulation is smaller than in the baseline model in which the distribution of unmanipulated transaction data is exogenous.

7 Conclusions and future research

We developed a simple model for the design of robust benchmark fixings in settings for which incentives to manipulate the benchmark arise from a profit motive related to investment positions that are valued according to the benchmark. We have restricted attention to fixings that are given by a size-dependent weighted average price, an important limitation. We characterize the optimal weight for each size of transaction. We showed that an optimal benchmark fixing must in general allow some amount of manipulation, puts very small weight on small transactions, and nearly equal weight on large transactions. When order-splitting cannot be detected or otherwise deterred, the volume-weighted average price (VWAP) emerges as the optimal design within the class of benchmark fixing methods that we consider.

An important advance would be to allow weights that depend on the prices of transactions. A simple example is the exclusion of "outlier" prices, as in the design of the LIBOR fixings, which discards the highest and lowest quartiles of the panel of reports. A more sophisticated approach would be to compute, for every transaction, the posterior probability that the transaction is manipulated, and to use this information to construct weights.

We have ignored collusion throughout.[31]

[31] For a given benchmark design, a collusive model of manipulation is suggested by Osler (2016).

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A The generalized statement of Theorem 1a-1b

We first define a generalization of ODE (4.1). The differential equation is indexed by two parameters: the starting point s_0 and a constant $\eta > 0$:

$$f''(s) = -\frac{\left[\eta - 2f(s)\sigma_U^2\right] H(R)g(s) + 2\gamma\sigma_M^2 h\left(\frac{\gamma}{f'(s)}\right)}{\left[2f(s)\sigma_M^2 - \eta\right]\left(-h'\left(\frac{\gamma}{f'(s)}\right)\right)\frac{\gamma^2}{(f'(s))^3}} \tag{A.1}$$

with boundary conditions $f(s_0) = (\gamma/R)s_0$, $f'(s_0) = \gamma/R$.

Theorem 1 *For any $R \in (0, \hat{R})$, there exists a unique optimal solution $f^\star$ to problem $\mathcal{P}(R)$. The optimal weighting function $f^\star$ is non-decreasing, concave, continuously differentiable everywhere, and given by*

$$f^\star(s) = \begin{cases} \frac{\gamma}{R}s & s \in [0,\, s_0] \\ \textit{solution to (A.1)} & s \in (s_0,\, s_1) \\ f^\star(s_1) & s \in [s_1,\, \bar{s}] \end{cases}.$$

The parameter η in (A.1) and the cutoff point s_1 are chosen so that $(f^\star)'(\bar{s}) = 0$: either $s_1 < \bar{s}$ in which case $\eta = 2f^\star(s_1)\sigma_U^2$, or $s_1 = \bar{s}$ in which case $\eta \in [2f^\star(s_1)\sigma_U^2,\, 2f^\star(s_0)\sigma_M^2)$ is chosen so that the solution to (A.1) on $[s_0,\, \bar{s}]$ satisfies $f'(\bar{s}) = 0$. Finally, the cutoff point $s_0 \in (0, \bar{s})$ is chosen to satisfy the constraint (3.5).[32]

Clearly, Theorem 1 implies Theorem 1a. To see that it also implies Theorem 1b, note that $f^\star$ described by Theorem 1b corresponds exactly to the first case described by Theorem 1: $s_1 < \bar{s}$ and $\eta = 2f^\star(s_1)\sigma_U^2$. Because the solution is unique, when $f^\star$ described by Theorem 1b exists, it must be optimal. In this case, the ODE (4.1) is obtained from (A.1) by plugging in the above expression for η, and dividing the numerator and the denominator by $2\sigma_U^2$.

The derivation of the optimal benchmark in Section 4.2 along with the proofs found in Appendix B establish the generalized version of Theorem 1a-1b described above.

[32] The existence of such η, s_0, s_1 and the existence of a solution to (A.1) will be proven.

B Proofs

B.1 Proof of Lemma 1

Let $V(f)$ be the value of the problem for a feasible function f, and let $V^\star$ be the value of the infimum in $\mathcal{P}(R)$ (it exists because the objective function is bounded by zero from below). By definition of an infimum, there exists a sequence of feasible functions f_n such that $\lim_n V(f_n) \to V^\star$. We have to prove that a subsequence of f_n converges to a well-defined and feasible limit f, and that $V(f) = V^\star$.

First, we define the limiting function f. Because each $f_n \in \mathcal{C}^{K,M}$, we can define $0 = s_1^n < s_2^n < ... < s_{K_{n-1}}^n < s_{K_n}^n = \bar{s}$ such that f_n is Lipshitz with constant M on each (s_i^n, s_{i+1}^n). Because $K_n \leq K$, and K is finite, there exists a subsequence (which we still denote by f_n) such that $K_n = L \leq K$ for all n, and $s_i^n \to s_i$. For any i, take a compact subset $A_i \subset (s_i, s_{i+1})$. Then, for large enough n, the sequence is uniformly bounded and equi-continuous on A_i, by assumption. By the Arzelá-Ascoli Theorem, we can find a subsequence that converges point-wise to some function $\tilde{f}^i$ on (s_i, s_{i+1}), and convergence is uniform on every compact subset. The limiting function $\tilde{f}^i$ preserves the Lipshitz constant M. Because the function is bounded and Lipshitz continuous, we can extend the function to $[s_i, s_{i+1}]$ in such a way that $\tilde{f}^i$ is continuous. Because there are finitely many i, we can find a subsequence of f_n such that the above properties hold for every interval (s_i, s_{i+1}). Finally, we define f to be a function that coincides with $\tilde{f}^i$ on every (s_i, s_{i+1}), and is equal to $\max\{\tilde{f}^{i-1}(s_i), \tilde{f}^i(s_i)\}$ for each s_i. The definition guarantees that f is upper semi-continuous, and thus belongs to the class $\mathcal{C}^{K,M}$. The chosen subsequence of f_n (which we will again denote by f_n) converges to f uniformly on every compact $A \subset [0, \bar{s}] \setminus \{s_1, ..., s_L\}$.

Second, we look at the properties of $d\Psi_f(s)$ – the distribution of manipulated trade sizes. In this paragraph, we use R_i to denote a generic positive exposure type. Let

$$S_f(R_i) = \mathrm{argmax}_{s\in[0,\,\bar{s}]}\{R_i f(s) - \gamma s\} \equiv \mathrm{argmax}_{s\in[0,\,\bar{s}]}\left\{f(s) - \frac{\gamma}{R_i}s\right\}$$

be the set of maximizers in the manipulator's problem. The function $f(s) - (\gamma/R_i)s$, defined on a lattice $[0, \bar{R}] \times [0, \bar{s}]$, is quasi-supermodular in s, and has a strict single crossing property in (s, R_i). It follows from Milgrom and Shannon (1994) that the set $S_f(R_i)$ is a complete sublattice, and any selection $s_f(R_i) \in S_f(R_i)$ is non-decreasing in R_i. In particular, this means that $S_f(R_i)$ is a singleton for almost all R_i. Define $\bar{s}_f(R_i) = \max S_f(R_i)$. Then, $\Psi_f(s) = \mathbb{P}_{R_i\sim H}(\bar{s}_f(R_i) \leq s)$.

