Newton's Shell Theorem

Abstract

One of the principal reasons Isaac Newton was motivated to invent the Calculus was to show that in applying his Law of Universal Gravitation to spherically-symmetric massive bodies (like planets, stars, and the like), one can regard these bodies as "mass points" with all of their mass concentrated at a point. The key ingredient in showing this is to show that for a thin mass shell, the gravitational force at a point outside this shell is the same as if all the mass of this shell is concentrated at its center.

Newton's Law of Universal Gravitation says that if we have two point masses m and M separated by a distance r, then the mutual force exerted on each is given by

$$F = G\frac{mM}{r^2},$$

where the universal constant is G has approximate value[1]

$$G \approx 6.67 \times 10^{-11} \text{N} \cdot \text{m}^2/\text{kg}^2.$$

Sometimes, it's more convenient to measure instead the **gravitational field** $\boldsymbol{E}$ resulting from a point with mass M; measured in units of Newtons per kilogram it measures the force on a point mass (of 1 kg) placed in this field. Therefore, $\boldsymbol{E}$ will be directed radially inward toward the initial point mass and have field strength

$$E = ||\boldsymbol{E}|| = \frac{GM}{r^2},$$

at a distance r (meters) away from the point with mass M.

For an extended massive object with mass M, not concentrated at a point, the determination of the resulting gravitational field at a given point requires that the contributions of each component particle of mass dM be integrated into a final answer. **Newton's Shell Theorem** states essentially two things, and has a very important consequence. First of all, it says that the gravitational field outside a spherical shell having total mass M is the same as if the entire mass M is concentrated at its center (center of mass). Secondly, it says that for the same sphere the gravitational field inside the spherical shell is identically 0. Proving Newton's Shell Theorem is the primary objective of this essay.

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[1] The value of G was first measured in 1798 by Henry Cavendish; this was already 71 years after Newton's death.

As a consequence of Newton's shell method, one can conclude immediately that for a spherical homogeneous solid having mass M, the resulting gravitational field is again the same as if the entire mass were concentrated at a point. A somewhat more esoteric consequence is that if the spherical homogeneous object has radius R, then the gravitational field inside the object as a distance $r < R$ from the center is the same as if total mass within a distance r from the center were concentrated at the object's center. (The mass outside the radius r can be ignored.)

In order to prove the first part of Newton's Shell Theorem we consider a spherical shell of total mass M and radius R; we shall compute the magnitude of the gravitational field at a point whose distance is r from the center of the spherical shell. We decompose the shell into thin circular rings, each at a (variable) distance s from the point at which E is to be computed. Since the mass density of the shell is $\sigma = \dfrac{M}{4\pi R^2}$ we see that the total mass of the ring (see Figure 1) is
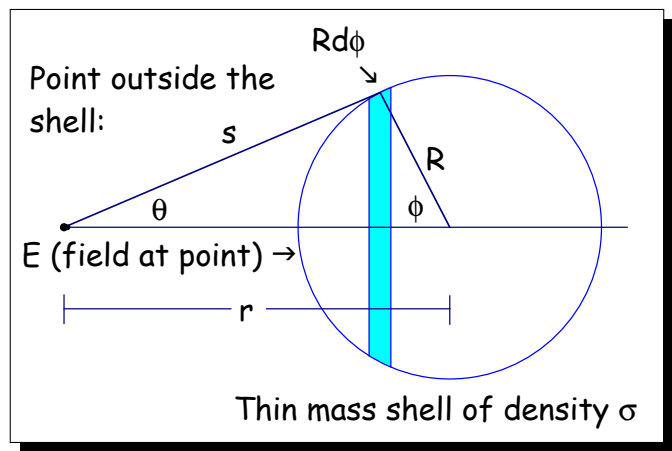

Figure 1: Point outside the shell

$$\begin{aligned} \text{total mass of ring} \ &= \ \sigma \times \text{area of ring} \\ &= \ \sigma \times 2\pi R \sin\phi \times R d\phi \\ &= \ \frac{1}{2} M \sin\phi \, d\phi \end{aligned}$$

Next, note that all of the mass is at a distance s from the point in question; however, since (by symmetry) the field direction is toward the center of the spherical shell, the field strength contribution from this thin ring must be

$$dE = \frac{GM \cos\theta \sin\phi \, d\phi}{2s^2} = -\frac{GM \cos\theta \, d(\cos\phi)}{2s^2}. \tag{1}$$

