André Lichnerowicz (1915–1998)

Marcel Berger, Jean-Pierre Bourguignon, Yvonne Choquet-Bruhat, Charles-Michel Marle, and André Revuz

André Lichnerowicz was born January 21, 1915, at Bourbon l'Archambault in France. His parents were teachers—one in the humanities, the other in mathematics. Early on he studied differential geometry extensively under the direction of Élie Cartan, and in 1939 he defended a thesis combining differential geometry and general relativity written under the direction of Georges Darmois.

I take the liberty of referring to my dear departed friend by the affectionate nickname that I have used for sixty-five years—Lichné. Lichné published more than 350 articles and books, and he had a great many thesis students of his own.

In 1941 Lichné was named maître de conférences in mechanics at the Faculté des Sciences in Strasbourg. The university continued to bear this name even though it had withdrawn to Clermont-Ferrand, and it was seen as quite evil by the occupying force. In November 1943 this force carried out a raid, in the course of which Lichné was arrested but, thank God, escaped.

At the end of the war the Faculté returned to Strasbourg, and it is in 1947 that he published his first treatise, *Algèbre et Analyse Linéaire*, presenting theories quite poorly taught in this period.

In 1949 Lichné was named to the Faculté des Sciences in Paris, where he established the diploma Mathematical Methods in Physics, and in 1952 he was named to a chair in mathematical physics at the Collège de France. He taught there until 1986, and he remained scientifically active until his death on December 11, 1998.

A particularly brilliant mathematician, Lichné was a singularity among his colleagues for not closing himself off in the ivory tower of mathematics, bewitching as it is, and for showing an active interest in the role of the sciences, and particularly mathematics, in the life of the city: without belonging to any political formation, he was in politics in the noblest sense of the word. He was in particular the organizer of conferences at Caen (1956) and Amiens (1960) whose object was to make people aware of the need for reform of the universities. From December 1966 to June 1973 he was the president of the famous Ministerial Commission on the Teaching of Mathematics, which everybody called the "Lichnerowicz Commission".[1]

The finest tribute that the whole mathematics community could render to Lichné would be to resume his efforts at educational reform, in the context of today, and, carefully measuring the formidable obstacles that it is necessary to confront (sociological inertia, administrative and corporate rigidity, narrowness of view, mathematical illiteracy of most of the population), to bring anew to mathematics teaching all the qualities that he wanted it to acquire.

—*André Revuz*

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[1] *Details of the work of this commission may be found in the* Gazette *articles from which the present article is extracted.*

Jean-Pierre Bourguignon

Although not his student, I met André Lichnerowicz early in my career and later many times, and he inspired me on several occasions. As far as I recall, I almost immediately said "tu" to him, in spite of the age difference between us, but he knew how to make one feel at ease to share opinions on mathematical or physical problems.

As a person André Lichnerowicz would radiate incredible energy, and one had to be impressed by the great variety of his interests. Talking with him was always stimulating. With his wife, who was born in Peru and taught Spanish in a Paris high school for many years, he formed an extremely interesting blend of different sensitivities. The two of them were sharp, remarkably cultivated, and open to many cultures. Nothing would escape their vigilance. An after-conference dinner with the Lichnerowicz couple (she often accompanied him on his scientific trips) was surely an enriching experience.

Historically, modern differential geometry has two main sources: on the one hand generalizing and deepening surface theory (in the spirit of Gauss and Darboux), and on the other hand looking for adequate spaces as mathematical models for reality (after Riemann, Helmholtz, Clifford, Minkowski, and Einstein). Lichnerowicz undoubtedly belonged to the second tradition, as testified by his remarkably prolific scientific production. I do not think that he ever produced a research paper in the spirit of the first source. His thesis in general relativity in the late 1930s is exemplary of his later work: bringing global geometric considerations in the context of general relativity thanks to formulas that allowed him to draw deep significant physical consequences.

The first time one talked to him one could be a bit set back by his approach to doing mathematics. He was always on to some new calculation, and he kept that urge up to the very end of his life. He had an exceptional ability to see geometric facts through formulas. In some sense he was the perfect antidote to the Bourbaki approach to mathematics: he kept being motivated by physical considerations and relied heavily on explicit computations as crystallizations of geometric facts.

His considerable scientific legacy shows his fascinating ability to put himself ahead of fashion: on topics like holonomy groups, transformation groups, harmonic maps, symplectic geometry, and deformations of algebras of observables, he made substantial contributions precisely when these topics were not considered as center-stage areas and were perhaps even marginal.

I owe to his encouragement my interest in the topic on which I have been concentrating for some years, namely, the study of spinors and Dirac operators. On several occasions we talked about developing a complete spinor geometry, a goal that has not yet been achieved. I recall his telling me on more than one occasion (the first one I recall being a meeting in the early 1980s in Metz) that the proper understanding of the Einstein equation will require looking at the equation with spinorial glasses. The proper angle of attack is yet to be found.

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Marcel Berger

Lichnerowicz was interested in very many aspects, or facets, of differential geometry. In a certain number of cases, I give some of his insights below; for these his contribution was relatively specific. In others, by contrast, he pursued his work over long periods; the case of Kähler geometry is one of the most striking. He kept a remarkable spirit through a wide-angle curiosity (had he not envisioned writing a thesis on detective stories?) and breadth of spectrum that partly explains his enormous influence as thesis advisor. The first four sections below seem to me to cover the areas that he explored at length and in depth. In the last section I will mention his research that is more localized in time. I shall ignore here his contributions in symplectic geometry, deformations, and quantization that are included in the segment by C.-M. Marle later in this article.