Third, we argue that $\lim_n V(f_n) = V(f)$, and that f satisfies the constraints of the problem

$\mathcal{P}(R)$. It is enough to prove that

$$\lim_n \int_0^{\bar{s}} f_n^k(s)g(s)ds = \int_0^{\bar{s}} f^k(s)g(s)ds,\ k \in \{1,\, 2\}, \tag{B.1}$$

and

$$\lim_n \int_0^{\bar{s}} f_n^k(s)d\Psi_{f_n}(s) = \int_0^{\bar{s}} f^k(s)d\Psi_f(s),\ k \in \{1,\, 2\}, \tag{B.2}$$

Showing (B.1) is straightforward – it follows from the Lebesgue dominated convergence theorem and the fact that f_n converges to f almost surely. We prove that (B.2) holds as well. Recalling that we are looking at the problem where agents with $R_i \geq R$ manipulate, we can write

$$\int_0^{\bar{s}} f^k(s)d\Psi_f(s) = \int_R^{\bar{R}} (f(\bar{s}_f(R_i)))^k \frac{dH(R_i)}{1-H(R)}.$$

It is therefore enough to prove that $f_n(\bar{s}_{f_n}(R_i)) \to f(\bar{s}_f(R_i))$ for almost all R_i. Intuitively, we have to show that the weight chosen by manipulators changes continuously with the weighting function, for almost all R_i. For some R_i, it is clear that the optimal choice can be discontinuous when the manipulator is indifferent between two transaction sizes, but as we saw in the second step of the proof, such situations are non-generic. It is enough to prove that $\bar{s}_{f_n}(R_i)$ converges to $\bar{s}_f(R_i)$ for almost all R_i. Indeed, if this is true, then the only scenario in which $f_n(\bar{s}_{f_n}(R_i))$ might fail to converge to $f(\bar{s}_f(R_i))$ is when $\bar{s}_{f_n}(R_i)$ approaches some s_i at which f has a jump, and convergence to s_i is from the side where f is lower – however, this would contradict the optimality of $\bar{s}_{f_n}(R_i)$. To show convergence of $\bar{s}_{f_n}(R_i)$ to $\bar{s}_f(R_i)$ for almost all R_i, it is enough to prove that the limit of $\bar{s}_{f_n}(R_i)$ is a solution to the agent's problem at f. Then, the conclusion follows from the fact that, by step 2 of the proof, the set of solutions is a singleton for almost all R_i.

Let $v(f) = \max_{s\in[0,\,\bar{s}]}(R_i f(s) - \gamma s)$, for a fixed R_i. Because the function $R_i f(s) - \gamma s$ is upper semi-continuous in s, it is enough to prove that $v(f_n) \to v(f)$. Let S_n be defined as $[0,\,\bar{s}] \setminus \left(\bigcup_{i=1}^{L}[s_i - \frac{1}{n},\, s_i + \frac{1}{n}]\right)$ – we removed each s_i with some small neighborhood from the domain. Then, f_n converges uniformly to f on S_n. We have, for large enough n,

$$\begin{aligned} |v(f_n) - v(f)| &= \left| \max_{s\in[0,\,\bar{s}]}(R_i f_n(s) - \gamma s) - \max_{s\in[0,\,\bar{s}]}(R_i f(s) - \gamma s) \right| \\ &\leq \frac{O(1)}{n} + \left| \max_{s\in S_n}(R_i f_n(s) - \gamma s) - \max_{s\in S_n}(R_i f(s) - \gamma s) \right| \leq \frac{O(1)}{n} + R_i \max_{s\in S_n} |f_n(s) - f(s)|, \end{aligned}$$

and the last expression goes to 0 by uniform convergence. Here, the term $O(1)$ denotes a constant, and the first inequality follows from the fact that all f_n are uniformly bounded and equi-continuous on each $(s_i^n,\, s_{i+1}^n)$ (intuitively, removing a small part of the domain cannot change the value of the optimization problem too much). This concludes the proof.

B.2 Proof of Lemma 2

Take a feasible function f and suppose it is not nondecreasing. We will prove the result by constructing a different feasible function $\bar{f}$ that improves the objective function (hence, f cannot be optimal). By assumption, there exist s_0 and s_1 such that $s_0 < s_1$, but $f(s_0) > f(s_1)$. Without loss of generality we can assume (making the interval smaller if necessary and using the regularity conditions on f) that either *(i)* f is strictly decreasing in $[s_0, s_1]$ or *(ii)* f has a jump discontinuity at s_0 and $f(s)$ is lower than $f(s_0)$ on $(s_0, s_1]$.[33]

Consider case *(i)*. By the choice of s_0 and s_1, there are no manipulations in (s_0, s_1), and this will continue to be true for any function f that is non-increasing in this interval. We can construct a non-increasing, Lipshitz continuous function $\hat{f}$ on $[s_0, s_1]$ with the following properties: $\hat{f}(s_0) = f(s_0)$, $\hat{f}(s_1) = f(s_1)$, $\int_{s_0}^{s_1} \hat{f}(s)g(s)\,ds = \int_{s_0}^{s_1} f(s)g(s)\,ds$ and there exists $s_2 \in (s_0, s_1)$ such that $\hat{f}(s) < f(s)$ for $s \in (s_0, s_2)$ and $\hat{f}(s) > f(s)$ for $s \in (s_2, s_1)$. We then define

$$\bar{f}(s) = \begin{cases} \hat{f}(s) & \text{if } s \in [s_0, s_1] \\ f(s) & \text{otherwise.} \end{cases}$$

By construction, $\bar{f}$ is feasible (in particular it satisfies the constraint that guarantees an unbiased estimator). The difference in the value of the administrator's objective function $\mathcal{P}$ under $\bar{f}$ and f is (using the fact that there are no manipulations in $[s_0, s_1]$ under $\bar{f}$),

$$\int_{s_0}^{s_1} \left(\bar{f}^2(s) - f^2(s)\right) \sigma_U^2 g(s)\,ds = \int_{s_0}^{s_1} \left(\bar{f}(s) - f(s)\right) \phi(s) g(s)\,ds,$$

where $\phi(s) \equiv \left(\bar{f}(s) + f(s)\right) \sigma_U^2$ is a strictly decreasing function. By the mean value theorem, there exists $x \in (s_0, s_1)$ such that

$$\int_{s_0}^{s_1} \left(\bar{f}(s) - f(s)\right) \phi(s) g(s)\,ds = \phi(s_0) \int_{s_0}^{x} \left(\bar{f}(s) - f(s)\right) g(s)\,ds.$$

But $\int_{s_0}^{x} \left(\bar{f}(s) - f(s)\right) g(s)\,ds < 0$ because the integrand is (strictly) negative on $[s_0, s_2)$, (strictly) positive on $(s_2, s_1]$ and $\int_{s_0}^{s_1} \left(\bar{f}(s) - f(s)\right) g(s)\,ds = 0$.

Therefore, $\bar{f}$ is feasible and yields a smaller value of the objective function than does f.

Now, consider case *(ii)*. We can choose s_1 so that $f(s) < f(s_0)$ for all $s \in (s_0, s_1)$ but not on any larger interval. By the choice of s_1, there cannot be any manipulations in (s_0, s_1) – this is because a manipulator would strictly prefer to choose s_0 over any s in that interval. Suppose that f is not (almost everywhere) constant on (s_0, s_1). Then, there is a way to improve on f by replacing it in this interval by a constant $\bar{f}(s) = \alpha$ with $\alpha(G(s_1) - G(s_0)) = \int_{s_0}^{s_1} \bar{f}(s)g(s)ds = \int_{s_0}^{s_1} f(s)g(s)ds$. Indeed, under both f and $\bar{f}$, there are

[33] We assume here that $s_0 < \bar{s}$. The opposite case is very easy to rule out.

only unmanipulated transactions in the interval $(s_0,\, s_1)$, so the objective function changes by

$$\sigma_U^2 \int_{s_0}^{s_1} (\bar{f}^2(s) - f^2(s)) g(s) ds < \sigma_U^2 [G(s_1) - G(s_0)] \left[\alpha^2 - \left(\frac{\int_{s_0}^{s_1} f(s) g(s) ds}{G(s_1) - G(s_0)} \right)^2 \right] = 0,$$

where the (strict) inequality follows from Jensen's Inequality and the fact that f is not (almost everywhere) constant. Thus, f could not be optimal.

Finally, consider the opposite case in which f is constant (almost everywhere) on $(s_0,\, s_1)$. By definition of s_1, we must in fact have $s_1 = \bar{s}$, and it is without loss of generality to assume that $f(s) = \beta$ for all $s > s_0$ for some β (in the opposite case there is a simple way to improve on f). Because the construction of $\bar{f}$ is similar to the previous cases, we only discuss it informally and omit a formal calculation. For $\epsilon > 0$ small enough, $\beta + \epsilon < f(s_0)$, so if we raise $f(s)$ from β to $\beta + \epsilon$ on $(s_0,\, \bar{s}]$, this has no influence on the distribution of manipulated trades. To preserve constraint (3.5), we can now lower f by δ on $[s_0 - \Delta,\, s_0]$. This might change the distribution of manipulated trades but only in the direction desired by the administrator – the manipulators are guaranteed to choose lower weights after the modification because, for small enough ϵ, trade sizes above s_0 are suboptimal. Define $\bar{f}$ as a function obtained by modifying f in a way described above with ϵ, δ, and Δ such that constraint (3.5) is preserved. Then, for small enough ϵ and δ, $\bar{f}$ achieves a strictly lower value of the objective function than does f. Hence, f could not be optimal.