Using the Law of Cosines, we have

$$R^2 = s^2 + r^2 - 2rs\cos\theta, \text{ and } s^2 = R^2 + r^2 - 2Rr\cos\phi.$$

Therefore,

$$\cos\theta = \frac{s^2 + r^2 - R^2}{2rs}, \text{ and } \cos\phi = \frac{R^2 + r^2 - s^2}{2Rr},$$

and so

$$-d(\cos\phi) = \frac{s}{Rr}\,ds.$$

Plugging into Equation (1) yields the field contribution from the thin ring:

$$dE = \frac{GM(s^2 + r^2 - R^2)}{4Rr^2 s^2}\,ds. \tag{2}$$

From Equation (2) we conclude that the total gravitational field induced by the spherical shell is the integral of the contributions of all of the rings:

$$\begin{aligned}
E &= \int_{s=r-R}^{s=r+R} dE = \frac{GM}{4Rr^2}\int_{s=r-R}^{s=r+R} \frac{s^2 + r^2 - R^2}{s^2}\,ds \\
&= \frac{GM}{4Rr^2}\left(s + \frac{R^2 - r^2}{s}\right)\Big|_{r-R}^{r+R} = \frac{GM}{4Rr^2}\times 4R = \frac{GM}{r^2},
\end{aligned}$$

proving the first part of Newton's Shell Theorem.

To prove the second part, namely that the gravitational field inside the spherical mass shell is 0, note that the element of field strength dE contributed by a typical thin ring does not change; see Figure 2. The only change is that the limits of integration for s are $s = R - s$ and $s = R + s$. Therefore,

$$\begin{aligned}
E &= \int_{s=R-r}^{s=R+r} dE = \frac{GM}{4Rr^2}\int_{s=R-r}^{s=R+r} \frac{s^2 + r^2 - R^2}{s^2}\,ds \\
&= \frac{GM}{4Rr^2}\left(s + \frac{R^2 - r^2}{s}\right)\Big|_{R-r}^{R+r} = 0,
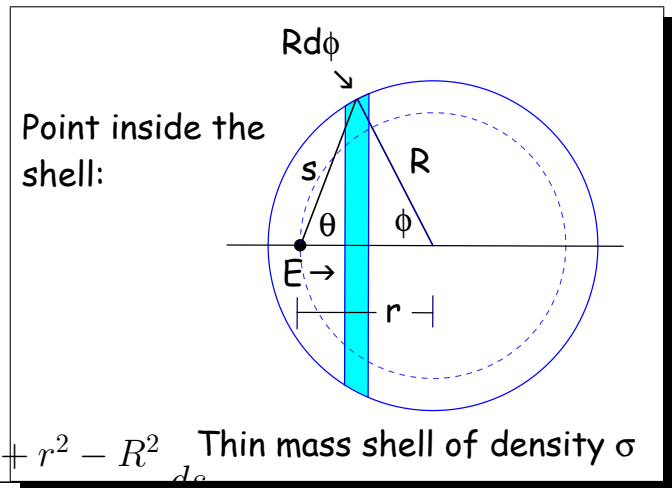\end{aligned}$$

Figure 2: Point inside the shell

exactly as predicted.

Finally, we shall determine the gravitational field induced by a solid homogeneous spherical mass (total mass $= M$), both at points inside and outside the masses. If $r \geq R$, i.e., the point at which we are to determine the gravitational field is outside (or on the surface of) the spherical mass. Denote the mass density by

$$\mu = \frac{3M}{4\pi R^3},$$

and, as above, let r be the distance of the point at which the field is to be computed to the center of the spherical mass. Next divide the sphere into concentric thin mass shells, each of thickness $d\rho$ and radius ρ, making the mass of each such shell

$$dM = 4\pi\rho^2\mu d\rho = \frac{3M\rho^2}{R^3}\,d\rho.$$

From the first part of Newton's Shell Theorem, we have that the field strength contribution from this shell is

$$dE = \frac{3GM\rho^2}{r^2 R^3}\,d\rho;$$

the total field strength is obtained as an integral:

$$E = \int_0^R dE = \int_0^R \frac{3GM\rho^2}{r^2 R^3}\,d\rho = \frac{GM\rho^3}{r^2 R^3}\bigg|_0^R = \frac{GM}{r^2},$$

in perfect agreement with our original contention.