It seems certain to me that his interest in "pure" differential geometry was motivated by his being at the same time a geometer and a mathematical physicist. "Pure" must not be understood negatively: for him, differentiable manifolds had a metric, curvature, and also a Laplacian; they could be homogeneous spaces, manifolds with a complex analytic structure, etc.

It is necessary to mention that after World War II he was one of the first to introduce into France a closer kind of direction of theses. Instead of the expectation of being told, "Here is a thesis topic. Come back and see me in eight years when you finish," one knew that one could go see him very often. He held also to a quality essential if one does not want to risk possibly causing grave damage with thesis students: verifying that the offered thesis topic carried no risk of being carried out sooner by someone else. To cite my own personal case, he said to me, "You can work on holonomy groups. I have just verified that Chevalley is no longer taking an interest in them."

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France between 1918 and 1955 was considerably backward mathematically, especially as regards teaching. Certainly there were Élie Cartan and Paul Lévy, but they were isolated from the great mass of widely diverse mathematics teaching. For example, there was, say between 1945 and 1955, not a single reasonably modern mathematics course at the Sorbonne, with the exception of the advanced graduate course of Paul Lévy. It seems that the cause nowadays is acknowledged: the absolute hemorrhage, except for Julia and Paul Lévy, from World War I. Books like those of van der Waerden, Courant-Hilbert, and many others were unknown by almost all professionals. Of course, Bourbaki was the great savior from isolation, but Lichnerowicz was another who participated widely in the same way. Through numerous courses and through the book [L47] that summarizes them, he worked tirelessly to spread throughout France the modern writing about tensor spaces and vector calculus, the notion of manifold and that of exterior differential form, Hilbert space, Fourier series and the Fourier transform, integral equations. These were a part of his teaching about mathematical physics in Strasbourg, and the book had considerable influence. The preface by Darmois is instructive in revealing French isolation in the area in question; all the works cited were German and were from before World War II.

Weitzenböck Formulas, Following Bochner

The relations between curvature and topology form a very natural topic in Riemannian geometry. The Bochner article [Boc] will remain an unavoidable cornerstone of transcendental methods linking the local geometry to global properties of the underlying space. Bochner calculated the Laplacian of the norm squared of a differential 1-form ω on a Riemannian manifold:

$$-\frac{1}{2}\Delta(\|\omega\|^2) = \|D\omega\|^2 + \langle D\omega, \omega\rangle + \mathrm{Ricci}(\omega, \omega).$$

Here Δ is the Laplacian on functions and also on 1-forms, D denotes the covariant derivative, and "Ricci" denotes the Ricci curvature. Since, as a consequence of the general Stokes theorem, the integral of a divergence (thus of a Laplacian in particular) on a compact manifold without boundary is always zero, Bochner deduced from his calculation that there exists no nonzero harmonic form (i.e., no form ω with $\Delta\omega = 0$) on any compact Riemannian manifold with positive Ricci curvature. Thanks to Hodge theory, this implies that the first real Betti number is zero. Before Bochner one hardly knew any results except for very weak ones, apart from the case of negative sectional curvature. In the direction of "positive curvature", Bochner opened a breach through which Lichnerowicz subsequently perceived the revealed horizon. Lichnerowicz used this insight in at least four directions, five if one includes the pure and apparently simple extension to the case of forms of degree greater than one, an extension developed by Bochner and Yano, as well as Lichnerowicz, and finished conclusively only in 1971 by D. Meyer. This kind of formula is now called a "Weitzenböck formula". In effect a certain number of the generalized formulas may be found in the 1923 book [W] of Weitzenböck as examples of "absolute" differential calculus. But Weitzenböck made absolutely nothing of them; moreover, he would have had difficulty making use of them, since one did not have at that time any theorem of Hodge-de Rham type.

André Lichnerowicz

Photograph courtesy of the Académie des Sciences de Paris.

In the 1950 article by Lichnerowicz for the International Congress of Mathematicians, one finds the calculation that furnishes the Laplacian of the norm squared of the complete curvature tensor R:

$$-\frac{1}{2}\Delta(\|R\|^2) = \|DR\|^2 + \mathrm{Univ}(R, R, R) + Q(D(\mathrm{Ricci})),$$

where $\mathrm{Univ}(R, R, R)$ is a universal cubic form in R and where Q is a universal quadratic form in the Ricci curvature. I am still surprised today at the few things that people have done with this extraordinary formula. In a 1986 paper of Tricerri and Vanhecke one sees in a few lines that the formula implies that locally symmetric spaces are characterized by the precise algebraic form of their curvature tensor R, a result that generalizes very partial results obtained previously in a very laborious way.

In [L58] appears the idea of applying the Bochner formula, no longer to a harmonic 1-form, but to the 1-form that is the differential df of an eigenfunction of the Laplacian: $\Delta f = \lambda f$. One finds, upon integrating, that

$$0 = \int_M \|\mathrm{Hess}\, f\|^2 - \lambda \int_M \|df\|^2 + \int_M \mathrm{Ricci}(df, df),$$

where Hess stands for Dd, the Hessian operator. From this follows easily the conclusion: the first eigenvalue λ_1 of the Laplacian of a compact Riemannian manifold M^d with Ricci curvature satisfying $\mathrm{Ricci} \geq d - 1$ satisfies $\lambda_1 \geq d$. It seems to me that this is the first relation historically between spectrum and curvature. Many followed. Under the same hypothesis he shows, moreover, that in the Kähler case one can obtain the stronger conclusion $\lambda_1 \geq 2d$. He points this out because he is in fact interested in holomorphic vector fields ξ; these satisfy $\Delta\xi = 2\,\mathrm{Ricci}(\xi)$, and one applies the above result to the eigenfunction f that is the divergence of ξ.

