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Fibrations for abstract multicategories

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Abstract. Building upon the theory of 2-dimensional fibrations and that of (abstract) multicategories, we present the basics of a theory of *fibred multicategories*. We show their intrinsic role in the general theory: a multicategory is representable precisely when it is covariantly fibrant over the terminal such. Futhermore, such fibred structures allow for a treatment of *algebras for operads* in the internal category setting. We obtain thus a conceptual proof of the 'slices of categories of algebras are categories of algebras' property, which is instrumental in setting up Baez-Dolan's opetopes.

1 Introduction

We introduce the notion of *fibration for multicategories*, the latter understood in their most general sense of (normal) lax algebras on bimodules, as we recall below. Given the space constraints, we limit ourselves to a brief introduction of the attendant theory of fibred multicategories, taking it as an opportunity to review some aspects of our work on 2-fibrations [Her99] and the theory of representable multicategories [Her00, Her01]. We omit most proofs, occassionally outlining interesting arguments.

In [Her00] we introduced the notion of **representable multicategory** as an alternative axiomatisation of the notion of **monoidal category**, *representability* being a *universal property* of a multicategory; it demands the existence of universal 'multilinear' morphisms, $\pi_{\vec{x}} : \vec{x} \to \otimes\, \vec{x}$ for every tuple of objects $\vec{x}$, whose codomain endows the underlying the underlying category of 'linear' morphisms with a 'tensor product'. The basics of this theory (axiomatics of universal morphisms, strictness, and coherence) were developed upon the heuristic

$$\textit{universal morphism} \sim \textit{cocartesian morphism} \tag{1}$$

so that the theory of representable multicategories should parallel that of (co)fibred categories *cf.*[Her00, Table I]. Subsequently, in [Her01] we gave a general treatment of the above transformation

$$\boxed{\text{monoidal category} \mapsto \text{representable multicategory}}$$

in the setting of psuedo-algebras for a cartesian monad M on a '2-regular 2-category' $\mathcal{K}$, *i.e.* a 2-category admitting a 'calculus of bimodules' (the 2-dimensional analogue of the calculus of relations available in a regular category), so that M induces a pseudo-monad $\mathsf{Bimod}(\mathsf{M}) : \mathsf{Bimod}(\mathcal{K}) \to \mathsf{Bimod}(\mathcal{K})$ on bimodules. Given these data, we constructed a 2-category $\mathcal{K}_\dashv$ (consisting of normal lax algebras for $\mathsf{Bimod}(\mathsf{M})$) equipped with a 2-monad $\mathsf{T}_\dashv$ such that

1. $\mathsf{T}_\dashv$ has the adjoint-pseudo-algebra property, *i.e.* a psuedo-algebra structure $x : \mathsf{T}_\dashv X \to X$ on an object X is a left-adjoint to the unit $\eta_X : X \to \mathsf{T}_\dashv X$.
2. The 2-categories of pseudo-algebras, strong morphisms and transformations of M and $\mathsf{T}_\dashv$ are equivalent.

This construction achieves the transformation

coherent structure $\mapsto$ universally characterised structure

which subsumes the case of monoidal categories above, as well as the classical Grothendieck transformation of psuedo-functors into (co)fibred categories (see Remark 1.1 below).

In [Her00, footnote p.169] we argued that in the analogy of representable multicategories with cofibred categories, the former lack a *base*. Here we rectify this statement, showing that representable multicategories are precisely those (covariantly) fibred over the *terminal* one (Theorem 4.1), thereby formalising the heuristic (1) above. Hence, when reasoning about certain categorical structures characterised by universal properties, we can soundly consider them as (covariantly) fibred structures. This correspondence provides yet another argument for the importance of fibred category theory in analysing categorical structure, complementary to the 'philosophical arguments' in [Bén85].

Note on Terminology: Fibrations in categories involve the 'lifting' of morphisms from codomain to domain (or target to source) and thus give rise to contravariant pseudo-functors. For the dual notion, working in the locally dual 2-category $\mathcal{C}at^{co}$, there are two conflicting terminologies in the literature: Gray [Gra66] advocates the use of the term *opfibration* while the Grothendieck school would use the term *cofibration*. Unfortunately, this latter clashes with the notion of cofibration of Quillen in the context of model structures on categories. Given that both notions would come to be used simultaneously in our subsequent work on coherence, it seems sensible to adopt a dissambiguating terminology right here: we shall refer to those fibred structures which give rise to covariant psuedo-functors as **covariant fibrations**, reserving the term **cofibration** for the algebraic topologists' stablished use.

In our foray into the theory of fibrations for multicategories, we would like the reader to keep in mind the following three levels of (increasing) generality and abstraction:

1. The ordinary $\mathcal{S}et$-based notion of multicategory introduced by Lambek [Lam69].
2. Multicategories as monads in a 'Kleisli bicategory of spans' $\mathbf{Spn}_{\mathsf{M}}(\mathbb{B})$, where $\mathsf{M} : \mathbb{B} \to \mathbb{B}$ is a cartesian monad on a category with pullbacks, as in [Her00](*cf.* [Bur71, Lei00]).