Finally, if the point at which we are to compute the field strength is **inside** the homogeneous spherical mass ($r < R$), then by the second part of Newton's Shell Theorem, we see that the field contribution by the concentric mass shell of radius ρ is given by

$$\begin{cases} dE = \dfrac{3GM\rho^2}{r^2 R^3}\,d\rho & \text{if} \quad 0 \leq \rho \leq r, \\ 0 & r \leq \rho \leq R. \end{cases}$$

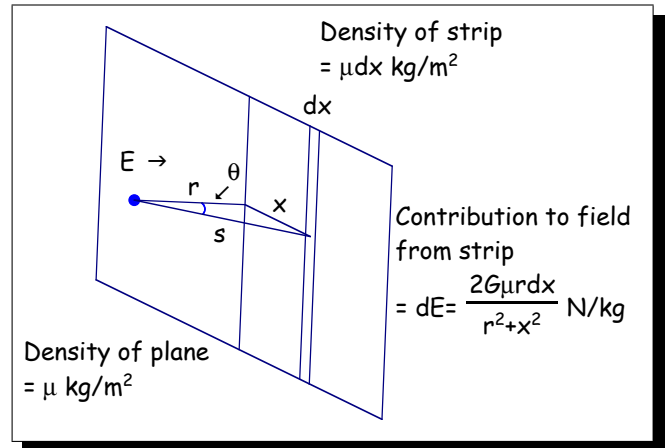Therefore, the total field contribution is the integral

$$E = \int_0^r dE = \int_0^r \frac{3GM\rho^2}{r^2 R^3}\,d\rho = \frac{GMr}{r^3 R^3} = \frac{GMr^3}{r^2 R^3} = \frac{G}{r^2}\times\text{total mass of sphere of radius } r,$$

and we're done!

Try these:

1. Let P be a point and let ℓ be an infinite line with mass density μ kg/m (and so ℓ has infinite mass). Assuming that P has distance r from this line, compute the gravitational field strength at the point P.[2]

2. Let P be a point and let Π be an infinite plane with mass density μ kg/m^2. Compute the gravitational field strength at the point P. (Hint: note first that by symmetry, the field vector will point from P toward to point on the plane Π closest to P. Next, by Problem #1 above, the contribution to the field in this direction is

$$\frac{2G\mu \cos \theta \, dx}{s^2}.$$

Now do an integration with respect to x.)[3]

Density of strip
= μdx kg/m^2

dx

E →

Contribution to field from strip

= dE= $\dfrac{2G\mu r dx}{r^2+x^2}$ N/kg
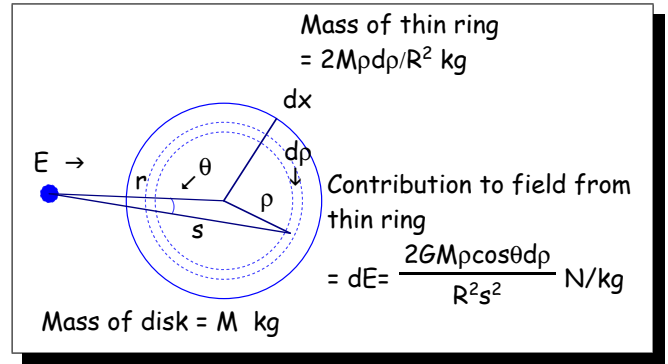
Density of plane
= μ kg/m^2

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[2]Show that

$$E = G\mu r \int_{-\infty}^{\infty} \frac{dx}{[r^2 + x^2]^{3/2}} = 2G\mu r \int_{0}^{\infty} \frac{dx}{[r^2 + x^2]^{3/2}} = \frac{2G\mu}{r} \;\; \text{N/kg}.$$

In evaluating the integral, use the trig substitution $x = r \cos \theta$.

[3]You should get $E = 2G\pi\mu$ N/kg.

3. Here's another approach to Problem #2, above. This time, start with a a circular disk of radius R and mass M at a distance r from the given point P.

(a) Show that the field strength at the given point is

Mass of thin ring
= 2Mρdρ/R² kg

dx

E →

r

θ

ρ

s

Contribution to field from thin ring

= dE= $\dfrac{2GM\rho\cos\theta d\rho}{R^2 s^2}$ N/kg

Mass of disk = M kg

$$E = \frac{2GM}{R^2}\left[1 - \frac{r}{\sqrt{r^2 + R^2}}\right].$$

(b) Show how letting $R \to \infty$ recovers the result of Problem #2.