B.3 Proof of Lemma 3

Take a feasible candidate solution f and suppose that it is not continuous. By the regularity condition and Lemma 2, it is enough to consider the case when f jumps up at some $s_0 \in (0,\, \bar{s})$. Consider lowering f by $\epsilon > 0$ in the interval $[s_0,\, \bar{s}]$, where ϵ is small. Note that after this modification, the distribution of manipulated transactions conditional on $\hat{s} \in [s_0,\, \bar{s}]$ does not change, but it is possible that some manipulators switch to choosing a size $\hat{s} < s_0$. However, we can ignore this in the calculations because, by Lemma 2, the function f is lower on $[0,\, s_0)$ than it is on $[s_0,\, \bar{s}]$ (and continues to be lower if ϵ is small enough) – hence, this can only improve the objective function. Next, notice that for small enough Δ and ϵ, there cannot be any manipulations in $(s_0 - \Delta,\, s_0)$ because the choice of any s in this interval is dominated by the choice of s_0. Let us define $\bar{f}$ in the following way

$$\bar{f}(s) = \begin{cases} f(s) & s \leq s_0 - \Delta \\ f(s) + \delta & s \in (s_0 - \Delta,\, s_0) \\ f(s) - \epsilon & s \geq s_0 \end{cases},$$

where δ is chosen so that the constraint (3.5) holds (as noted before, we can ignore the manipulated transactions):

$$\int_{s_0-\Delta}^{s_0} \delta g(s)ds = \int_{s_0}^{\bar{s}} \epsilon g(s)ds. \tag{B.3}$$

Because of (B.3), the function $\bar{f}$ is feasible, so we only have to prove that $\bar{f}$ achieves a lower value of the objective function. We have

$$\begin{aligned}&\int_0^{\bar{s}} \sigma_U^2 \bar{f}^2(s)g(s)ds - \int_0^{\bar{s}} \sigma_U^2 f^2(s)g(s)ds \\ &\qquad = \sigma_U^2 \left(\delta^2 \int_{s_0-\Delta}^{s_0} g(s)ds + 2\delta \int_{s_0-\Delta}^{s_0} f(s)g(s)ds + \epsilon^2 \int_{s_0}^{\bar{s}} g(s)ds - 2\epsilon \int_{s_0}^{\bar{s}} f(s)g(s)ds\right).\end{aligned}$$

The terms multiplied by ϵ^2 and δ^2 can be ignored because they are negligibly small compared to other terms once ϵ and δ are small enough (they cannot reverse a strict inequality). Using equality (B.3), it is enough to prove that

$$\epsilon \frac{1-G(s_0)}{G(s_0)-G(s_0-\Delta)} \int_{s_0-\Delta}^{s_0} f(s)g(s)ds - \epsilon \int_{s_0}^{\bar{s}} f(s)g(s)ds < 0.$$

or equivalently,

$$\frac{\int_{s_0-\Delta}^{s_0} f(s)g(s)ds}{G(s_0)-G(s_0-\Delta)} < \frac{\int_{s_0}^{\bar{s}} f(s)g(s)ds}{1-G(s_0)}.$$

This means that we are done because

$$\frac{\int_{s_0-\Delta}^{s_0} f(s)g(s)ds}{G(s_0)-G(s_0-\Delta)} < \frac{f(s_0^-)+f(s_0)}{2} < \frac{\int_{s_0}^{\bar{s}} f(s)g(s)ds}{1-G(s_0)},$$

where $f(s_0^-)$ is the left limit of f at s_0, and the inequality follows from the fact that f is globally non-decreasing, and that there is a jump at s_0.

B.4 Proof of Lemma 4

Take a feasible f and suppose it is not concave. By Lemma 2 and Lemma 3, we can assume that f is continuous and non-decreasing. This means that we can find an affine increasing function $\varphi(s) = a + bs$ and an interval $[s_0, s_1]$ such that $\varphi(s_0) = f(s_0)$, $\varphi(s_1) = f(s_1)$ and $\varphi(s) \geq f(s)$ for all $s \in (s_0, s_1)$, with a strict inequality for at least some $\tilde{s} \in (s_0, s_1)$. We first prove that there can be no manipulations[34] in (s_0, s_1). It's enough to show that for generic

[34] Strictly speaking, the measure of manipulations is zero.

R, and for all $s \in (s_0,\, s_1)$,

$$Rf(s) - \gamma s < \max\left\{Rf(s_0) - \gamma s_0,\, Rf(s_1) - \gamma s_1\right\}.$$

We have

$$\max\left\{Rf(s_0) - \gamma s_0,\, Rf(s_1) - \gamma s_1\right\} = \begin{cases} Ra + (Rb - \gamma)\, s_1 & \text{if } Rb > \gamma, \\ Ra + (Rb - \gamma)\, s_0 & \text{if } Rb < \gamma. \end{cases}$$

Take the case $Rb > \gamma$. Then we have, for all $s \in (s_0,\, s_1)$,

$$Rf(s) - \gamma s \leq Ra + (Rb - \gamma)\, s < Ra + (Rb - \gamma)\, s_1.$$

Similarly, for $Rb < \gamma$ and for all $s \in (s_0,\, s_1)$,

$$Rf(s) - \gamma s \leq Ra + (Rb - \gamma)\, s < Ra + (Rb - \gamma)\, s_0.$$

This conclusion depended only on the fact that f lies below the affine function φ. Thus, if f cannot be improved upon by another feasible function $\bar{f}$, it must be the case that f restricted to the interval $[s_0,\, s_1]$ arises as a solution to the following optimal control problem:

$$\min_{u \geq 0} \int_{s_0}^{s_1} \tilde{f}^2(s) g(s)\, ds \tag{B.4}$$

subject to

$$\int_{s_0}^{s_1} \tilde{f}(s) g(s)\, ds = \int_{s_0}^{s_1} f(s) g(s)\, ds,$$

$$\tilde{f}'(s) = u(s),$$

$$\tilde{f}(s_0) = f(s_0),$$

$$\tilde{f}(s_1) \leq f(s_1),$$

$$\tilde{f}(s) \leq \varphi(s).$$

Here, the first derivative plays the role of the control variable, and the weighting function is the state variable. Notice that this is a problem mathematically equivalent to that considered in Proposition 2. A standard application of optimal control techniques (see, for example, the Arrow's Theorem on page 107 of Seierstad and Sydsaeter, 1987) yields the conclusion that the optimal $\tilde{f}(s)$ is equal to $\varphi(s)$ up to some $s_2 \in (s_0,\, s_1)$, and is constant equal to $\tilde{f}(s_2)$ on $(s_2,\, s_1]$, where s_2 is chosen to satisfy the constraint $\int_{s_0}^{s_1} \tilde{f}(s) g(s)\, ds = \int_{s_0}^{s_1} f(s) g(s)\, ds$. Note that $s_2 < s_1$ because, by assumption, f lies strictly below φ for at least some points. Define the function $\bar{f}$ that coincides with f outside of the interval $(s_0,\, s_1)$ and is equal to the optimal $\tilde{f}$ otherwise. Then, $\bar{f}$ achieves a weakly lower value of the objective function than f, and has a jump discontinuity at s_1. By Lemma 3, $\bar{f}$ can be (strictly) improved upon, and hence f cannot be optimal either.

B.5 Proof of Lemma 5

We will first find a solution to the relaxed problem (4.2) - (4.3), and then prove that it satisfies the properties listed in Theorem 1, a generalization of Theorem 1a-1b found in Appendix A. This will establish Theorem 1, and thus Lemma 5 and Theorem 1a-1b as a special case.

We fix an $R \in (0,\, \hat{R})$ which guarantees that the set of functions $f \in \mathcal{F}$ that satisfy the constraints of problem (4.2) - (4.3) is non-empty.