In [L61], motivated by mathematical physics, Lichnerowicz discovered that there exist natural Laplacians for objects other than just exterior differential forms. One has not finished exhausting that line of investigation; for example, one can apply it fruitfully to the case of symmetric differential forms of order two in order to prove that the set of Einstein structures on a compact manifold is finite dimensional; see the book [Bes] by Besse. The text [L61] is important also for other aspects; one finds in it a complete set of formulas for the various curvatures of one-parameter variations of Riemannian or Lorentzian structures and as an application some results on the variations of the equations of general relativity.

Last but not least, still motivated by physical considerations, Lichnerowicz in [L63] applied the Bochner technique to the spinor fields on a spin manifold and found the formula, astounding in its simplicity,

$$\mathrm{Dir}^2 = D^*D + \frac{\mathrm{scal}}{4},$$

where Dir is the Dirac operator, D the covariant derivative, and scal the scalar curvature of the matrix. The index theorem had just been proved by Atiyah and Singer, and one could identify the index of the Dirac operator with the $\hat{A}$-genus of Borel-Hirzebruch. From these things follows a topological restriction for the ultra-weak condition "admitting a metric with positive scalar curvature" (this was the only known topological restriction, even under the much stronger hypothesis of existence of a metric with positive sectional curvature, until a result of Gromov in 1981). Ramifications of this formula continue to be exploited. It has spread into a good part of the present-day literature. In books one can appreciate it in Berline-Getzler-Vergne [BGV] and Lawson-Michelsohn [LaM], for example. Lichnerowicz continued to work on spinor geometry up to the last minute.

The Kähler Kingdom

This kingdom, discovered by Kähler in 1933, remained for a very long time practically unexplored; see one of its rare historical analyses in [Bou]. But it seems to me that what launched the subject was the book [Ho] by Hodge in 1941, for Hodge recognized that the notion of a Kähler manifold was broader than that of a nonsingular algebraic variety. Into this gap after World War II rushed Chern in 1946, Weil in 1949, and many others. Lichnerowicz was one of them. One of his motivations was Élie Cartan's long-standing question, Is every bounded homogeneous domain in $\mathbb{C}^n$ symmetric? This was a question on which he worked furiously (one knows today that there are counterexamples). An essential question was to know what remained, under the hypothesis Kähler only, of the results of Lefschetz on nonsingular algebraic varieties. Similarly, what remained under the hypothesis symplectic only or almost-Kähler? It was Lichnerowicz who established the equivalence of the following properties of a Riemannian manifold: the metric is Kähler, the holonomy group is contained in the unitary group $U(n)$, and there exists an exterior 2-form of maximum rank whose covariant derivative is zero. Although this would appear "elementary" today, it was not so at that time. He established also the equivalence, in the Kähler context, between the vanishing of the Ricci curvature and the inclusion of the holonomy group in the special unitary group $SU(n)$. All this permitted people to begin to see more clearly what remained of Lefschetz's results.

In a 1969 paper of Lichnerowicz, one finds for the first time the generalization to Riemannian manifolds of arbitrary dimension, of the notion of Albanese variety and the Jacobi mapping on this variety, notions considered previously only in the case of Riemann surfaces. An important result on this topic can be found in a 1971 paper of his: If the Ricci curvature (alias the first Chern class) of a compact Kähler manifold is nonnegative, then the Jacobi mapping is a holomorphic fibration.

Holonomy Groups

The holonomy group attached to a covariant derivative is the group formed by parallel transport along all loops issuing from a point. This notion, essential today in mathematical physics as well as in "pure" mathematics in the case of the Calabi conjecture dealing with complex manifolds with vanishing first Chern class and with hyper-Kähler manifolds and quaternion-Kähler manifolds, was created by Élie Cartan in 1925. For simplicity, in what follows we restrict our considerations to loops that are contractible to a point; these form what is called the restricted holonomy group. Holonomy groups remained in limbo for a very long time before being decisively removed from there by A. Borel and Lichnerowicz [BoL] in 1952. These two showed the astounding result that the restricted holonomy group of a Riemannian manifold, even a local one (i.e., an open one), is always a compact Lie group. It is surprising that this group is compact for manifolds that are open and arbitrarily small. The hope of Élie Cartan was to classify Riemannian manifolds according to their holonomy groups. One knows today that holonomy groups are too coarse an invariant for such a classification, except for the very special case of symmetric spaces. For the restricted holonomy groups of nonsymmetric spaces besides the exceptional ones, namely, G_2 and Spin(9), there are only the following cases, except for the special orthogonal group: Kähler, Kähler with Ricci curvature zero, quaternion-Kähler, and hyper-Kähler. Thanks to the solution of the Calabi conjecture in the special case of vanishing Ricci curvature due to Shing-Tung Yau, this explains the importance of the special manifolds mentioned above. We mentioned earlier how Lichnerowicz realized the tie between having an inclusion of the holonomy group in $U(n)$ and having the metric be Kähler, more so in the case of Ricci curvature zero. He did not fail to use holonomy groups in an essential way also for the results of the following section.

