

# QFT Seminar

Bethes calculation of the Lamb shift

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# OVERVIEW

- Historical background
- Greens function
- Propagators
- Feynman graphs
- Bethes calculation
- Outlook on further developments

# The Shelter Island conference

June 1947

- leading physicist of the US gathered
- discussion of present problems with Quantum Electrodynamics (QED)
- direction for the next years
- electron mass calculated to be infinite interpretation?
- experiment presented by Lamb and Retherford no explanation within QED

# Shelter Island conference



Richard Feynman explains a point during the 1947 Shelter Island Conference. Standing (from left to right) are Willis Lamb, Karl Darrow, Victor Weisskopf, George Uhlenbeck, Robert Marshak, Julian Schwinger, and David Bohm. Seated are Robert Oppenheimer, Abraham Pais, Feynman, and Herman Feshbach. (Courtesy of the National Academy of Sciences.)

The problem of the divergent mass of the electron was addressed by Hendrik Kramers (non US) with a technique called “renormalization”.

observed mass = bare mass + self mass

With the bare mass and the self mass infinite (and one negative), the observed mass should come out finite.

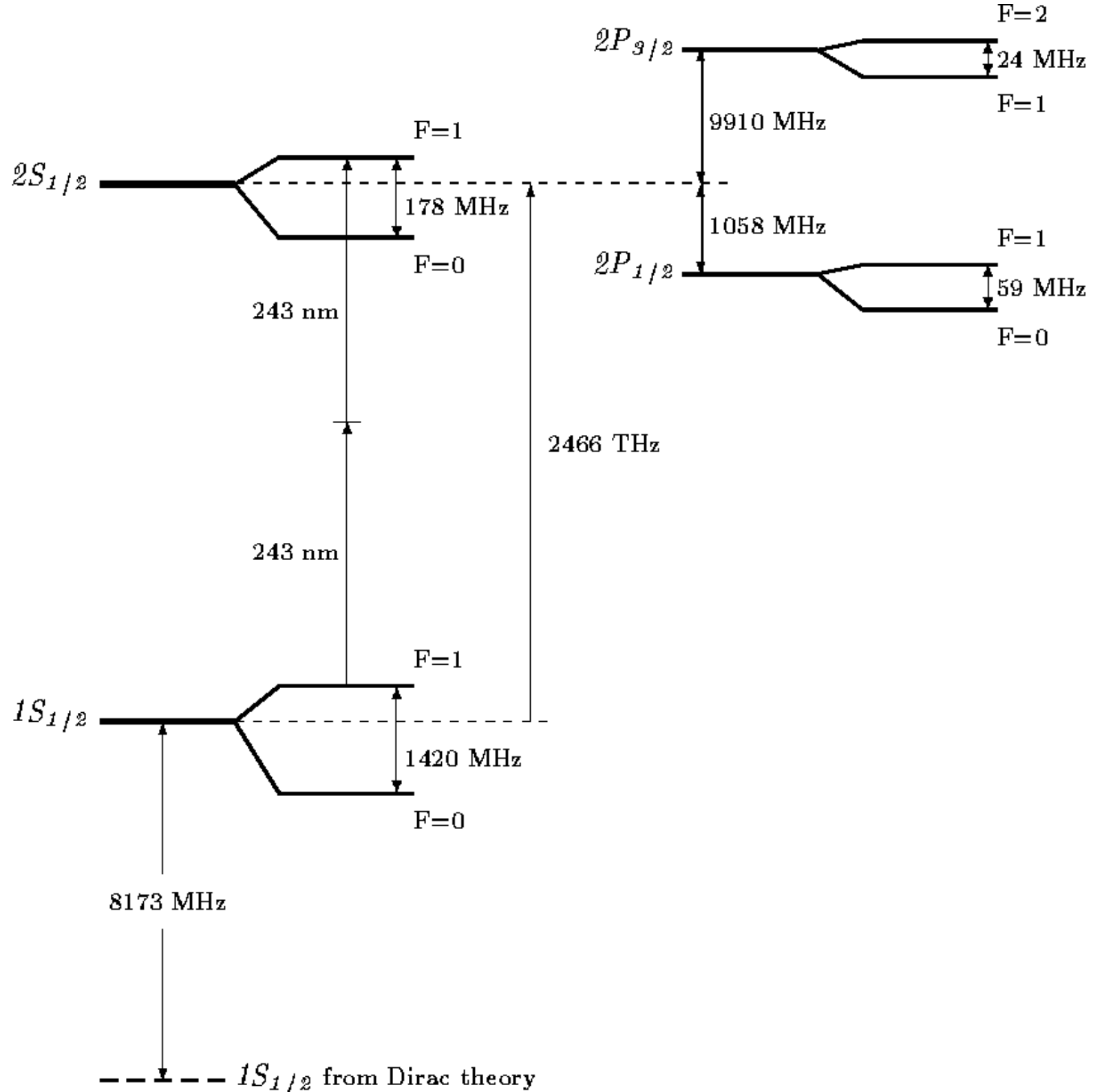
It looked promising, but he could not calculate it for a real problem.