3. Multicategories as normal lax algebras in $\mathsf{Bimod}(\mathcal{K})$, the bicategory of bimodules in a 2-regular 2-category $\mathcal{K}$ ([Her02]), with respect to the pseudo-monad $\mathsf{Bimod}(\mathsf{M})$ induced by a 2-monad $\mathsf{M} : \mathcal{K} \to \mathcal{K}$, compatible with the calculus of bimodules, as in [Her01].

1.1. Remark. Since the Grothendieck correspondence between covariant psuedo-functors and cofibrations (covariant fibrations)

$$\mathsf{Ps}\text{-}[\mathbb{C}, \mathcal{C}at] \simeq \mathcal{C}o\mathcal{F}ib/\mathbb{C}$$

on a category $\mathbb{C}$ is not exhibited explicitly in [Her01, §11] as a consequence of the general theory, we outline here how it can be achieved. This example shows that our 'abstract multicategories' may not look like multicategories at first sight.

Recall that a bimodule $\alpha\colon X \not\rightarrow Y$ is called *reprepresentable* when it is of the form $x_{\#} = x{\downarrow}id$ (the span given by the two projections out of the comma-object) for a functor $x : X \to Y$. We observe that pseudo-functors out of $\mathbb{C}$ correspond to pseudo-algebras in the 2-category $[C_0, \mathcal{C}at]$, where C_0 is the object-of-objects of $\mathbb{C}$ (we are in an internal category setting $[C_0, \mathcal{C}at] \cong \mathcal{C}at[C_0, \mathcal{S}et]$). The appropriate 2-monad is $\mathsf{M} = \mathcal{C}at(\mathbb{C} \star \_)$, whose (1-dimensional) cartesian monad $\mathbb{C} \star \_ : [C_0, \mathcal{S}et] \to [C_0, \mathcal{S}et]$ expresses the free action of the (morphisms of) $\mathbb{C}$ on a C_0-family of sets. Remark 11.3 of *ibid.* shows that

$$\mathsf{Ps}\text{-}[\mathbb{C}, \mathcal{C}at] \simeq \mathit{Representable}\text{−}\mathsf{Lax}_{rep}[\mathbb{C}, \mathsf{Bimod}(\mathcal{C}at)]$$

where $\mathsf{Lax}_{rep}[\mathbb{C}, \mathsf{Bimod}(\mathcal{C}at)]$ is the 2-category of lax functors into bimodules and representable transformations between them (that is, induced by functors), obtained by our transformation process out of the 2-monad M and the 2-category $[C_0, \mathcal{S}et]$. The *Representable−* qualificative means the adjoint pseudo-algebras over such with respect to the 2-monad induced by the 'envelope' adjunction, which we recall in §2.1. See also Remark 2.2.(1) below for a reminder of the characterisation of such pseudo-algebras.

We want to show that the new basis of aximotisation is equivalent to the 2-category $\mathcal{C}at/\mathbb{C}$. Theorem 8.2 of *ibid.* entails :

$$\mathsf{Lax}_{rep}[\mathbb{C}, \mathsf{Bimod}(\mathcal{C}at)] \simeq \mathcal{M}ulticat_{\mathbb{C}\star\_}([C_0, \mathcal{S}et])$$

where C_0 is the object-of-objects of $\mathbb{C}$. This equivalence means that since we are working in an internal category setting we can simplify from bimodules to spans (and their easy composition via pullbacks). We now appeal to the well-known (and easy) equivalence $[C_0, \mathcal{S}et] \simeq Set/C_0$, which expresses the two canonical ways of viewing a family of sets. We obtain the following equivalences of 2-categories:

$$\mathcal{M}ulticat_{\mathbb{C}\star\_}([C_0, \mathcal{S}et]) \simeq \mathcal{M}ulticat_{\mathbb{C}\star\_}(\mathcal{S}et/C_0) \simeq \mathcal{C}at/\mathbb{C}$$

the last equivalence resulting from a mere inspection of the diagrams involved. Hence, 'multicategories' in this situation amount to functors into the category $\mathbb{C}$. Remark 11.2 of *ibid.* completes the argument, in the sense that the resulting adjoint pseudo-algebras over $\mathcal{C}at/\mathbb{C}$ obatined by our transformation are indeed cofibrations (covariant fibrations):

$$\mathit{Representable}\text{−}\mathcal{C}at/\mathbb{C} \simeq \mathcal{C}o\mathcal{F}ib/\mathbb{C} \quad \Box$$

In [Str73], Street develops some basic aspects of the theory of fibrations internally in a 2-category $\mathcal{K}$ in a representable fashion, *i.e.* a morphism $p : E \to B$ is a fibration iff $\mathcal{K}(X,p) : \mathcal{K}(X,E) \to \mathcal{K}(X,B)$ is a fibration in $\mathcal{C}at$, for every object X. Starting with the concrete setting 1) above, we expect a fibration of multicategories to involve a lifting of (multi)morphisms. Here we ran into the problem that in the 2-category $\mathcal{M}ulticat$, the 2-cells refer only to *linear* morphisms, *i.e.* those morphisms whose domain lies in the image of the unit of the monad M (a 'singleton sequence'). Hence the representable definition is unsuitably weak for multicategories.