The Kingdom of Homogeneous Spaces

Always motivated by Élie Cartan's question about bounded homogeneous domains, Lichnerowicz was constantly haunted by homogeneous spaces. Geometries, in the sense of Klein, are always homogeneous. The whole book [L58] is devoted to them. Starting from 1953, Lichnerowicz classified almost completely in the semisimple case the compact homogeneous Kähler spaces, a classification finished by Borel in 1954; see the book [Bes]. But he did not stop there; a 1990 paper by Lichnerowicz finishes completely the question for the case of Kähler groups.

Insights

• **Harmony of Spheres**. In 1943 Lichnerowicz showed that all "harmonic spaces" of dimension four are symmetric. Harmonic spaces are Riemannian manifolds for which the Laplacian admits an elementary solution depending only on the distance. They may be defined by many other equivalent conditions: for example, the value of a harmonic function at any point is equal to its mean value over balls centered at the point. All the symmetric spaces of rank 1 are obviously symmetric, since their isotropy subgroup is transitive on directions. One knows now that harmonicity, in the presence of compactness, implies symmetry. By contrast, there are counterexamples in the noncompact case.

• **Well before Riemannian Submersions**. In 1949 Lichnerowicz considered fiber bundles from a Riemannian point of view. This was well before their systematic study by B. O'Neill in 1966. Lichnerowicz's idea was to see what one could do with geometric conditions on the fibers using Hodge-de Rham theory. He obtained topological results on the bundle when the fibers are minimal submanifolds of the total space.

• **Harmonic versus Holomorphic**. Several times, typically for the Jacobi "fibration" above, where he used it in a major way, Lichnerowicz was interested in relationships between holomorphic mappings and harmonic mappings. In fact, he was one of the first to realize the importance of harmonic mappings created by Eells and Sampson in 1964. For example, Lichnerowicz wrote a fine study of this relationship in 1970, and in particular he extended known results to the almost-Kähler case. There one again finds his concern for extending as many results as possible to the symplectic context.

Yvonne Choquet-Bruhat

I was still only a high school student when I heard my father[2] speak highly of a young mathematician, André Lichnerowicz, who combined with his mathematical gifts a profound sense of physics and a remarkable teaching ability. The only son of brilliant parents, a literary father who was secretary-general of the *Alliance française* and a mathematician mother from the École Normale Supérieure de Sèvres, André Lichnerowicz was an immensely cultured man, interested throughout his life in the most varied problems, scientific or philosophical, and in their impact on the world in which we live. His mind was brilliant, clear, rapid, and tirelessly active. A great intellectual, Lichnerowicz was also very human. He had a strong desire to communicate his ideas and a sure loyalty to his friends. Lichnerowicz considered himself responsible for everyone who had been his student, and he had many of them. He carried for them unfailing support, particularly when they had difficulties in their professional or private lives. Lichnerowicz knew how to choose for each a thesis topic appropriate to the person's tastes and capacities, a topic that would permit the person, encouraged and helped as much as necessary, almost certainly to obtain the looked-for diploma. This diversity of choice offered by Lichnerowicz to his students came from the variety of his own interests. I shall speak only of Lichnerowicz's works in general relativity; others are better able to report on other areas.

The first and fundamental contribution of Lichnerowicz to general relativity was in 1939 in his thesis defended under the direction of Georges Darmois, namely, to provide a global differential geometric point of view to general relativity: every relativistic model is a differentiable manifold equipped with a metric of hyperbolic signature satisfying the Einstein equations, with or without sources. He made explicit in the appropriate general context the linking conditions given by Darmois in particular coordinates: these are the necessary and sufficient conditions for a metric to be a global classical solution of the Einstein equations with the second member possibly discontinuous. Lichnerowicz's methodology has been used in the construction of numerous models and could be extended without real difficulty to weak solutions, whose concept was developed later.

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[2] *Editor's Note: Georges Bruhat was one of the most influential French physicists in the period before World War II, in particular through a series of advanced textbooks notable for their clarity.*

The global point of view adopted by Lichnerowicz permitted him in 1939 to prove in complete generality a fundamental result obtained in particular cases by Einstein and Pauli. It earned him their admiration. Lichnerowicz established, thanks to his mastery of tensor calculus, two fundamental identities that permitted him to show that there exists no gravitational soliton—that is, no stationary solution nontrivial in the sense of having nonzero curvature—to the Einstein equations for a vacuum on a manifold of type $S \times \mathbb{R}$, where S is spacelike and is asymptotically Euclidean and complete or compact. Lichnerowicz extended this result, with his student Y. Thiry, to the 5-dimensional unitary theory (gauge group $U(1)$) in the case, as he emphasized, in which the circle bundle is trivial. Some thirty years later E. Witten constructed a counterexample in the case where this bundle is not trivial. It has been necessary to wait until this last decade for a proof that solitons exist when the gauge group is nonabelian. The Einstein equations are invariant under diffeomorphisms. As with gauge theories, the integration of these equations divides into a problem of evolution and a problem of constraints, a constraint called Hamiltonian and a momentum constraint—equations that must be satisfied by the initial data. In 1944 Lichnerowicz used the relationship between the scalar curvatures of two conformal metrics to transform the momentum constraint into a linear system independent of the conformal factor when the initial manifold is a maximal submanifold of space-time, and the Hamiltonian constraint into an elliptic semilinear equation for the conformal factor. This equation, called since then the Lichnerowicz equation, still plays an essential role in solving the problem of constraints.