We can simplify the objective function (4.2): Applying integration by parts for the Riemann-Stieltjes Integral, and using the fact that f is absolutely continuous, we obtain

$$\begin{aligned}\int_0^{\bar{s}} f^2(s)\, dH\left(\frac{\gamma}{f'(s)}\right) &= f^2(\bar{s}) - 2\int_0^{\bar{s}} f(s)f'(s)H\left(\frac{\gamma}{f'(s)}\right)\, ds\\ &= 2\int_0^{\bar{s}} f(s)f'(s)\left(1 - H\left(\frac{\gamma}{f'(s)}\right)\right) ds.\end{aligned}$$

Therefore, the objective function (4.2) becomes

$$\int_0^{\bar{s}} \left[f^2(s)\sigma_U^2 H(R)g(s) + 2f(s)f'(s)\sigma_M^2\left(1 - H\left(\frac{\gamma}{f'(s)}\right)\right)\right] ds.$$

Applying the same method, we get

$$\int_0^{\bar{s}} f(s)\, dH\left(\frac{\gamma}{f'(s)}\right) = \int_0^{\bar{s}} f'(s)\left(1 - H\left(\frac{\gamma}{f'(s)}\right)\right) ds,$$

which allows us to express the constraint (4.3) as

$$\int_0^{\bar{s}} \left[f(s)H(R)g(s) + f'(s)\left(1 - H\left(\frac{\gamma}{f'(s)}\right)\right)\right] ds = \frac{1}{n}.$$

Moreover, we can transform the problem into an unconstrained one by defining an auxiliary state variable Γ by

$$\Gamma(s) = \int_0^{s} \left[f(t)H(R)g(t) + f'(t)\left(1 - H\left(\frac{\gamma}{f'(t)}\right)\right)\right] dt, \;\; s \in [0,\, \bar{s}].$$

This means that

$$\Gamma'(s) = f(s)H(R)g(s) + f'(s)\left(1 - H\left(\frac{\gamma}{f'(s)}\right)\right)$$

with $\Gamma(0) = 0$ and $\Gamma(\bar{s}) = 1/n$.

We thus have the following optimal control problem:

$$\max_{u:\, u(s)\in[0,\,\gamma/R]} - \int_0^{\bar{s}} \left[f^2(s)\sigma_U^2 H(R)g(s) + 2f(s)u(s)\sigma_M^2 \left(1 - H\left(\frac{\gamma}{u(s)}\right)\right)\right] ds \tag{B.5}$$

subject to

$$f'(s) = u(s), \quad f(0) = 0, \quad f(\bar{s}) \ - \text{free}, \tag{B.6}$$

$$\Gamma'(s) = f(s)H(R)g(s) + u(s)\left(1 - H\left(\frac{\gamma}{u(s)}\right)\right), \quad \Gamma(0) = 0, \quad \Gamma(\bar{s}) = \frac{1}{n}. \tag{B.7}$$

The Hamiltonian corresponding to the problem is

$$\begin{aligned}\mathcal{H}(f(s),\, u(s),\, s) = &- \left[f^2(s)\sigma_U^2 H(R)g(s) + 2f(s)u(s)\sigma_M^2 \left(1 - H\left(\frac{\gamma}{u(s)}\right)\right)\right] \\ &+ p_1(s)u(s) + p_2(s)\left[f(s)H(R)g(s) + u(s)\left(1 - H\left(\frac{\gamma}{u(s)}\right)\right)\right],\end{aligned} \tag{B.8}$$

where $p_i(s)$, for $i = 1,\, 2$, are the multipliers on the two state variables f and Γ.

The lemma below gives sufficient conditions for optimality and uniqueness of a candidate solution.

Lemma 6 *Let $(f(s),\, u(s))$ be a feasible pair for the problem (B.5) - (B.7). If there exists a continuous and piecewise continuously differentiable function $p(s) = (p_1(s),\, p_2(s))$ such that the following conditions are satisfied*

1. $p_1'(s) = \left[2f(s)\sigma_U^2 - \eta\right] H(R)g(s) + 2u(s)\sigma_M^2 \left(1 - H\left(\frac{\gamma}{u(s)}\right)\right)$;
2. $p_2'(s) = 0$;
3. *$u(s)$ maximizes $\mathcal{H}(f(s),\, u,\, s)$ over $u \in [0,\, \gamma/R]$ for all $s \in [0,\, \bar{s}]$;*
4. *$p_1(\bar{s}) = 0$;*
5. *$\hat{H}(f,\, s) = \max_{u\in[0,\,\gamma/R]} \mathcal{H}(f,\, u,\, s)$ exists and is concave in f for all s,*

then $(f(s),\, u(s))$ solve the problem (B.5) - (B.7). If $\hat{H}(f,\, s)$ is strictly concave in f for all s, then f is the unique solution.

Proof: By direct application of the Arrow Sufficiency Theorem (Theorem 5 on page 107 of Seierstad and Sydsaeter, 1987). □

Before we proceed, we state a simple lemma that will be used throughout.

Lemma 7 *Suppose X is a nonnegative random variable with a finite variance and a continuously differentiable decreasing density h on $(0, \infty)$. Then $\lim_{x\to\infty} h(x)x^2 = \lim_{x\to\infty} h'(x)x^3 = 0$.*

Proof: The first claim follows directly from the definition of variance, and the second can be obtained by applying integration by parts. □

We will construct the functions p_1, p_2, and show that the conditions of Lemma 6 all hold with (f, f') as described in Theorem 1. (We omit the superscript in $f^\star$ and write f throughout.) We let $\eta = p_2(s)$ for all s, for some constant $\eta > 0$. We conjecture that $\eta \in [2f(s_0)\sigma_U^2, 2f(s_0)\sigma_M^2)$ (we will verify that conjecture later). This definition of $p_2(s)$ satisfies condition 2 of Lemma 6.

Consider the interval $[0, s_0]$, where $f(s) = (\gamma/R)s$, and $u(s) = f'(s) = \gamma/R$. We want to make sure that condition 3 of Lemma 6 is satisfied:

$$\frac{\gamma}{R} \in \operatorname*{argmax}_{u\in[0,\,\frac{\gamma}{R}]} \left\{ - \left[2f(s)\sigma_M^2 - \eta\right] u \left(1 - H\left(\frac{\gamma}{u}\right)\right) + p_1(s)u \right\}.$$

It is enough to show that the derivative of the objective function with respect to u is non-negative, for all $u \in [0, \gamma/R]$:

$$\left[\eta - 2\frac{\gamma}{R}s\sigma_M^2\right]\left[1 - H\left(\frac{\gamma}{u}\right) + \frac{\gamma}{u}h\left(\frac{\gamma}{u}\right)\right] + p_1(s) \geq 0. \tag{B.9}$$

Notice that the second derivative with respect to u is given by

$$\left[\eta - 2\frac{\gamma}{R}s\sigma_M^2\right]\left[\frac{\gamma^2}{u^3}\left(-h'\left(\frac{\gamma}{u}\right)\right)\right]$$

which, by the assumption that h is decreasing, is non-negative if and only if $\eta \geq 2(\gamma/R)s\sigma_M^2$. Thus, the Hamiltonian is convex in the control variable u (implying a boundary solution) for all s such that $\eta \geq 2(\gamma/R)s\sigma_M^2$, and is concave otherwise. In either case, it is enough to show that (B.9) holds for $u = \gamma/R$, i.e., that

$$\left[\eta - 2\frac{\gamma}{R}s\sigma_M^2\right][1 - H(R) + Rh(R)] + p_1(s) \geq 0. \tag{B.10}$$

To satisfy condition 2 of Lemma 6, we set

$$p_1'(s) = \underbrace{\left[2\frac{\gamma}{R}s\sigma_U^2 - \eta\right] H(R)g(s)}_{\leq 0} + 2\frac{\gamma}{R}\sigma_M^2(1 - H(R))$$

in the interval $s \in [0,\, s_0]$, using the assumption that $\eta \geq 2f(s_0)\sigma_U^2$. Thus, we can write

$$p_1(s) = p_1(0) + 2\frac{\gamma}{R}s\sigma_M^2(1 - H(R)) + \int_0^s \left[2\frac{\gamma}{R}\tau\sigma_U^2 - \eta\right] H(R)g(\tau)d\tau.$$