# the Lamb shift

The Lamb shift is the energy shift between the  $2S_{1/2}$  and the  $2P_{1/2}$  energy levels of the hydrogen atom.

With the Dirac equation, the energies come out equal, but Lamb measured a energy difference of about 1000 megacycles (Mhz) or  $4.3 \mu\text{eV}$ .

The  $2S_{1/2}$  and the  $2P_{1/2}$  energy levels are at approximately 10.2 eV.



Schwinger, Weisskopf and Oppenheimer already suggested that the self-energy of the electron might cause that shift.

General opinion was, that the calculation of the self-energy required a complete relativistic quantum field theory (QED) approach to the problem, which was very difficult.

Only Hans Bethe tried a non-relativistic calculation, which was far easier.

This calculation restored the belief in QED.



# Hans Bethe

(\* 2<sup>nd</sup> of July 1906 † 6<sup>th</sup> of March 2005)



Hans in his office at Cornell, circa 1935–1936. (Courtesy of Rose Bethe.)

## Bethe and his slide rule



**Fig. 2** The slide rule was Bethe's calculational tool of choice. In this photo, taken at Cornell University around 1986, Bethe analyzes computer output. (Photograph by Kurt Gottfried, courtesy of AIP Emilio Segrè Visual Archives.)

# Green's function for the electric potential

Poisson equation  $\nabla^2 V(\vec{r}) = -\rho(\vec{r})$

define  $\nabla^2 G(\vec{r}, \vec{r}') = -\delta(\vec{r} - \vec{r}')$

and  $V(\vec{r}) = \int G(\vec{r}, \vec{r}') \rho(\vec{r}') d\vec{r}'$

to solve the Poisson equation

$$\nabla^2 V(\vec{r}) = \int \nabla^2 G(\vec{r}, \vec{r}') \rho(\vec{r}') d\vec{r}' = -\rho(\vec{r})$$

compare  $V(\vec{r}) = \int G(\vec{r}, \vec{r}') \rho(\vec{r}') d\vec{r}'$

and  $V(\vec{r}) = \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d\vec{r}'$

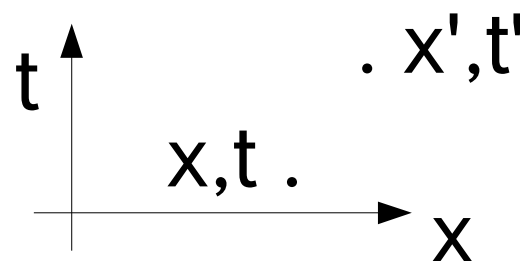
to see  $G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|}$

thus it is easy to see that

$$\nabla^2 \frac{1}{|\vec{r} - \vec{r}'|} = -\delta(\vec{r} - \vec{r}')$$

In QFT the Greens functions in Minkowski space (3+1D) play an important role and are called propagators. Each type of particle has its own propagator.

A propagator propagates an amplitude in the Minkowski space from one point  $(x,t)$  to the point  $(x',t')$ .