The expected lifting of (multi)morphisms is obtained, abstractly, by means of the 'fundamental' monadic adjunction

$$\mathsf{Lax\text{-}Bimod_T}(\mathcal{K})\text{-}\mathsf{alg} \underset{U}{\overset{\mathsf{Env}}{\underset{\perp}{\rightleftarrows}}} \mathsf{T\text{-}alg}$$

so that the left adjoint 'envelope' 2-functor *reflects (covariant) fibrations* (Theorem 2.4). This point of view allows us to reduce the situation to the ordinary case of (representable) fibrations in a 2-category mentioned above. We could draw an analogy with modules for a Lie algebra $\mathfrak{G}$, which correspond to ordinary modules for its universal envelope $U\mathfrak{G}$.

This incipient theory of fibred structures in a multicategory scenario has several foreseeable applications besides our motivational correspondence with representability above. We illustrate this by elucidating some aspects of the theory of *algebras for operads* (in the non-Σ case), giving a conceptual proof of the fact that the 'slices' of such a category of algebras are categories of algebras for a multicategory (Theorem 5.2). Among the topics we have left out for lack of space are the pseudo-functorial (or 'indexed') version of the fibred structures, the 'comprehensive' factorisation system associated to (discrete) covariant fibrations and a related Yoneda structure, which we will present elsewhere.

As for related work, we should mention that quite independently of our developments, Clementino, Hofmann and Tholen have used V-enriched multicategories (relative to a monad) as an abstract setting for categorical topology, with emphasis on the theory of descent [CT01, CH02, CHT03]. In particular, their analysis of exponentiability involves liftings of factorisations of 'multimorphisms', analogous to Giraud's characterisation of exponentiability in $\mathcal{C}at$ [Gir64] (so that covariant fibrations in our sense are exponentiable, just like in $\mathcal{C}at$). While their setting of V-enriched modules cannot deal with internal structures (which has been our emphasis), their developments and ours can be put into a common framework of 'abstract proarrows' in the sense of [Woo85]. In fact, as we pointed out in [Her01, §2], the theory developed therein is essentially based on such an axiomatic of 'bicategories of bimodules'.

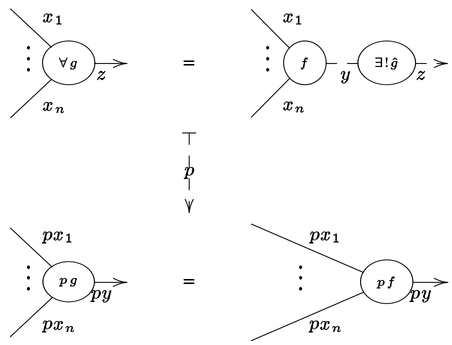
2 Fibrations for multicategories

We start concretely with $\mathcal{S}et$-based multicategories and introduce the elementary definitions of (covariant) fibrations for them. The covariant situation features more prominently in our applications than the contravariant one.

2.1. Definition. Let $p : \mathbb{T} \to \mathbb{B}$ be a morphism of multicategories.

- A morphism $f : \langle x_1 \dots x_n \rangle \to y$ in $\mathbb{T}$ is (p-)**cocartesian** iff every morphism $g : \langle x_1 \dots x_n \rangle \to z$, with the same image on the base $pg = pf$, admits a *unique* factorisation $g = \hat{g} \circ f$, with $\hat{g} : y \to z$ a *vertical* morphism ($p\hat{g} =$

$id_{py} = id_{pz}$). Diagrammatically,



- The morphism $p : \mathbb{T} \to \mathbb{B}$ is a **covariant fibration** if the following hold:
  1. for every list of objects $\vec{x} = \langle x_1, \ldots, x_n \rangle$ in $\mathbb{T}$ and every morphism $u : p\vec{x} \to j$ in $\mathbb{B}$, there is a cocartesian morphism $\underline{u} : \vec{x} \to u_!\vec{x}$ over u ($p\underline{u} = u$).
  2. Cocartesian morphisms are closed under composition.

2.2. Remarks.

1. Just like in the ordinary categorical situation, we could have phrased the definition of covariant fibration appealing to a stronger notion of cocartesian morphism (so that its universal property holds with respect to morphisms which factorise through its projection) and thereby dispense with the composition requirement above. But one of our basic results [Her01, Theorem 5.4] shows that the given formulation is more fundamental: a lax algebra $\alpha\colon MA \not\rightarrow A$ admits an adjoint pseudo-$\mathsf{T}_{\dashv}$-algebra structure iff
   (a) the bimodule α is representable (which in our context amounts to the existence of cocartesian morphisms), and
   (b) the structural 2-cell $\mu : \alpha \bullet \alpha \Rightarrow \alpha$ is an isomorphism (cocartesian morphisms are closed under multicategory composition).
2. We shall distinguish between fibration of multicategories, as defined above, and *fibrations in $\mathcal{M}$ulticat* (in the representable sense), which have cartesian liftings of *linear* morphisms only.