Subsequently, until the 1970s and from time to time later on, Lichnerowicz tackled most of the fundamental problems tied to general relativity. He gave a systematic treatment of them without being disheartened by calculations that were sometimes very complicated. He published clear and detailed articles, including mention of contributions of his students, which have often served as a basis for later works. Lichnerowicz was interested throughout his career in the representation of material sources in relativity. He was the first to obtain, in 1940 in collaboration with R. Marrot, a coherent mathematical formulation of the relativistic kinetic theory. Starting in the 1950s, he found good extensions of various general theorems in the classical mechanics of fluids. In the 1970s, on the occasion of a course at the Collège de France and in the United States, he took up again work on hydrodynamics and relativistic magnetohydrodynamics, including thermodynamic considerations due to Taub and Pichon. His original study of shock waves in magnetohydrodynamics represents a considerable work. As in many of the other works of Lichnerowicz, he succeeded after very complex calculations that others would have been unable to do. This work has clear, physically significant conclusions.

I shall now cite the works of Lichnerowicz on gravitational radiation, spinor fields, quantization of fields on curved space-time—all of which were a prelude to his later works on quantization tied to the theory of deformations.

In 1960 in a memoir [L60] of almost one hundred pages, Lichnerowicz gave a complete theoretical study first of electromagnetic radiation on curved space-time, then of what is called gravitational radiation tied to the curvature tensor, finally to the coupling of these two quantities. This article, like many others of the same author, remains a basic reference used in later developments in the subject.

In his article on gravitational radiation, Lichnerowicz was already studying quantization. In an important article [L64a] published in 1964, he introduced tensor propagators which generalize to curved space-time the propagator of Jordan-Pauli. He used them in the construction of the quantized commutator on curved space-time first of the electromagnetic field and then of the variation of the gravitational field. This work is a great classic and contains numerous intermediate results that have been used many times. He introduced in particular the equations called of higher order. These equations and the Bel tensor, studied by Luis Bel in his thesis, are fundamental for the a priori estimates used in the past few years in research on solutions valid for all time. The fine results of Lichnerowicz on the quantization of bosonic fields on curved space-time have naturally led to the problem of quantization of spinor fields on curved space-time. In two other long articles [L64b] and [L64c] in 1964, Lichnerowicz got completely under way the theory of spinors on a pseudo-Riemannian manifold. He gave the intrinsic definition of the operators in current use in physics, charge conjugation and Dirac adjoint. The formulas that he established have been essential for the success of the theories of supergravity. The important contributions of Lichnerowicz to Riemannian geometry that are tied to the theory of spinors have been noted in the above segment by M. Berger. The last article of Lichnerowicz, which appeared some weeks before his death, treated the Dirac operator on a Kähler manifold.

Lichnerowicz founded in 1957, with the American J. A. Wheeler and the Russian V. Fock, the International Society for General Relativity and Gravitation. At the beginning this was a kind of club having a relatively small number of members who met at a conference every two and then every three years. There reigned among the relativists a warm convivial atmosphere, as was always the case in the company of Lichnerowicz. Since these distant beginnings, the number of interactions through general relativity of physics and mathematics has grown, and the number of relativists has increased considerably. However, the points of view of global differential geometry introduced by Lichnerowicz have been adopted by all, and his name is always cited with admiration.

Charles-Michel Marle

Symplectic geometry, which has close ties with mechanics and, more generally, with the mathematical representation of the physical universe, aroused the interest of André Lichnerowicz, who was both a geometer and a physicist. His work in this area is vast and important; I am going to make an effort to present some aspects (the ones I know the best), without claiming that they are exhaustive.

Poisson Manifolds

A symplectic manifold (W, F) is a differentiable manifold W, of even dimension $2n$, equipped with a differentiable 2-form F, closed in the sense that $dF = 0$ and everywhere having maximum rank $2n$, i.e., satisfying $F^n = F \wedge \cdots \wedge F \neq 0$. The mapping $\mu : TW \to T^*W$ defined by $\mu(X) = -i(X)F$ is an isomorphism of vector bundles that extends to the exterior powers, thereby allowing us to consider the field of contravariant antisymmetric 2-tensors $\Lambda = \mu^{-1}(F)$. To simplify, we are going to say *k-tensor* instead of *field of contravariant antisymmetric tensors of degree k* in what follows.