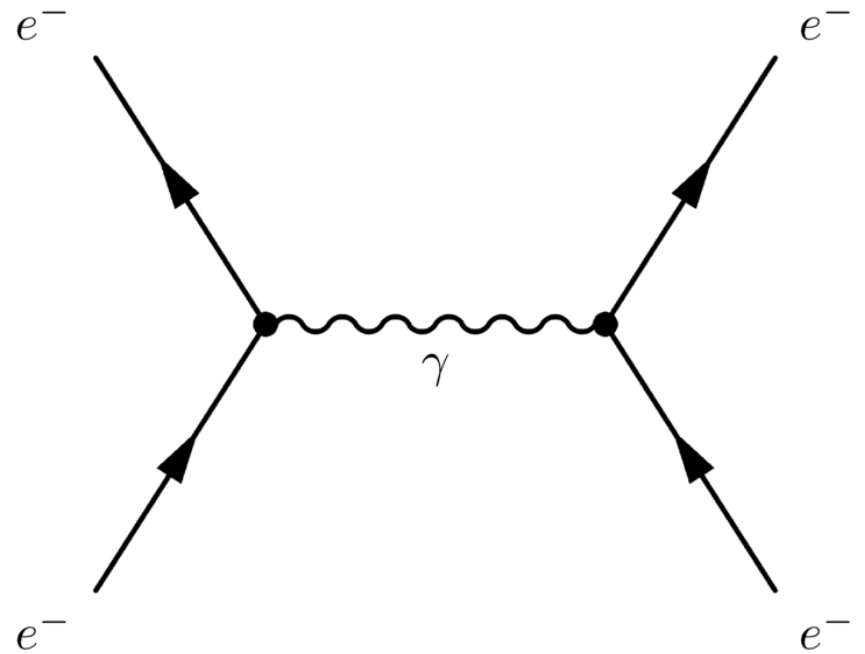


Propagators are used to construct the equations from Feynman graphs.

# Feynman graphs

Feynman graphs are used to construct the amplitude, e.g. the self-energy, of particle and field interaction by perturbation theory.

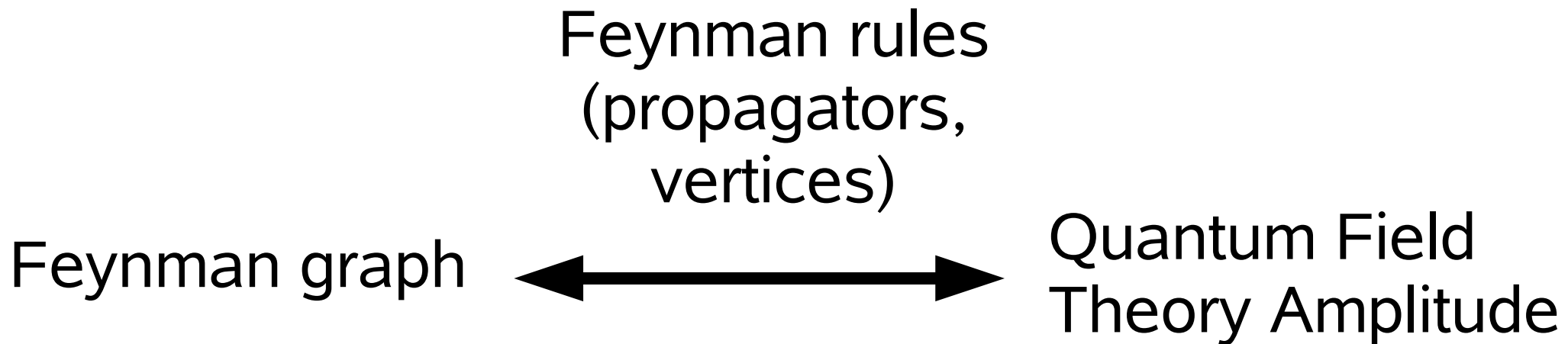
particles are depicted by lines  
lines intersect at vertices,  
depicting interactions