Dually, we have a notion of (p-)cartesian morphism and of fibration (contravariant lifting).

2.3. Examples.

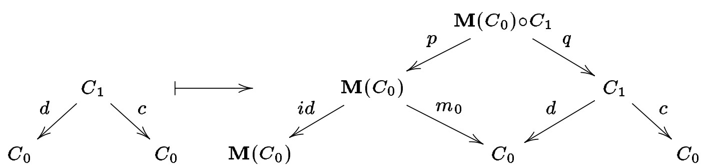
1. Given a functor $q : \mathbb{E} \to \mathbb{C}$ in $\mathcal{C}at$, consider the induced morphism of multicategories $q_\blacktriangleright : \mathbb{E}_\blacktriangleright \to \mathbb{C}_\blacktriangleright$ (the multicategories of discrete cocones [Her00, Example 2.2(2)]):
   - If q is a fibration in $\mathcal{C}at$, $q_\blacktriangleright$ is a fibration of multicategories (cartesian cocones consist of cartesian morphisms in $\mathbb{E}$). In particular the multicategory $\mathbb{C}_\blacktriangleright$ is fibred over the terminal multicategory.
   - If q is a covariant fibration, and $\mathbb{E}$ has cofibred coproducts (coproducts in the fibres preserved by direct images), $q_\blacktriangleright$ is a covariant fibration of multicategories: given a list of objects $\langle x_1, \ldots, x_n \rangle$ of $\mathbb{E}_\blacktriangleright$ and a morphism $\langle u^1 : qx_1 \to j, \ldots, u^n : qx_n \to j \rangle$ we obtain a cocartesian

lifting by considering individual cocartesian liftings $\underline{u^i} : qx_i \to u^i_!(x_i)$ and forming the coproduct $\vec{u}_!(\vec{x}) = \coprod_i u^i_!(x_i)$ in $\mathbb{E}_j$ with coproduct injections $\kappa^i : u^i_!(x_i) \to \vec{u}_!(\vec{x})$. The composite cocone $\langle \kappa^i \circ \underline{u^i} \rangle_i$ is a cocartesian lifting of $\vec{u}$.

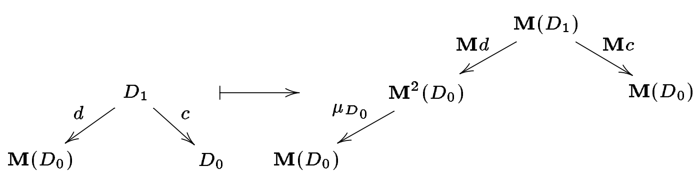
2. Let Rng be the category of commuatitive rings with unit and Rng_m the corresponding multicategory of multilinear maps: $\mathsf{Rng}_m(\langle R_1, \ldots, R_n \rangle, S) = \mathsf{Rng}(R_1 \otimes \ldots \otimes R_n, S)$. Let Mod_m be the multicategory whose objects are pairs (R, M), with R a ring and M an R-module. A morphism in $\mathsf{Mod}_m(\langle (R_1, M_1) \ldots (R_n, M_n) \rangle, (S, N))$ consists of a pair of morphisms (h, a), with $h : R_1 \otimes \ldots \otimes R_n \to S$ in Rng and a $R_1 \otimes \ldots \otimes R_n$-module morphism $a : M_1 \otimes \ldots \otimes M_n \to N$ (the tensor product of the abelian groups M_is has componentwise action by the tensor product of the rings). The evident forgetful functor $U : \mathsf{Mod}_m \to \mathsf{Rng}_m$ is a covariant fibration of multicategories: a cocartesian lifting of $\langle (R_1, M_1) \ldots (R_n, M_n) \rangle$ at $h : R_1 \otimes \ldots \otimes R_n \to S$ is the direct image $(M_1 \otimes \ldots \otimes M_n) \otimes_{R_1 \otimes \ldots \otimes R_n} h^*(S)$, where $h^*(S)$ is S regarded as a $(R_1 \otimes \ldots \otimes R_n)$-module via h.

   A sophisticated variation of this example is explored in [Sny02], where the total multicategory has 'multilinear maps with singularities' and the (implicit) base category consists of the full subcategory of Rng on the tensor powers of a Hopf algebra H. It provides a framework for *vertex algebras*.