The condition of maximality imposed on the rank of F turns out to be too restrictive for a number of applications, notably to mechanics. Numerous authors have also considered the notion of *presymplectic manifold*, namely, a differentiable manifold W equipped with a closed differentiable 2-form, but not necessarily of maximum rank. Despite the ingenuity of the researchers who have been interested in these objects, the results have been disappointing and poorly adapted to the applications considered, except perhaps in the very particular case where the rank of F is constant. André Lichnerowicz was, to my knowledge, the first to see clearly that a fruitful generalization of symplectic manifolds should use the contravariant tensor Λ rather than the 2-form F [L77]. He considered a pair (W, Λ), where W is a differentiable manifold and Λ is a 2-tensor on W. The *Poisson bracket* of two functions u and v in $N = C^\infty(W, \mathbb{R})$ is then defined by $\{u, v\} = i(\Lambda)(du \wedge dv)$, and the *Hamiltonian vector field* X_u associated to a function $u \in N$ is the field such that for every $v \in N$, $i(X_u)\, dv = \{u, v\}$. Lichnerowicz showed that the Poisson bracket of functions satisfies the Jacobi identity if and only if the tensor Λ satisfies the condition $[\Lambda, \Lambda] = 0$, the latter bracket being the "Schouten-Nijenhuis bracket", whose precise definition we do not need. (It is discussed in [Kos].) When this condition is satisfied, the space N of differentiable functions on W, with Poisson bracket as composition law, is a Lie algebra, and the mapping $u \mapsto X_u$ is a homomorphism of Lie algebras. One then says that (W, Λ) is a *Poisson manifold*, for which Λ is the *Poisson tensor*. The 2-tensor Λ allows one to define a morphism of vector bundles $\Lambda^\sharp : T^*M \to TM$ by putting $\langle \Lambda^\sharp \alpha, \beta \rangle = i(\Lambda)(\alpha \wedge \beta)$, where α and β are two elements of the same fiber of T^*W. This morphism extends to exterior powers. Of course, a Poisson manifold (W, Λ) of even dimension $2n$ whose Poisson tensor is everywhere of rank $2n$ is a symplectic manifold: the morphism $\Lambda^\sharp$ is then an isomorphism, and the symplectic 2-form is $F = (\Lambda^\sharp)^{-1}(\Lambda)$.

Jacobi Manifolds

A *contact form* on a differentiable manifold W of odd dimension $2n + 1$ is a differentiable 1-form ω such that $\omega \wedge (d\omega)^n$ is a volume form. With Lichnerowicz we will say then that (W, ω) is a *Pfaffian manifold*. Like that of a symplectic manifold, the structure of a Pfaffian manifold can be defined by means of contravariant objects instead of by a covariant object (the contact 1-form ω). But while for a symplectic manifold a single contravariant object (the tensor Λ) is sufficient, two contravariant objects are now necessary (corresponding roughly to the 1-form ω and its exterior derivative $d\omega$), namely, a vector field E called a *Reeb field* (because it was considered for the first time by G. Reeb in 1952) and a 2-tensor Λ. Lichnerowicz proved that these two objects satisfy the identities

$$(*) \qquad [E, \Lambda] = 0, \qquad [\Lambda, \Lambda] = 2E \wedge \Lambda,$$

the bracket figuring in these expressions being the Schouten-Nijenhuis bracket. More generally he considered a differentiable manifold W equipped with a vector field E and a 2-tensor Λ. By means of the following formulas he defined the *Jacobi bracket* $\{u, v\}$ of two differentiable functions u and v in $N = C^\infty(W, \mathbb{R})$, and he associated to every differentiable function $u \in N$ a vector field X_u, called the *Hamiltonian field* associated to u:

$$(**) \qquad \begin{aligned} \{u, v\} &= i(\Lambda)(du \wedge dv) + \langle u\, dv - v\, du, E \rangle, \\ X_u &= \Lambda^\sharp(du) + uE. \end{aligned}$$

Lichnerowicz showed that the Jacobi bracket satisfies the Jacobi identity if and only if E and Λ satisfy the identities $(*)$. When this is the case, (W, Λ, E) is a *Jacobi manifold* [L78]; the space N of differentiable functions on W, equipped with a Jacobi bracket, is a Lie algebra, and the mapping $u \mapsto X_u$ is a homomorphism of Lie algebras. When W is of odd dimension $2n + 1$ and the tensor $E \wedge \Lambda^n$ is nowhere vanishing, the manifold W is in

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fact a Pfaffian manifold whose contact 1-form ω can be expressed in terms of E and Λ. Moreover, a Jacobi manifold whose Reeb field is identically zero is a Poisson manifold. Jacobi manifolds therefore generalize at the same time symplectic manifolds, Pfaffian manifolds, and Poisson manifolds. The introduction of *conformal Jacobi manifolds* permitted Lichnerowicz to include as well contact manifolds (whose structure is defined by the datum of a subbundle of rank 1 of the cotangent bundle generated locally, in a neighborhood of each point, by a contact 1-form) and locally conformal symplectic manifolds.

Poisson and Jacobi Geometries

The importance of Poisson manifolds was recognized rapidly, notably by A. Weinstein (1983), who studied their local properties. Let us make note also of other work that has permitted a new vision of Jacobi manifolds. A *local Lie algebra* on a differentiable manifold W is a vector bundle (V, π, W) with base W whose space of differentiable sections is equipped with a composition law $(s_1, s_2) \mapsto \{s_1, s_2\}$ making it into a Lie algebra, this composition law being local in the sense that the support of $\{s_1, s_2\}$ is contained in the intersection of the supports of s_1 and s_2. This notion, introduced by Shiga (1974), has been studied by A. Kirillov (1976) in the case where the dimension of the fibers of V is 1. It turns out to be equivalent to that of a conformal Jacobi manifold. When $V = W \times \mathbb{R}$ and $\pi : V \to W$ is the first projection, the space of differentiable sections of the bundle $(W \times \mathbb{R}, \pi, W)$ is identified with the space $N = C^\infty(W, \mathbb{R})$ of differentiable functions on W. The datum of a composition law on this space is equivalent with that of a composition law on N. When this law is local and satisfies the Jacobi identity, Kirillov showed that there exists on W a tensor Λ and a vector field E such that, for every pair (u, v) of differentiable functions on W, the bracket $\{u, v\}$ is given by the first formula of $(**)$ above. Since this bracket satisfies the Jacobi identity, Λ and E satisfy the identities $(*)$. In other words, (W, Λ, E) is a Jacobi manifold.