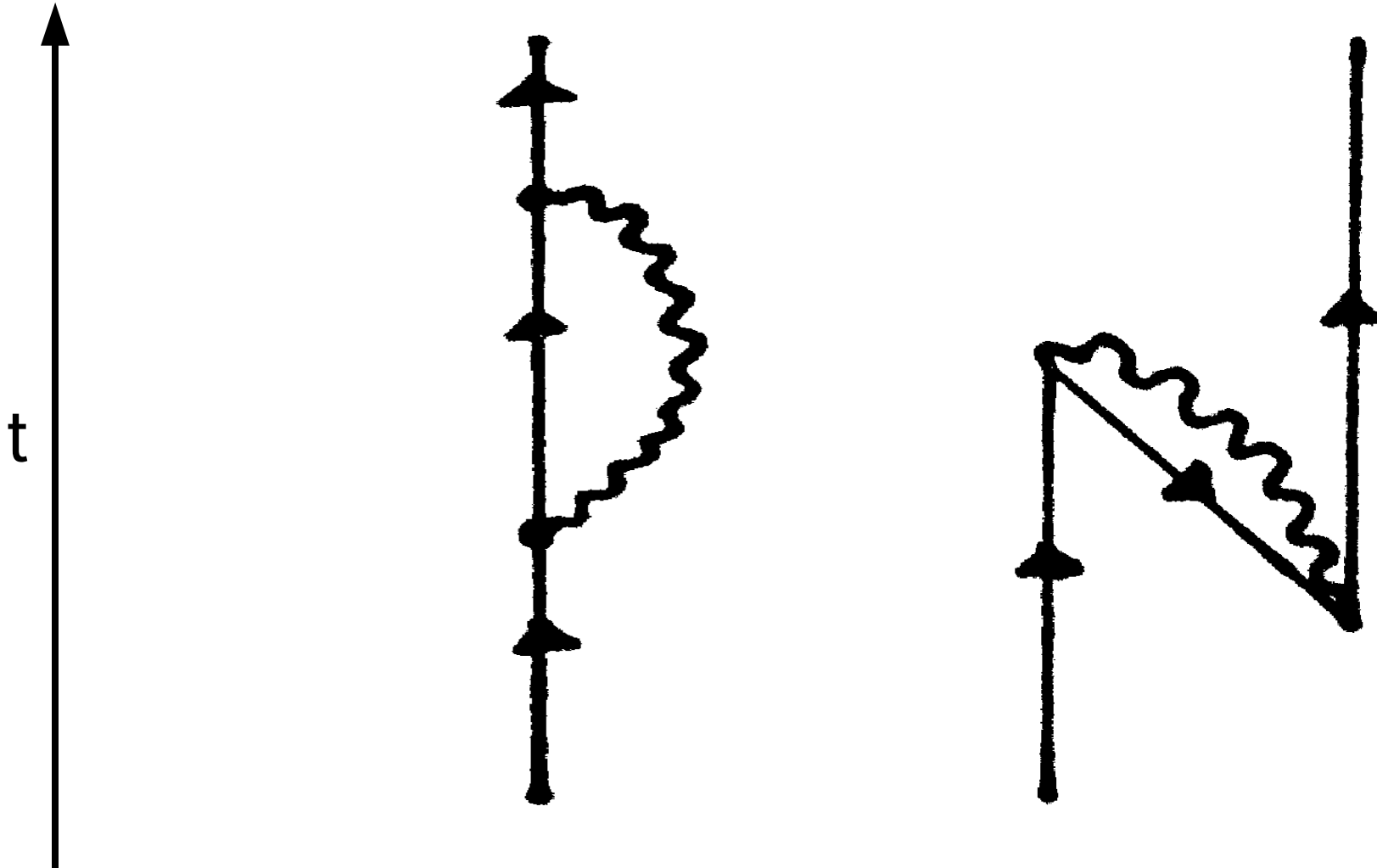
from Wikipedia

The different graphical objects are directly equivalent to certain parts of the amplitude.