2.1 Fibrations and the enveloping adjunction. We recall that the monadic adjunction $F \dashv U : \mathcal{MonCat} \to \mathcal{Multicat}$ (for our second view of multicategories (2) as monads in in the Kleisli bicategory of spans $\mathbf{Spn}_{\mathsf{M}}(\mathbb{B})$) acts as follows: given a monoidal category $\mathbb{C}$ with objects C_0 and arrows C_1, $U\mathbb{C}$ is



where $\mathbf{M}$ is the free-monoid monad in the ambient category (in $\mathcal{S}et$, $\mathbf{M}X = X^*$ the monoid of sequences under concatenation). Given a multicategory $\mathbb{D}$ with objects D_0 and arrows D_1, the free monoidal category $F\mathbb{D}$ is



For more details of how this construction works, see [Her00, §8.3]. A more involved construction (via a lax colimit for a monad in $\mathsf{Bimod}(\mathcal{K})$ [Her01, §2.2]) yields $F \dashv U : \mathbf{M}\text{-}\mathsf{alg} \to \mathsf{Lax\text{-}Bimod}_{\mathbf{M}}(\mathcal{K})\text{-}\mathsf{alg}$, with the same intuitive content: a generalised 2-cell (or 'morphism') of Fx is a 'tuple' of 'morphims' of x (generalised 2-cells of the top object of the bimodule x) whose domain is the 'concatenation' of the domains of its components. The adjunction induces a cartesian 2-monad $\mathsf{T}_{\dashv} : \mathsf{Lax\text{-}Bimod}_{\mathbf{M}}(\mathcal{K})\text{-}\mathsf{alg} \to \mathsf{Lax\text{-}Bimod}_{\mathbf{M}}(\mathcal{K})\text{-}\mathsf{alg}$.

The basic relationship between the notions of fibrations for multicategories and for categories is the following:

2.4. Theorem. *Let $p : \mathbb{T} \to \mathbb{B}$ be a morphism of multicategories. Tfae:*

1. *p is a (covariant) fibration of multicategories.*
2. *$Fp : F\mathbb{T} \to F\mathbb{B}$ is a (covariant) fibration of categories.*
3. *$Fp : F\mathbb{T} \to F\mathbb{B}$ is a (covariant) fibration of monoidal categories.*
4. *$\mathsf{T}_\dashv p : \mathsf{T}_\dashv\mathbb{T} \to \mathsf{T}_\dashv\mathbb{B}$ is a (covariant) fibration in $\mathcal{M}ulticat$ (in the sense of Remark 2.2.(2)).*

A fibration of monoidal categories in (3) above is the one-object case of a 2-fibration in the sense of [Her99]. We remind the reader that a 2-functor $P : \mathscr{E} \to \mathscr{B}$ is a *2-fibration* if it is a fibration at the 1-dimensional level and every 'local hom' functor $P_{X,Y} : \mathscr{E}(X, Y) \to \mathscr{B}(PX, PY)$ is a fibration, whose cartesian (2-)cells are preserved by precomposition with 1-cells (they have a pointwise nature). Hence, a fibration of strict monoidal categories amounts to a fibration of categories whose cartesian morphisms are closed under tensoring.

Notice that both (2) and (4) in the above characterisation make sense for our abstract multicategories as lax algebras in $\mathsf{Bimod}(\mathcal{K})$ and either of them can be adopted as a *definition* of (covariant) fibration in this setting.

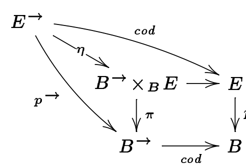
From the above characterisation theorem, since both F and $\mathsf{T}_\dashv$ preserve pullbacks (because $\mathbf{M}$ is a cartesian monad), we deduce a change-of-base result for fibrations of multicategories, *i.e.* they are stable under pullback. Let $\mathcal{F}ib(\mathcal{M}ulticat)$ denote the 2-category whose objects are fibrations of multicategories, morphisms are commuting squares where the top morphism preserves cartesian morphisms of the total multicategories, and the evident 2-cells (*cf.* [Her99]).

2.5. Proposition. *The forgetful 2-functor base : $\mathcal{F}ib(\mathcal{M}ulticat) \to \mathcal{M}ulticat$ taking a fibration to its base multicategory, is a 2-fibration.*

Hence all the basic results of 2-fibrations of [Her99] (factorisation of adjunctions, construction of Kleisli objects, *etc.*) carry through to the setting of multicategories.

2.2 Adjoint characterisation. In [Str73] Street gave a characterisation of fibrations internal to a 2-category admitting comma-objects; the existence of cartesian liftings amounts to the existence of a right adjoint to the unit mapping a morphism (object over B) to its free fibration. We reformulate this characterisation using $\mathsf{Hom}\_$ (cotensors with the $\to$ category) and pullbacks, which we can fruitfully reinstantiate in the setting of lax algebras on bimodules.

2.6. Lemma. *A functor $p : \mathbb{E} \to \mathbb{B}$ in $\mathcal{C}at$ is fibration iff the functor η canonically induced into the pullback*



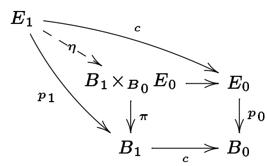
admits a right adjoint in $\mathcal{C}at/(\mathbb{B}^{\to})$.