To show (B.10), we need to prove that

$$\left[\eta - 2\frac{\gamma}{R}s\sigma_M^2\right][1 - H(R) + Rh(R)] + p_1(0) + 2\frac{\gamma}{R}s\sigma_M^2(1 - H(R)) + \int_0^s \left[2\frac{\gamma}{R}\tau\sigma_U^2 - \eta\right] H(R)g(\tau)d\tau \geq 0.$$

This is equivalent to

$$\eta\,[1 - H(R)] + \left[\eta - 2\frac{\gamma}{R}s\sigma_M^2\right][Rh(R)] + p_1(0) + \int_0^s \left[2\frac{\gamma}{R}\tau\sigma_U^2 - \eta\right] H(R)g(\tau)d\tau \geq 0.$$

The derivative of the left hand side is equal to

$$-2\frac{\gamma}{R}\sigma_M^2 Rh(R) + \left(2\frac{\gamma}{R}s\sigma_U^2 - \eta\right) H(R)g(s) \leq 0,$$

as long as $2(\gamma/R)s\sigma_U^2 \leq \eta$ which is true by conjecture when $s \leq s_0$. Thus, we can choose $p_1(0)$ to satisfy the inequality (B.10) at $s = s_0$, and then it will hold on the entire interval $[0,\, s_0]$. We can define $p_1(0)$ so that the inequality binds at s_0 which gives us

$$p_1(s) = \left[2\frac{\gamma}{R}s\sigma_M^2 - \eta\right][1 - H(R)] - \left[\eta - 2\frac{\gamma}{R}s_0\sigma_M^2\right][Rh(R)] - \int_s^{s_0} \left[2\frac{\gamma}{R}\tau\sigma_U^2 - \eta\right] H(R)g(\tau)d\tau.$$

We have thus shown that conditions 1-3 of Lemma 6 all hold in the interval $[0,\, s_0]$.

Next, consider the interval $[s_0,\, \bar{s}]$. In this interval, we want to have an interior maximizer $u(s)$ of the Hamiltonian (B.8). Because $\eta \leq 2f(s_0)\sigma_M^2$, the Hamiltonian is concave in u, and thus it is enough that the first-order condition holds at $u = u(s)$:

$$-\left[2f(s)\sigma_M^2 - \eta\right]\left[1 - H\left(\frac{\gamma}{u(s)}\right) + \frac{\gamma}{u(s)}h\left(\frac{\gamma}{u(s)}\right)\right] + p_1(s) = 0. \tag{B.11}$$

Taking the derivative over s, and using the fact that the equality holds at $s = s_0$, this is equivalent to

$$-2f'(s)\sigma_M^2\left[1 - H\left(\frac{\gamma}{u(s)}\right) + \frac{\gamma}{u(s)}h\left(\frac{\gamma}{u(s)}\right)\right] + \left[2f(s)\sigma_M^2 - \eta\right]\frac{\gamma^2}{u^3(s)}h'\left(\frac{\gamma}{u(s)}\right)u'(s) + p_1'(s) = 0. \tag{B.12}$$

Using the fact that $u(s) = f'(s)$, so that $u'(s) = f''(s)$, and combining (B.12) with the differential equation from condition 1 of Lemma 6 for p_1, we obtain

$$\left[2f(s)\sigma_M^2-\eta\right]\frac{\gamma^2}{u^3(s)}h'\left(\frac{\gamma}{u(s)}\right)f''(s)=\left[\eta-2f(s)\sigma_U^2\right]H(R)g(s)+2\gamma\sigma_M^2 h\left(\frac{\gamma}{u(s)}\right).$$

This means that it is enough to show that ODE (A.1) holds whenever $u(s)>0$, and that $\eta=2f(s)\sigma_U^2$ whenever $u(s)=0$.

Notice that from the first-order condition (B.11), we have

$$p_1(s)=\left[2f(s)\sigma_M^2-\eta\right]\left[1-H\left(\frac{\gamma}{u(s)}\right)+\frac{\gamma}{u(s)}h\left(\frac{\gamma}{u(s)}\right)\right],$$

and thus condition 4 of Lemma 6 will hold as long as $u(\bar{s})=0$. Moreover, to show that $p_1(s)$ is continuous and piecewise continuously differentiable, it is enough to prove that $u(s)$ is continuous. All of that is accomplished by the following lemma.

Lemma 8 *There exists $\eta\in[2f(s_0)\sigma_U^2,\,2f(s_0)\sigma_M^2)$, and a non-decreasing, concave solution f of class C^1 to the ODE*

$$f''(s)=\phi(s,\,f,(s)\,f'(s))\equiv\begin{cases}-\dfrac{\left[\eta-2f(s)\sigma_U^2\right]H(R)g(s)+2\gamma\sigma_M^2 h(\frac{\gamma}{f'(s)})}{\left[2f(s)\sigma_M^2-\eta\right]\left(-h'(\frac{\gamma}{f'(s)})\right)\frac{\gamma^2}{[f'(s)]^3}} & \text{if } f'(s)>0\\ 0 & \text{if } f'(s)\le 0\end{cases}\tag{B.13}$$

on an interval $[s_0,\,\bar{s}]$ with boundary conditions $f(s_0)=\frac{\gamma}{R}s_0$ and $f'(s_0)=\frac{\gamma}{R}$ such that $f'(\bar{s})=0$. Moreover, if $f'(s_1)=0$ for some $s_1<\bar{s}$, then $\eta=2f(s_1)\sigma_U^2$ (in the opposite case, $\eta\ge 2f(\bar{s})\sigma_U^2$).

Proof: In the proof, we will rely on Lemma 7 which implies that the denominator of the ODE (B.13) goes to 0 as $f'(s)\to 0$. Fix a small $\epsilon>0$. We will work with a modified ODE

$$f''(s)=\phi_\epsilon(s,\,f,(s),\,f'(s))\equiv\min\left\{0,\,\phi(s,\,f,(s),\,\min\{\epsilon,\,f'(s)\})\right\}.\tag{B.14}$$

With this modification, the function ϕ_ϵ is uniformly Lipshitz continuous in f and f' (using the assumption that the density h is twice continuously differentiable). By the Picard-Lindelöf Theorem, there exists a unique solution of class C^1 which we will denote by $f_{\eta,\,\epsilon}(s)$; moreover, the solution depends on η in a continuous way. Because the second derivative of $f_{\eta,\,\epsilon}(s)$ is non-positive by definition of ϕ_ϵ, we know that $f_{\eta,\,\epsilon}(s)$ is concave.

Next, we will choose η such that $f'_{\eta,\,\epsilon}(\bar{s})=0$. When ϵ is small enough, and we take η to be close enough to $2(\gamma/R)s_0\sigma_M^2$, we have $f''_{\eta,\,\epsilon}(s_0)\to-\infty$, so the function $f'_{\eta,\,\epsilon}(s)$ will hit zero for some $s<\bar{s}$, and we will have $f'_{\eta,\,\epsilon}(\bar{s})<0$. On the other hand, if we take η low enough, in particular $\eta<2(\gamma/R)s_0\sigma_U^2$, then $\phi_\epsilon(s,\,f_{\eta,\,\epsilon}(s),\,f'_{\eta,\,\epsilon}(s))=0$, and hence $f_{\eta,\,\epsilon}(s)$ will coincide with $(\gamma/R)s$. In this case $f'_{\eta,\,\epsilon}(\bar{s}) > 0$. Thus, there exists an intermediate value η such that $f'_{\eta,\,\epsilon}(\bar{s}) = 0$: Let $f_\epsilon = f_{\eta,\,\epsilon}$ for this η. Thus, we have found a solution f_ϵ to the modified ODE (B.14) with the property that $f'_\epsilon(\bar{s}) = 0$.