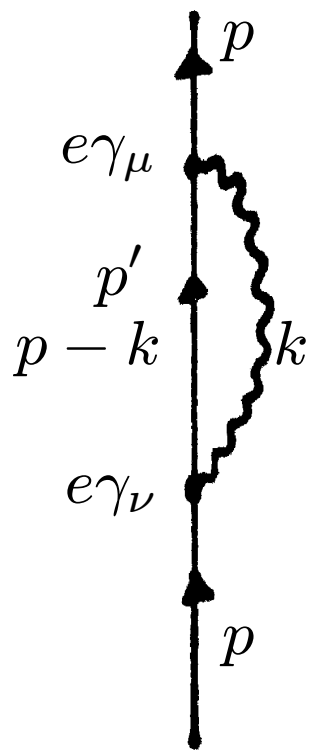
Given a Feynman graph, one can write the complete amplitude for the interaction studied using the Feynman rules.



the second order Feynman graph  
for the self energy



from the feynnman graph, the self-energy is constructed to be



$$\int d^4 k \frac{\bar{u}(p) (-ie\gamma^\mu) i \cdot (-ig_{\mu\nu}) (-ie\gamma^\nu) u(p)}{(\not{p} - \not{k} - m + i\epsilon) \cdot (k^2 + i\epsilon)}$$

carrying out the integration over  $k_0$  using the residue theorem, we can approximate the result by

$$4\pi e^2 \int \frac{d^3 k}{(2\pi)^3} \frac{\bar{u}(p) \gamma^\mu u(p - k) g_{\mu\nu} \bar{u}(p - k) \gamma^\nu u(p)}{\left(E_p - E_{p-k} - |\vec{k}|\right) \cdot \left(2|\vec{k}|^2\right)} + \dots$$

$\vec{k}$  is a 3-vector in space



# Bethes calculation

The normal electron mass  $m_e$  consists of the bare mass  $m_0$  and the self energy in the vacuum.

$$m_e = m_0 + \Sigma_{\text{vacuum}}$$

In the bound state of the atom, the self energy in the vacuum has to be replaced by the self energy of the bound state.

$$E = m_0 + \Sigma_{\text{atom}} + E_{\text{coulomb}}$$

$$E = m_e \underbrace{-\Sigma_{\text{vacuum}} + \Sigma_{\text{atom}}}_{\text{Lamb-shift}} + E_{\text{coulomb}}$$

# ansatz

$$4\pi e^2 \int \frac{d^3 k}{(2\pi)^3} \frac{\bar{u}_m \gamma^\mu u_n g_{\mu\nu} \bar{u}_n \gamma^\nu u_m}{(E_n - E_m - |\vec{k}|) \cdot (2k^2)}$$

for the free electron  $E_m, E_n \ll k$

neglecting coulomb photons

$$g_{\mu\nu} \rightarrow \epsilon_\mu \epsilon_\nu \quad (\text{projection on transversal photons})$$

$$u_m \gamma^\mu \epsilon_\mu(k) u_n \rightarrow \langle m | \hat{v} | n \rangle \vec{\epsilon}(k)$$

$$W_0 = -\frac{e^2}{2\pi} \int \frac{d^3 k}{(2\pi)^3} \frac{\sum_{\lambda} |\langle m | \hat{v} | n \rangle \vec{\epsilon}(k)|^2}{k^2}$$

integration over  $d\theta$  and  $d\phi$  carried out gives

$$W_0 = -\frac{2e^2}{3\pi} \int k dk \frac{\vec{v}^2}{k}$$

for the bound state, we can not assume

$$E_n - E_m \ll k$$

and depending on the quantum states,  $E_n$  can have many values (no momentum conservation due to nucleus) which are summed up we get

$$W = -\frac{2e^2}{3\pi} \int k dk \sum_n \frac{|\vec{v}_{mn}|^2}{E_n - E_m + k}$$

$$(\vec{v}^2)_{mm} = \langle m | \hat{v}^2 | m \rangle = \sum_n \langle m | \hat{v} | n \rangle \langle n | \hat{v} | m \rangle = \sum_n |\vec{v}_{mn}|^2$$

$$\begin{aligned}
 W' = W - W_0 = & \quad + \frac{2e^2}{3\pi} \int_0^K k dk \sum_n \frac{|\vec{v}_{mn}|^2}{k} \\
 & - \frac{2e^2}{3\pi} \int_0^K k dk \sum_n \frac{|\vec{v}_{mn}|^2}{E_n - E_m + k}
 \end{aligned}$$