With F. Guédira, Lichnerowicz carried out a deep study of local Lie algebras and their relations to Poisson manifolds [GuL]. They notably showed that the total space V of a Jacobi bundle (V, π, W) whose fibers are of dimension 1 is canonically equipped with a homogeneous Poisson structure, the Poisson bracket of two homogeneous functions on V corresponding to the bracket of the two sections with which they are canonically associated.

Let (W, Λ, E) be a Jacobi manifold. The field of directions generated by the vector field E and by the image of the morphism $\Lambda^\sharp$ is called the *characteristic field*. It is not in general a vector subbundle of TW because its rank is not necessarily constant. However, Kirillov proved that the characteristic field is, in a generalized sense, completely integrable. That field determines a *Stefan foliation* of W, namely, a partition of W into maximal connected immersed submanifolds, called *leaves*, whose tangent space at each point is the value at this point of the characteristic field. The leaves are not necessarily all of the same dimension; those of even dimension are symplectic manifolds, and those of odd dimension are Pfaffian manifolds. When the Jacobi manifold under consideration is in fact a Poisson manifold (W, Λ), the leaves, all of even dimension, are called *symplectic leaves*. This result highlights the fact that the singularities of the pair (Λ, E), that is, the points in the neighborhood of which the rank of the characteristic field is not constant, organize themselves into immersed submanifolds and are therefore much simpler than the singularities of the rank and of the class of a Pfaff form. Similarly for a Poisson manifold (W, Λ), the singularities of the Poisson tensor Λ, namely, the points in a neighborhood of which the rank of the morphism $\Lambda^\sharp$ is not constant, are much nicer than the singularities of presymplectic forms. It is perhaps for this reason that Poisson manifolds are much better adapted for applications to mechanics and physics than are presymplectic manifolds.

Poisson-Lichnerowicz Cohomology

Let (W, Λ) be a Poisson manifold. From his first publication on Poisson manifolds [L77], Lichnerowicz noticed that the operator ∂_Λ that associates to each p-tensor P the $(p+1)$-tensor $\partial_\Lambda P = [\Lambda, P]$ (this bracket being the Schouten-Nijenhuis bracket) has square 0. Thus ∂_Λ allows one to define a cohomology on W by using as p-cochains the contravariant antisymmetric p-tensors. This cohomology, commonly called *Poisson cohomology*—but it will be more judicious to call it *Poisson-Lichnerowicz cohomology*—is in general complicated because it reflects certain topological properties of the manifold W and of the Stefan foliation formed by its symplectic leaves. Lichnerowicz began the study of it; at the present time this study is being very actively pursued by numerous researchers (for a recent account, see for example the books by Vaisman [Va] and by Cannas da Silva and Weinstein [CaW]). Notably, Lichnerowicz showed that the morphism of vector bundles $\Lambda^\sharp : T^*W \to TW$, extended to exterior powers, is such that for any differentiable p-form η on W, $\Lambda^\sharp(d\eta) = \partial_\Lambda(\Lambda^\sharp \eta)$. Consequently, $\Lambda^\sharp$ determines a homomorphism of de Rham cohomology into Poisson-Lichnerowicz cohomology. When the Poisson manifold under consideration is in fact a symplectic manifold, this homomorphism is an isomorphism.

Let us mention again an important property of Poisson manifolds, although its discovery (made independently by several authors, including

B. Fuchssteiner, F. Magri and C. Morosi, A. Weinstein, P. Dazord) is not attributable to André Lichnerowicz: the cotangent bundle T^*W of a Poisson manifold (W, Λ) possesses a *Lie algebroid* structure whose *anchor* is the morphism of vector bundles $\Lambda^\sharp : T^*W \to TW$. This means that there exists, on the space of differentiable sections of this bundle (that is, on the space of differential 1-forms on W), a composition law denoted $(\zeta, \eta) \mapsto [\zeta, \eta]$ that makes a Lie algebra satisfying, for every pair (ζ, η) of differential 1-forms and every differentiable function f on W,

$$\Lambda^\sharp[\zeta, \eta] = [\Lambda^\sharp \zeta, \Lambda^\sharp \eta],$$
$$[\zeta, f\eta] = \Big(\mathcal{L}(\Lambda^\sharp \zeta)f\Big)\eta + f[\zeta, \eta].$$

The bracket of two exact 1-forms du and dv is tied to the Poisson bracket $\{u, v\}$ by the relation $[du, dv] = d\{u, v\}$.

In 1985 J.-L. Koszul [Kos] proved that the bracket of differential 1-forms on a Poisson manifold (W, Λ) extends to a composition law on the graded vector space of differential forms of all degrees, making this space into a graded Lie algebra. The morphism $\Lambda^\sharp$, extended to exterior powers, is a homorphism of graded Lie algebras (the graded vector space of contravariant antisymmetric tensors being equipped with the Schouten-Nijenhuis bracket as composition law). Certain of these properties have since been extended to general Lie algebroids, which appear closely tied to Poisson manifolds. Thus, for example, the total space of the dual bundle of a Lie algebroid possesses a canonical homogeneous Poisson structure. Thus, the importance of Poisson manifolds is confirmed!