Moreover, by the boundary conditions, we can write $f_\epsilon(s) = \frac{\gamma}{R}s_0 + \int_{s_0}^{s} f'_\epsilon(t)dt$, and we know that $\eta \geq 2f_\epsilon(s)\sigma_U^2$ for all $s \geq s_0$, for ϵ small enough. Indeed, if this last claim was not true, then by the properties of the function ϕ_ϵ, we could show that as f'_ϵ goes to 0, ϕ becomes positive, and thus ϕ_ϵ becomes 0. This, however, contradicts the fact that $f'_\epsilon(\bar{s}) = 0$. When $\eta \geq 2f_\epsilon(s)\sigma_U^2$ for all $s \geq s_0$, and $\eta < 2f_\epsilon(s_0)\sigma_M^2$, then $\phi(s,\, f_\epsilon(s),\, f'_\epsilon(s)) \leq 0$, so f_ϵ is a solution to the ODE

$$f''(s) = \phi(s,\, f,(s),\, \min\{\epsilon,\, f'(s)\}). \tag{B.15}$$

This means that we can write $f'_\epsilon(s)$ as a fixed point of the following operator

$$f'_\epsilon(s) = \Lambda_\epsilon(f'_\epsilon(s)) \equiv \max\left\{0,\, \frac{\gamma}{R} - \int_{s_0}^{s} \phi_\epsilon\left(t,\, \frac{\gamma}{R}s_0 + \int_{s_0}^{t} f'_\epsilon(\tau)d\tau,\, f'_\epsilon(t)\right) dt\right\}.$$

We want to prove that $f'(s) = \lim_{\epsilon\to 0} f'_\epsilon(s)$ exists, and that f' is a fixed point of the limit operator $\Lambda = \lim_{\epsilon\to 0}\Lambda_\epsilon$. By Tychonoff's Theorem, we can obtain $f'(s)$ which is a pointwise limit of a subsequence of $f'_\epsilon(s)$ because $f'_\epsilon \in [0,\, \gamma/R]$. We prove that the limiting function f' is in fact continuous. The only point at which continuity of f' might fail is a point s_1 at which f'' diverges to $-\infty$ (at such a point, f' could have a jump discontinuity from a positive level to 0). Because $h \in C^2$, we can find a number $B > 0$ such that $f''_\epsilon(s) \geq -B/f'_\epsilon(s)$ uniformly in ϵ and s. Intuitively, $f''_\epsilon(s)$ cannot be highly negative unless $f'_\epsilon(s)$ is close to 0. But this means that $f'_\epsilon(s) \leq -B/f''(s)$ and in particular $f'_\epsilon(s) \to 0$ when $f''_\epsilon(s) \to -\infty$. Therefore, $f'(s_1) = 0$ if f'' diverges to $-\infty$ at s_1, and hence f' is continuous at s_1. When a sequence of non-decreasing continuous functions converges pointwise to a continuous (non-decreasing) function, the convergence is uniform. Therefore, we have proven that $f'_\epsilon \rightrightarrows f'$. Because the convergence is uniform, f' is also a fixed point of the limiting functional Λ. Thus, we have obtained a continuous f' such that

$$f'(s) \equiv \max\left\{0,\, \frac{\gamma}{R} - \int_{s_0}^{s} \phi\left(t,\, \frac{\gamma}{R}s_0 + \int_{s_0}^{t} f'(\tau)d\tau,\, f'(t)\right) dt\right\},$$

and $f'(\bar{s}) = 0$. In particular, this means that whenever $f' > 0$, f is a solution to the ODE (B.13) (and hence (A.1)).

To finish the proof, we argue that $\eta \geq 2f(\bar{s})\sigma_U^2$, and if $f'(s_1) = 0$ for some $s_1 < \bar{s}$, then $2f(s_1)\sigma_U^2 = \eta$. The first claim is a consequence of the inequality $\eta \geq 2f_\epsilon(s)\sigma_U^2$ for every $\epsilon > 0$ (proven earlier). Suppose that the second claim is not true. Then, because we proved uniform convergence of f'_ϵ to f', for any $\delta > 0$, we can find $\bar{\epsilon} > 0$ such that for all $\epsilon < \bar{\epsilon}$,

$$\max_{s \in [s_1,\, \bar{s}]} |f'_\epsilon(s)| < \delta. \tag{B.16}$$

However, when $2f(s_1)\sigma_U^2 > \eta$, (so that $2f_\epsilon(s)\sigma_U^2$ is bounded away from η on $[s_1,\, \bar{s}]$), this implies that $-f''_\epsilon$ gets arbitrarily large as δ gets small. This is a contradiction with f'_ϵ being a solution to ODE (B.15) that at the same time satisfies (B.16). □

Given Lemma 8, the proof of Lemma 5 is immediate. By taking η whose existence is guaranteed by Lemma 8, we satisfy conditions 1-4 of Lemma 6. The functions $p_1(s)$ and $p_2(s)$ are continuous and continuously differentaible by construction (and Lemma 8 which guarantees that $u(s)$ is continuous everywhere). The function $\hat{H}(f,\, s)$ is strictly concave in f for all s because the Hamiltonian $\mathcal{H}$ is a quadratic (strictly concave) function of f. Finally, we can choose s_0 such that the corresponding f is feasible, that is, satisfies constraint (B.7), or equivalently, (3.5). Indeed, *(i)* f depends on s_0 in a continuous way, *(ii)* choosing $s_0 = \bar{s}$ yields $f(s) = (\gamma/R)s$ which gives $\Gamma(\bar{s}) > 1/n$ because $R < \hat{R}$, and *(iii)* when $s_0 \to 0$, the corresponding $f(s)$ also converges to zero pointwise, so $\Gamma(\bar{s}) < 1/n$. By the intermediate value theorem, there exists $s_0 \in (0,\, \bar{s})$ such that $\Gamma(\bar{s}) = 1/n$, that is, constraint (3.5) holds.

This implies that the constructed f is the unique solution to the problem (B.5) - (B.7). Because this function is feasible for the original problem $\mathcal{P}(R)$, it is also the unique solution to $\mathcal{P}(R)$.

B.6 Proof of Proposition 3

We will show that the optimal benchmark fixing with $R \in \{0,\, \hat{R}\}$ is dominated by choosing a weighting function of the form

$$f_\beta(s) = \frac{\gamma}{R(\beta)} \max\{s,\, \beta\},$$

for some $\beta \in [0,\, \bar{s}]$, where $R(\beta)$ is chosen to make f_β feasible, that is, to satisfy (3.5):

$$\frac{H(R(\beta))}{R(\beta)} \left(\int_0^\beta \gamma\tau g(\tau) d\tau + \gamma\beta(1 - G(\beta)) \right) + \frac{1 - H(R(\beta))}{R(\beta)} \gamma\beta = \frac{1}{n}.$$

As noted in the discussion of Theorem 1a-1b, the optimal weighting function for $R = 0$ is $f(s) = 1/n$, and the optimal weighting function for $R = \hat{R}$ is $f(s) = (\gamma/\hat{R})s$. Importantly, these two functions are the limits of the family f_β as β varies from 0 to $\bar{s}$.[35] Moreover,

[35]Formally, we have $f_{\bar{s}}(s) = (\gamma/\hat{R})s$ and $\lim_{\beta \to 0} f_\beta(s) \to 1/n$ for all $s > 0$.

$R(\beta) \in (0, \hat{R})$ for all $\beta \in (0, \bar{s})$. Let

$$V(\beta) = \frac{H(R(\beta))}{R^2(\beta)} \left[\int_0^\beta \gamma^2 \tau^2 g(\tau) d\tau + \gamma^2 \beta^2 (1 - G(\beta)) \right] \sigma_U^2 + \frac{1 - H(R(\beta))}{R^2(\beta)} \gamma^2 \beta^2 \sigma_M^2$$

denote the value of the objective function (4.2) at f_β. Then, $V(0)$ corresponds to the value attained by the optimal weighting function with $R = 0$, and $V(\bar{s})$ corresponds to the value attained by the optimal weighting function with $R = \hat{R}$. Because V is continuous and differentiable, to prove Proposition 3, it is enough to show that $V'(0) < 0$, and $V'(\bar{s}) > 0$.