W and  $W_0$  are linear divergent but W is only logarithmic divergent

$$= + \frac{2e^2}{3\pi} \int_0^K dk \sum_n \frac{|\vec{v}_{mn}|^2 (E_n - E_m)}{E_n - E_m + k}$$

Integration over  $k$  yields the logarithmic function.

$$W' = \frac{2e^2}{3\pi} \sum_n |\vec{v}|^2 (E_n - E_m) \ln \frac{K}{|E_n - E_m|}$$

The logarithm is approximately independent of  $n$ , leaving

$$\sum_n |\vec{v}_{mn}|^2 (E_n - E_m)$$

to be calculated.

$$\begin{aligned}
& \sum_n |\vec{v}_{mn}|^2 (E_n - E_m) \\
= & \sum_n \left\{ \langle m | \frac{1}{i} \vec{\nabla} H | n \rangle \langle n | \frac{1}{i} \vec{\nabla} | m \rangle \right. \\
& - \frac{1}{2} \langle m | \frac{1}{i} \vec{\nabla} | n \rangle \langle n | \frac{1}{i} \vec{\nabla} H | m \rangle \\
& \left. - \frac{1}{2} \langle m | H \frac{1}{i} \vec{\nabla} | n \rangle \langle n | \frac{1}{i} \vec{\nabla} | m \rangle \right\} \\
= & \langle m | \vec{\nabla} H \vec{\nabla} - \frac{1}{2} \nabla^2 H - \frac{1}{2} H \nabla^2 | m \rangle \\
& = \langle m | \frac{1}{2} \left[ \vec{\nabla}, [H, \vec{\nabla}] \right] | m \rangle
\end{aligned}$$

$$\implies \langle m | \frac{1}{2} \left[ \vec{\nabla}, \left[ H, \vec{\nabla} \right] \right] | m \rangle = - \langle m | (\nabla^2 V) | m \rangle$$

With the Greens function and the potential, the following was derived:

$$V = -\frac{4\pi e^2 Z}{r} \implies \nabla^2 V = 4\pi e^2 Z \delta(\vec{r})$$

finally we get:

$$\begin{aligned} \sum_n |\vec{v}_{mn}|^2 (E_n - E_m) &= \langle m | \frac{1}{2} (\nabla^2 V) | m \rangle \\ &= \int \psi_m^* 2\pi e^2 Z \delta(\vec{r}) \psi_m d\vec{r} = 2\pi e^2 Z \psi_m^2(0) \end{aligned}$$



Since the wave function vanishes at the nucleus for all except the S states, there is obviously no Lamb shift for the  $2P_{1/2}$  state.

For the S state we have:

$$\psi_m^2(0) = \frac{Z^3}{n^3 a^3 \pi}$$

finally, the Lamb shift is

$$W' = \frac{8}{3\pi} e^6 Ry \frac{Z^4}{n^3} \ln \frac{K}{\langle E_n - E_m \rangle_{Av}}$$

Since relativity theory is expected to provide a cutoff energy for the involved photons,  $K$  is set to  $m_e c^2$ . The average excitation energy of the 2s state was calculated numerically to be 17.8 Ry.

$$\begin{aligned} W' &= \frac{8}{3\pi} e^6 Ry \frac{Z^4}{n^3} \ln \frac{K}{\langle E_n - E_m \rangle_{Av}} \\ &= 5.6 \cdot 10^{-7} eV \cdot 7.65 = 4.28 \cdot 10^{-6} eV \\ &= 1040 \text{ MHz} \end{aligned}$$

# further development of the calculation

fully relativistic and convergent calculation  
published Oct. 1948 by Kroll and Lamb  
(1051 Mhz)

complete non-relativistic calculation also  
converges, but with the wrong result an 1134  
MHz, which corresponds to a cutoff at  $2m_e c^2$

# further development of QED calculations

further calculations, have been carried out by  
Toichiro Kinoshita up to many orders of  
perturbation theory

recent publication (March 2006, with Makiko Nio)

Tenth-order QED contribution to the lepton  $g-2$ : Evaluation of dominant  $\alpha^5$  terms of  
muon  $g-2$

# further development of QFTs

- Electroweak interaction (Glashow-Weinberg-Salam theory)
- Quantum Chromodynamic (QCD) (strong interaction)