While in $\mathcal{C}at$ the situation for covariant fibrations is entirely dual (simply replacing *cod* by *dom* and *right* by *left* above), the asymmetry between the domain and codomain of (multi)morphisms means that we have to state the characterisations of covariant and contravariant fibrations of multicategories separately:

2.7. Proposition (Adjoint characterisation of fibrations of multicategories).

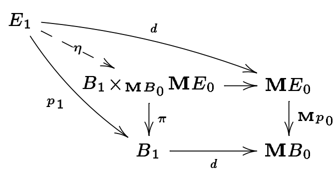
Consider a morphism $p : E \to B$ *in* $\mathsf{Lax}\text{-}\mathsf{Bimod}_{\mathbf{M}}(\mathcal{K})\text{-}\mathsf{alg}$, *with lax algebras on bimodules* $E = \mathbf{M}E_0 \overset{d}{\leftarrow} E_1 \overset{c}{\rightarrow} E_0$ *and* $B = \mathbf{M}B_0 \overset{d}{\leftarrow} B_1 \overset{c}{\rightarrow} B_0$

1. *p is a fibration iff* $p_0 : E_0 \to B_0$ *is a fibration (in* $\mathcal{K}$*) and the canonical morphism* η *into the pullback*



   admits a right adjoint in $\mathcal{K}/B_1$.

2. *p is a covariant fibration iff* $p_0 : E_0 \to B_0$ *is a covariant fibration (in* $\mathcal{K}$*) and the canonical morphism* η *into the pullback*



   admits a left adjoint in $\mathcal{K}/B_1$.

2.8. Corollary. *Given algebras* $x : \mathbf{M}X \to X$ *and* $y : \mathbf{M}Y \to Y$, *and a morphism* $f : x \to y$, *such that the underlying morphism* $f : X \to Y$ *is a covariant fibration in* $\mathcal{K}$, *the induced morphism* $Uf : x_\# \to y_\#$ *between lax algebras is a cofibration*

Proof Applying Proposition 2.7.(2), the corresponding left adjoint is obtained from the given one characterising f as a covariant fibration in $\mathcal{K}$, by pulling this latter back along the algebra structure $x : \mathbf{M}X \to X$.

□

3 Coherence for fibrations of multicategories

Fixing a base multicategory $\mathbb{B}$, let $\mathcal{F}ib/\mathbb{B}$ denote the fibre over $\mathbb{B}$ of the 2-fibration *base* : $\mathcal{F}ib(\mathcal{M}ulticat) \to \mathcal{M}ulticat$ and similarly let *Split*$(\mathcal{F}ib/\mathbb{B})$ the corresponding sub-2-category of split fibrations (*i.e.* those with a choice of cartesian liftings closed under composition and identities) and morphisms between such preserving the splittings. Using the usual coherence theorem for fibrations of categories, the characterisation Theorem 2.4, and the fact that the unit of the monadic adjunction $F \dashv U$ is cartesian, we deduce the following coherence result:

3.1. Theorem. *The inclusion Split*$(\mathcal{F}ib/\mathbb{B}) \hookrightarrow \mathcal{F}ib/\mathbb{B}$ *has a left biadjoint whose unit is a pseudo-natural equivalence (with a section). Thus, every fibration is equivalent to a split one.*

The dual statement for covariant fibrations also holds.

3.2. Remark. An equivalence with a section is a split-epi at the 'object' level and thus both a covariant and (contravariant) fibration in any 2-category (in the representable sense). By Theorem 2.4, the same holds for equivalences with sections between abstract multicategories.

4 Covariant fibrations and representability

When the 2-category $\mathcal{K}$ has a terminal object $\mathbf{1}$, it bears a unique M-algebra structure, which makes it the terminal object in M-alg. Consequently, $U1$ is the terminal object in $\mathsf{Lax\text{-}Bimod_M}(\mathcal{K})$-alg. For any M-algebra $x : \mathsf{M}X \to X$, the unique morphism $! : X \to \mathbf{1}$ is (rather trivially) a (covariant) fibration, since the terminal object is discrete. For multicategories, the situation is interestingly different (we work in the framework of a 2-regular 2-category in the sense of [Her02]):

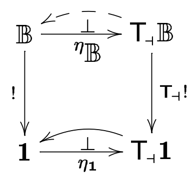
4.1. Theorem. *A multicategory $\mathbb{B}$ (qua lax algebra) is representable iff the unique morphism $! : \mathbb{B} \to \mathbf{1}$ is a covariant fibration of multicategories.*

Proof

($\Leftarrow$) Since the unit $\eta : id \Rightarrow \mathsf{T}_\dashv$ of the adjunction

$$F \dashv U : \mathsf{M\text{-}alg} \to \mathsf{Lax\text{-}Bimod_M}(\mathcal{K})\text{-}\mathsf{alg}$$

is cartesian with respect to (representable) covariant fibrations (this is where the axioms of 2-regularity come into play), the following square is a pullback

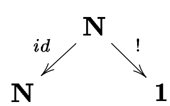


The existence of a left adjoint to the bottom morphism ($\mathbf{1}$ is clearly representable, since it comes from a M-algebra), the lifting of adjoints in a 2-fibration ([Her99, Lemma 4.1]) implies the existence of the 'dashed' left adjoint on top, which shows that $\mathbb{B}$ is representable.