Using the implicit function theorem, we can write $R(\beta)$ as a function of β with

$$\lim_{\beta \to 0} \frac{R(\beta)}{\beta} = \gamma n,$$

and

$$R'(\beta) = \frac{R(1 - H(R(\beta))G(\beta))}{\beta - (H(R(\beta)) - h(R(\beta))R(\beta)) \left(\beta G(\beta) - \int_0^\beta \tau g(\tau) d\tau \right)}.$$

We can calculate the derivative of $V(\beta)$ at $\beta = \bar{s}$ directly:

$$\begin{aligned} V'(\bar{s}) = {} & \frac{h(\hat{R})\hat{R}^2 - 2H(\hat{R})\hat{R}}{\hat{R}^3} \frac{1 - H(\hat{R})}{\bar{s} - \left(H(\hat{R}) - h(\hat{R})\hat{R} \right) \left(\bar{s} - \int_0^{\bar{s}} \tau g(\tau) d\tau \right)} \left[\int_0^{\bar{s}} \tau^2 g(\tau) d\tau \sigma_U^2 - \bar{s}^2 \sigma_M^2 \right] \\ & - \left[\frac{2H(\hat{R})}{\hat{R}^2} \bar{s} - \frac{2}{\hat{R}^2} \bar{s}^2 \frac{1 - H(\hat{R})}{\bar{s} - \left(H(\hat{R}) - h(\hat{R})\hat{R} \right) \left(\bar{s} - \int_0^{\bar{s}} \tau g(\tau) d\tau \right)} + \frac{2}{\hat{R}^2} \bar{s} \right] \sigma_M^2. \end{aligned}$$

If we let $\lambda = \sigma_M^2 / \sigma_U^2$, then $V'(\bar{s}) > 0$ is equivalent, after some simplifications, to

$$\left[2H(\hat{R}) - h(\hat{R})\hat{R} \right] \left[\int_0^{\bar{s}} \left(\frac{\tau}{\bar{s}} \right)^2 g(\tau) d\tau \right] < \lambda \left[h(\hat{R})\hat{R} + 2 \left(H(\hat{R}) - h(\hat{R})\hat{R} \right) \int_0^{\bar{s}} \left(\frac{\tau}{\bar{s}} \right) g(\tau) d\tau \right].$$

We know that the density h is decreasing, and because a density is integrable, we must have $\lim_{R \to 0} h(R)R = 0$. It follows that $H(R) > h(R)R$ for all $R > 0$ because

$$\frac{d}{dR}[H(R) - h(R)R] = h(R) - h(R) - h'(R)R > 0.$$

Therefore, $V'(\bar{s}) > 0$ is equivalent to

$$\lambda > \frac{\left[2H(\hat{R}) - h(\hat{R})\hat{R} \right] \frac{\mathbb{E}s_1^2}{\bar{s}^2}}{h(\hat{R})\hat{R} + 2 \left(H(\hat{R}) - h(\hat{R})\hat{R} \right) \frac{\mathbb{E}s_1}{\bar{s}}}. \tag{B.17}$$

Next, we have

$$\frac{\left[2H(\hat{R})-h(\hat{R})\hat{R}\right]\frac{\mathbb{E}s_1^2}{\bar{s}^2}}{h(\hat{R})\hat{R}+2\left(H(\hat{R})-h(\hat{R})\hat{R}\right)\frac{\mathbb{E}s_1}{\bar{s}}} \leq \frac{\left[2H(\hat{R})-h(\hat{R})\hat{R}\right]\frac{\mathbb{E}s_1}{\bar{s}}}{h(\hat{R})\hat{R}+2\left(H(\hat{R})-h(\hat{R})\hat{R}\right)\frac{\mathbb{E}s_1}{\bar{s}}} \leq 1,$$

where the last inequality follows from the fact that, by direct calculation of the derivative, the middle expression is increasing in $\mathbb{E}s_1/\bar{s}$. This proves that (B.17) always holds because its left hand side is strictly greater than 1, while the right hand side is less than 1.

Now, we will show that $V'(0) < 0$. We have

$$V'(0) = \lim_{\beta\to 0} V'(\beta) = \left[\sigma_U^2 - \sigma_M^2\right]\frac{\gamma^2 h(0)}{n} < 0.$$

This ends the proof.

B.7 Proof of Theorem 2

We will first show that $f^\star(s) = (\gamma/\hat{R})s$ solves problem $\mathcal{P}$. It follows that $f^\star$ solves problem $\mathcal{P}(R)$ for any $R \leq \hat{R}$, because $f^\star$ is feasible for $\mathcal{P}(R)$ for any $R \leq \hat{R}$.

By arguments analogous to the ones used in the proof of Lemma 2 and Lemma 3, the optimal function f is continuous and non-decreasing. By Bruckner and Ostrow (1962), Assumption 1 is equivalent to the following condition when $f(0) = 0$, and f is non-decreasing and continuous:

$$f'(s^-) \geq \frac{f(s)}{s}, \text{ for all } s \in (0, \bar{s}],$$

where $f'(s^-)$ denotes the left Dini derivative of f at s. Because $f \in \mathcal{C}^{K,M}$ together with continuity of f implies that f is absolutely continuous, we can write that condition as

$$f'(s) \geq \frac{f(s)}{s}, \text{ for a.e. } s \in (0, \bar{s}). \tag{B.18}$$

We first prove that under Assumption 1, all manipulators ($R_i \geq R$) choose $\hat{s}_i = \bar{s}$. We have

$$\frac{d}{ds}\left(Rf(s) - \gamma s\right) = Rf'(s) - \gamma \geq R\frac{f(s)}{s} - \gamma = s(Rf(s) - \gamma s).$$

This implies that if there exists any $s > 0$ at which a manipulator can make positive profits, then that manipulator maximizes profits by choosing $\hat{s}_i = \bar{s}$. This implies that the problem to solve is

$$\inf_{f\in\mathcal{C}^{K,M}} \int_0^{\bar{s}} f^2(s)\sigma_U^2 H(R_f)g(s)ds + f^2(\bar{s})\sigma_M^2(1 - H(R_f)) \tag{B.19}$$

subject to (B.18), and

$$f(s) \le \frac{\gamma}{R_f} s,\ \forall s \in [0,\ \bar{s}], \tag{B.20}$$

$$\int_0^{\bar{s}} f(s) H(R_f) g(s) ds + f(\bar{s})(1 - H(R_f)) = \frac{1}{n}. \tag{B.21}$$

Similarly as for the baseline model, we will parameterize the above problem by $R = R_f$, and solve it first for any fixed $R \le \hat{R}$.

To simplify the objective function, note that

$$f^2(\bar{s}) = 2\int_0^{\bar{s}} f(s) f'(s) ds.$$

Similarly, we can express condition (B.21) with $R = R_f$ as

$$H(R)\int_0^{\bar{s}} f(s) g(s) ds + (1 - H(R))\int_0^{\bar{s}} f'(s) ds = \frac{1}{n}.$$

To incorporate condition (B.18) into the problem, we will redefine the control variable $u(s)$ relative to the baseline model. Instead of $f'(s) = u(s)$, we let $u(s) = f'(s) - f(s)/s$. Constraint (B.18) can now be expressed as $u(s) \ge 0$. Thus, the full problem can be written as

$$\min_{u \ge 0} \int_0^{\bar{s}} \left[f^2(s)\sigma_U^2 g(s) + 2 f(s) \sigma_M^2 \left(u(s) + \frac{f(s)}{s} \right) \right] ds \tag{B.22}$$

subject to

$$f'(s) = u(s) + \frac{f(s)}{s},\ f(0) = 0,\ f(\bar{s}) - \text{free} \tag{B.23}$$

$$\Gamma'(s) = H(R) f(s) g(s) + (1 - H(R)) \left(u(s) + \frac{f(s)}{s} \right),\ \Gamma(0) = 0,\ \Gamma(\bar{s}) = \frac{1}{n} \tag{B.24}$$

$$f(s) \le \frac{\gamma}{R} s. \tag{B.25}$$

We conjecture that the constraint $f(s) \le (\gamma/R)s$ is slack. We want to prove that the optimal f is linear: $f(s) = \alpha s$ for some $\alpha \le \gamma/R$. There exists a unique α under which a linear f satisfies the constraint (B.21) (or B.24), namely, $\alpha = \gamma/\hat{R}$. Such f satisfies constraint (B.25), and thus if it solves the relaxed problem, it also solves the original problem.