Deformations of the Algebra of Functions on a Manifold

In papers in 1976 to 1982 with M. Flato, D. Sternheimer, F. Bayen, and C. Fronsdal, Lichnerowicz applied the theory of deformations of algebraic structures (initiated by M. Gerstenhaber in 1964) to the associative and Lie algebra structures of the space of functions on a symplectic (or contact) manifold. Let us indicate briefly the point of departure of these works by considering, as an example, the space $N = C^\infty(W, \mathbb{R})$ of functions on a Poisson manifold (W, Λ). Let $(u, v) \mapsto u *_\nu v$ be a bilinear mapping of $N \times N$ into the space $E(N, \nu)$ of formal series in a parameter ν with coefficients in N, of the form $u *_\nu v = uv + \sum_{r=1}^{\infty} \nu^r C_r(u, v)$. The $C_r : N \times N \to N$ are bilinear mappings called *cochains*. One says that $(u, v) \mapsto u *_\nu v$ is a *formal deformation* of the associative algebra structure of N (briefly, a *$*_\nu$-product*) if the associative property $(u *_\nu v) *_\nu w = u *_\nu (v *_\nu w)$ is satisfied formally. Two formal deformations, denoted $(u, v) \mapsto u *_\nu v$ and $(u, v) \mapsto u *'_\nu v$, are said to be *equivalent* if there exists a formal endomorphism $T_\nu = \mathrm{id}_N + \sum_{s=1}^{\infty} \nu^s T_s$, where the T_s are linear endomorphisms of N such that one has formally $T_\nu(u *'_\nu v) = (T_\nu u) *_\nu (T_\nu v)$.

Lichnerowicz and his coworkers immediately saw that it is convenient to choose $C_1(u, v) = \{u, v\}$, the Poisson bracket. They showed that when one tries to determine successively the cochains C_r for $r = 2, 3, \ldots$, one encounters, at each order of the development in formal series, an obstruction represented by an element in a certain cohomology space. That cohomology space is the third "Hochschild cohomology" space. For each integer $p \geq 0$, the space of p-cochains of Hochschild cohomology, used for the construction of the p-th cohomology space, is the space of p-multilinear mappings of N^p into N. The vanishing of this class is the necessary and sufficient condition for pushing the development in formal series to the next higher order. Similarly, the study of the equivalence of two deformations of the associative algebra structure of N makes appear, at each order, an obstruction represented by an element in the second Hochschild cohomology space.

In an analogous manner, one can define and study the formal deformations of the Lie algebra structure of N. The role above played by the identity, expressing the associativity, is played by the Jacobi identity. The obstructions are then classes of another cohomology, the *Chevalley cohomology*, whose p-cochains are the alternating p-multilinear mappings of N^p into N. From each formal deformation of the associative algebra structure of N, one can deduce by antisymmetrization a formal deformation of the Lie algebra structure.

Lichnerowicz and his coworkers showed that the formal deformations of the associative algebra N of differentiable functions on a symplectic (or Poisson) manifold offer a *method of quantization* of classical Hamiltonian systems, different from the method based on the geometric quantization of B. Kostant and J.-M. Souriau.

Ever since the works of H. Weyl (1931) and J. Moyal (1949), one has known an example of a nontrivial deformation of the associative algebra of differentiable functions on $\mathbb{R}^{2n}$ equipped with its canonical symplectic structure, called the *Moyal-Weyl bracket*. Numerous researchers have studied the existence of formal deformations of the associative or Lie algebra structure of the space of differentiable functions on a general symplectic (or Poisson) manifold. The first results are due to J. Vey (1975) for the Lie algebra structure and to O. Neroslavsky and A. Vlassov (1981) for the associative structure under a topological hypothesis (the vanishing of the third Betti number). This hypothesis has been dropped first by M. Cahen and S. Gutt (1982) in the case of the cotangent bundle, then by M. de Wilde and P. Lecomte (1983) in the case of an arbitrary symplectic manifold. Simpler proofs of the existence theorem were subsequently given by several authors—notably Karasev and

Photographs courtesy of Charles-Michel Marle.

Aix en Provence, 1990. Left to right: André Lichnerowicz, Claude Itzykson, and Jean-Marie Souriau.

Maslov [KM] in 1993, Omori-Maeda-Yoshioka (1991), and Fedosov (1994). Recently M. Kontsevich obtained, as a consequence of a conjecture which he formulated in 1993 and proved in 1997, a very deep result: on any differentiable manifold, there is an equivalence between the classes of formal deformations of the associative algebra of differentiable functions and the classes of formal deformations of the trivial Poisson structure (whose Poisson tensor is identically zero).

By Way of Conclusion

For lack of space I had to give up trying to present many other aspects of the work of André Lichnerowicz in symplectic geometry that deserve a detailed description: the study of Lie algebras associated to symplectic manifolds, contact manifolds, Poisson manifolds, Jacobi manifolds; the geometry of canonical transformations; homogeneous contact spaces; … .

I had the privilege of being a student of André Lichnerowicz and receiving his help and encouragement. I have the deepest respect for his human qualities as well as for his scientific achievements. When in the 1960s I took his courses at the Collège de France, I admired his exceptional virtuosity in calculation and the perfect arrangement of difficult proofs, which he always explained in a complete manner. With more hindsight I realize now that the most admirable of his mathematical skills was the depth of his vision, which permitted him to abstract key concepts of today's and tomorrow's mathematics.

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