($\Rightarrow$) For an M-algebra X, the unique $! : X \to \mathbf{1}$ is a covariant fibration, and so is $U! : UX \to U\mathbf{1}$ (Corollary 2.8). By coherence [Her01, Thm.7.4], any representable $\mathbb{B}$ is equivalent (via a covariant fibration, see Remark 2.8) to some UX (a strict M-algebra). The composite $\mathbb{B} \to UX \to \mathbf{1}$ is then a covariant fibration, as required. □

4.2. Remark. In the setting of $\mathcal{S}et$-based multicategories, the above theorem has the following concrete interpretation:

- The terminal multicategory has underlying multigraph



  where $\mathbf{N}$ is the set of natural numbers. Thus we have a unique arrow $n \succ \bullet$ for every n. Notice that this object is 'discrete' with respect to 2-cells into it, but it has non-trivial (multi)morphisms.
- Universal morphisms in a multicategory $\mathbb{C}$, $\pi_{\vec{x}} : \vec{x} \to \otimes\, \vec{x}$, are precisely the cocartesian morphisms for $! : \mathbb{C} \to \mathbf{1}$ over $|\vec{x}| \succ \bullet$ (see our heuristic (1) in §1).

Of course, in this simple setting, the above correspondence can be seen by mere inspection of the definitions involved. The general proof however requires vastly different methods. It is worth emphasizing that the ($\Rightarrow$) argument is quintessentially 2-fibrational. In the opposite direction, we have used coherence for adjoint pseudo-algebras. Although this use of coherence is not strictly necessary, the given argument does show several pieces of the theory at work.

4.3. Corollary. *The assignment $\mathbb{M} \mapsto (! : \mathbb{M} \to \mathbf{1})$ yields an equivalence of the 2-categories of representable multicategories and covariant fibrations over the terminal:*

$$\mathcal{R}ep\mathcal{M}ulticat \cong \mathcal{C}o\mathcal{F}ib/\mathbf{1}$$

Notice that the coherence theorem 3.1 and the above corollary do allow us to recover the coherence theorem for (abstract) representable multicategories.

The relationship between representability and covariant fibrations is completed by the following two results:

4.4. Proposition. *Let $p : \mathbb{T} \to \mathbb{B}$ be a morphism of multicategories.*

- *If both $\mathbb{T}$ and $\mathbb{B}$ are representable and p preserves universals, then p is a covariant fibration of multicategories iff p is a covariant fibration in $\mathcal{M}ulticat$.*
- *If p is a covariant fibration of multicategories and $\mathbb{B}$ is representable, then $\mathbb{E}$ is representable and p preserves universals.*

The first item above means that a covariant fibration between representable multicategories is the same thing as a *fibration of categories* (assuming that the induced functor is strong monoidal). The second result has the following logical interpretation: as we have argued in [Her99] and the references therein, the notion of *logical relation* between models of (various kinds of) type-theories can be fruitfully understood in terms of categorical structure in the total category of a fibration (over the base 'models'). The above result shows how to obtain a *logical tensor* in a *multicagory of predicates*, which is one covariantly fibred over a representable multicategory, that is, a base which admits a 'tensor'. For instance, in Example 2.3.(2), since the base multicategory Rng_m is representable, so is the multicategory Mod_m.

5 Operads and algebras

As an application of the theory of covariant fibrations, we show their role in the theory of *operads* and their *algebras*. From their origin in algebraic topology [KM95], these tools have found their way into various approaches to higher-dimensional category theory ([Bat98, BD98, Lei00]).

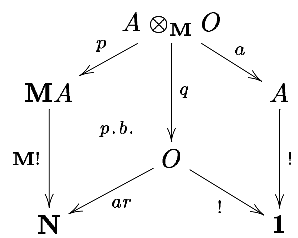
The basic setting of [KM95] is a (symmetric) monoidal category $\mathbb{C}$. In order to treat these notions with our multicategorical formulation, we would assume that $\mathbb{C}$ has finite limits and admits a free-monoid cartesian monad M, so that we consider multicategories as monads in $\mathbf{Spn}_{\mathsf{M}}(\mathbb{C})$. We have the following identification:

one-object multicategory $\equiv$ (non-permutative) operad

Indeed, a one-object multicategory amounts to an $\mathbf{N}$-indexed family $O = \{O_n\}$ (elements of O_n should be thought of as n-ary operations) closed under composition and identities. Thus, an operad is a structure which groups together the *operations* of a (restricted) algebraic theory. On the other hand, a *monad* describes the result of applying such operations to some generators, thereby describing the *free algebras* of the theory. With these identifications in mind, it is easy to see that an operad O gives rise to a monad $(\_) \otimes_{\mathsf{M}} O$: the category $\mathbb{C}$ embeds in $\mathbf{Spn}_{\mathsf{M}}(\mathbb{C})(\mathbf{1},\mathbf{1})$ ($J : \mathbb{C} \to \mathbf{Spn}_{\mathsf{M}}(\mathbb{C})(\mathbf{1},\mathbf{1})$ regards an object X as a span $\mathsf{M}\mathbf{1} \stackrel{\eta}{\leftarrow} \mathbf{1} \leftarrow X \to \mathbf{1}$), while taking the top object of the span gives a functor $D : \mathbf{Spn}_{\mathsf{M}}(\mathbb{C})(\mathbf{1},\mathbf{1}) \to \mathbb{C}$. Given an operad O as an object in $\mathbf{Spn}_{\mathsf{M}}(\mathbb{C})(\mathbf{1},\mathbf{1})$, we set $(\_) \otimes_{\mathsf{M}} O = D \circ (\_) \bullet O \circ J$, which inherits the monoid structure from O, thereby yielding a monad $(\_) \otimes_{\mathbf{M}} O : \mathbb{C} \to \mathbb{C}$. Notice that the composite $(\_) \bullet O$ is the application of the operations to generators, the latter suitably reinterpreted as a family of operations. We can identify algebras for the operad O with those of its associated monad:

$$\boxed{O\text{-algebras} \equiv (\_ \otimes_{\mathbf{M}} O)\text{-}\mathsf{alg}}$$

In concrete terms, an O-algebra amounts to an object A of $\mathbb{C}$ endowed with actions $a(o) : A^n \to A$ for every n-ary operation o, associative and unitary (with respect to the monoid structure on O). By inspection of the resulting diagrams, we notice that such actions can be equivalently phrased in terms of discrete covariant fibrations:



where $\mathbf{N} = \mathbf{M1}$ and the left-hand square is a pullback. The top span is then a multicategory (because the actions are associative and unitary), which we write $(A, a)^+$, and $! : A \to \mathbf{1}$ is a covariant fibration. The pullback square means that the fibres are discrete, so that a (multi)morphism of $(A, a)^+$ is uniquely determined by its source and its image in O. We arrive to the following:

5.1. Proposition.

$$\boxed{O\text{-}\textit{algebras} \equiv \textit{discrete covariant fibrations over the multicategory } O}$$

This identification indicates that we can consider more generally a notion of **algebra for a multicategory** as a discrete covariant fibration over it. From the algebraic-theory perspective above, operads correspond to single-sorted restricted theories (equations must involve the same variables in the same order on both sides [CJ95], in the non-permutative case), while multicategories correspond to the many-sorted version.

We now consider slice categories of algebras. In order to see that such slice categories are themselves categories of algebras, we appeal to the following two 'fibrational facts':

- Given covariant fibrations of multicategories $p : \mathbb{A} \to \mathbb{B}$ and $q : \mathbb{C} \to \mathbb{B}$, a morphism of covariant fibrations $h : p \to q$ is a covariant fibration in $\mathcal{C}o\mathcal{F}ib/\mathbb{B}$ iff it is a covariant fibration of multicategories $h : \mathbb{A} \to \mathbb{C}$. More concisely

  $$\{\mathcal{C}o\mathcal{F}ib - in(\mathcal{C}o\mathcal{F}ib/\mathbb{C})\}/q \equiv \mathcal{C}o\mathcal{F}ib/\mathbb{C}$$

  This property can be deduced from the corresponding one for ordinary fibrations in a 2-category, via the 2-fibrational argument in [Her99, §4.3] and the adjoint characterisation of covariant fibrations in Proposition 2.7.
- In the same situation, if the base covariant fibration q has discrete fibres, any morphism into it is a covariant fibration:

  $$\{\mathcal{C}o\mathcal{F}ib - in(\mathcal{C}o\mathcal{F}ib/\mathbb{C})\}/q \cong (\mathcal{C}o\mathcal{F}ib/\mathbb{C})/q$$

Combining these two facts we obtain the following slicing result:

5.2. Theorem. *For a multicatgory O, any slice of the category of O-algebras is again a category of algebras:*

$$\boxed{O\text{−algebras}/(A,a) \equiv (A,a)^{+}\text{−algebras}}$$

5.3. Remark. For brevity we do not deal here with permutative operads (those whose operations have symmetric-group actions), which we will take us into the more involved setting of lax algebras on bimodules rather than spans (we would work with $\mathsf{Lax\text{-}Bimod_S}(\mathcal{K})\text{-}\mathsf{alg}$, where S is the free-symmetric-monoidal-category monad). This is the set-up of [BD98]: an operad in their sense is a lax algebra. One technical subtlety of this extension is that S is not quite compatible with the calculus of bimodules and the resulting gadget $\mathsf{Bimod_S}(\mathcal{K})$ is only a *lax* bicategory. Nevertheless, the notion of monad applies equally to this setting and the conceptual identifications above regarding algebras carry through. In particular Theorem 5.2 gives an alternative (fibrational) view of the slicing result claimed in *ibid.* See [Che00] for a more detailed account of the slicing process.

We conclude pointing out that the consideration of algebras for an operad O via the *endomorphism operad* $\mathsf{End}(A,A)$ of an object A (O-algebra structure on $A \equiv$ operad morphism $O \to \mathsf{End}(A,A)$) is available in our setting if the ambient category $\mathbb{C}$ is locally cartesian closed, but we forego the details for lack of space.

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