The Hamiltonian corresponding to the relaxed problem (B.22) - (B.24) is

$$\begin{aligned}\mathcal{H}(f(s),\, u(s),\, s) = &- \left[f^2(s)\sigma_U^2 g(s) + 2\sigma_M^2 f(s) \left(u(s) + \frac{f(s)}{s} \right) \right] \\ &+ p_1(s) \left(u(s) + \frac{f(s)}{s} \right) + p_2(s) \left(H(R) f(s) g(s) + (1 - H(R)) \left(u(s) + \frac{f(s)}{s} \right) \right). \end{aligned} \tag{B.26}$$

We state sufficient conditions for a function f to be optimal, using Arrow's Theorem.

Lemma 9 *Let $(f(s),\, u(s))$ be a feasible pair for the problem (B.22) - (B.24). If there exists a continuous and piecewise continuously differentiable function $p(s) = (p_1(s),\, p_2(s))$ such that the following conditions are satisfied*

1. $p_1'(s) = 2f(s)\sigma_U^2 g(s) + 2\sigma_M^2 \left(u(s) + 2\frac{f(s)}{s}\right) - p_1(s)\frac{1}{s} - p_2(s)\left(H(R)g(s) + (1 - H(R))\frac{1}{s}\right);$
2. $p_2'(s) = 0;$
3. *$u(s)$ maximizes $\mathcal{H}(f(s),\, u,\, s)$ over $u \geq 0$ for all $s \in [0,\, \bar{s}]$;*
4. *$p_1(\bar{s}) = 0$;*
5. *$\hat{H}(f,\, s) = \max_{u\in[0,\, \gamma/R]} \mathcal{H}(f,\, u,\, s)$ exists and is concave in f for all s,*

then $(f(s),\, u(s))$ solve the problem (B.22) - (B.24). If $\hat{H}(f,\, s)$ is strictly concave in f for all s, then f is the unique solution.

Proof: By direct application of the Arrow Sufficiency Theorem (Theorem 5 on page 107 of Seierstad and Sydsaeter, 1987). □

Since we want to prove that $f(s) = \alpha s$ is optimal, we have $u(s) = 0$ for all $s \in [0,\, \bar{s}]$. The Hamiltonian is maximized at $u = 0$ across feasible $u \geq 0$ when

$$-2\sigma_M^2 f(s) + p_1(s) + p_2(s)(1 - H(R)) \leq 0.$$

We can set $p_2(s) = \eta$ for some constant η for all s (this will satisfy condition 2 of Lemma 9). The Hamiltonian is strictly concave in f. Thus, to satisfy all conditions of Lemma 9, it is enough to prove that there exists a continuously differentiable $p(s)$ (we abuse notation slightly by dropping the subscript from $p_1(s)$) and a constant η such that

$$p(s) + \eta(1 - H(R)) \leq 2\sigma_M^2 \alpha s, \tag{B.27}$$

$$p(\bar{s}) = 0, \tag{B.28}$$

$$p'(s) + p(s)\frac{1}{s} = 2\alpha s\sigma_U^2 g(s) + 4\sigma_M^2\alpha - \eta H(R)g(s) - \eta(1 - H(R))\frac{1}{s}, \tag{B.29}$$

for all $s \in [0,\, \bar{s}]$. Solving the ODE (B.29), we obtain

$$p(s) = \frac{1}{s}\left(\kappa + \int_0^s \left[2\alpha\tau^2\sigma_U^2 g(\tau) + 4\sigma_M^2\alpha\tau - \eta H(R)g(\tau)\tau - \eta(1 - H(R))\right] d\tau\right),$$

for all $s > 0$, and some constant κ. With the final condition (B.28), we obtain

$$p(s) = -\frac{1}{s}\int_s^{\bar{s}} \left[2\alpha\tau^2\sigma_U^2 g(\tau) + 4\sigma_M^2\alpha\tau - \eta H(R)g(\tau)\tau - \eta(1 - H(R))\right] d\tau.$$

This means in particular that $p(s)$ is well defined and continuously differentiable for all $s \in (0, \bar{s}]$. To guarantee that we can define $p(0)$ so that $p(s)$ is continuous at $s = 0$, we need

$$\int_0^{\bar{s}} \left[2\alpha\tau^2\sigma_U^2 g(\tau) + 4\sigma_M^2\alpha\tau - \eta H(R)g(\tau)\tau - \eta(1 - H(R))\right] d\tau = 0. \tag{B.30}$$

Condition (B.30) is also sufficient: By d'Hosptial rule, if (B.30) holds, then the limit $\lim_{s\searrow 0} p(s)$ exists and is finite. Condition (B.30) pins down a unique candidate for η:

$$\eta = \frac{\int_0^{\bar{s}} \left[2\alpha\tau^2\sigma_U^2 g(\tau) + 4\sigma_M^2\alpha\tau\right] d\tau}{\int_0^{\bar{s}} \left[H(R)g(\tau)\tau + (1 - H(R))\right] d\tau}.$$

With η defined this way, and after simplifying the expressions, (B.27) becomes equivalent to

$$\begin{aligned}\frac{\int_0^{\bar{s}} \left[\tau^2\sigma_U^2 g(\tau) + 2\sigma_M^2\tau\right] d\tau}{\int_0^{\bar{s}} \left[H(R)g(\tau)\tau + (1 - H(R))\right] d\tau}&\left[\int_s^{\bar{s}} \tau H(R)g(\tau)d\tau + (1 - H(R))\bar{s}\right] \\ &\leq \sigma_M^2\bar{s}^2 + \int_s^{\bar{s}} \tau^2\sigma_U^2 g(\tau)d\tau,\end{aligned}$$

for all $s \in [0, \bar{s}]$. Equivalently, after some simplifications,

$$\begin{aligned}&H(R)\left(\int_0^{\bar{s}} \tau g(\tau)d\tau\right)\left(\int_0^{s} \tau^2\sigma_U^2 g(\tau)d\tau\right) + (1 - H(R))\bar{s}\int_0^{s} \tau^2\sigma_U^2 g(\tau)d\tau \\ &\qquad \leq \sigma_M^2(1 - H(R))\bar{s}^2\int_0^{s} \tau g(\tau)d\tau + H(R)\left(\int_0^{\bar{s}} \tau^2\sigma_U^2 g(\tau)d\tau\right)\left(\int_0^{s} \tau g(\tau)d\tau\right),\end{aligned}$$

for all $s \in [0, \bar{s}]$. Because the above expression is linear in $H(R)$, and R does not appear anywhere else in the expression, it is enough to show that it holds for both $H(R) = 0$ and $H(R) = 1$. That is, it is enough to show that

$$\frac{\int_0^{\bar{s}} \tau g(\tau)d\tau}{\int_0^{\bar{s}} \tau^2\sigma_U^2 g(\tau)d\tau} \leq \frac{\int_0^{s} \tau g(\tau)d\tau}{\int_0^{s} \tau^2\sigma_U^2 g(\tau)d\tau}, \tag{B.31}$$

and

$$\int_0^{s} \tau^2\sigma_U^2 g(\tau)d\tau \leq \bar{s}\int_0^{s} \tau\sigma_M^2 g(\tau)d\tau, \tag{B.32}$$

for all $s \in [0, \bar{s}]$. Inequality (B.32) is clearly true because $\sigma_U^2 < \sigma_M^2$ by assumption. To prove inequality (B.31), it is enough to show that

$$\frac{\int_0^s \tau g(\tau) d\tau}{\int_0^s \tau^2 \sigma_U^2 g(\tau) d\tau}$$

is decreasing in s. By calculating the derivative, we can show that a sufficient condition is $\int_0^s [\tau - s] \, \tau g(\tau) d\tau \leq 0$ for all s, which is clearly satisfied. This ends the proof that conditions (B.27) – (B.29) all hold.

Therefore, all conditions of Lemma 9 also hold, and thus we have proven that $f(s) = (\gamma/\hat{R})s$ is the unique solution to the relaxed problem (B.22) - (B.24) for any $R \leq \hat{R}$, and hence also the problem (B.22) - (B.25). It follows that the same f solves the problem $\mathcal{P}$ and $\mathcal{P}(R)$ for any $R \leq \hat{R}$.