Addenda to "Canonical models for fragments of the Axiom of Choice"

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Abstract

We continue [3], producing a model with a Hamel basis but no injection from ω_1 into $\mathbb{R}$, and showing that all sets of reals are Lebesgue measurable in the generic model with an improved MAD family.

1 Introduction

In [3], we introduced technology for obtaining a certain type of consistency result in Choice-less set theory, showing that various consequences of the Axiom of Choice are independent of each other. We briefly review the terminology introduced there.

Definition 1.1. A Σ^2_1 sentence Φ is *tame* if it is of the form

$$\exists A \subseteq \omega^\omega \ ((\forall \vec{x} \in (\omega^\omega)^{<\omega} \ \exists \vec{y} \in A^{<\omega} \ \phi(\vec{x}, \vec{y})) \wedge (\forall \vec{x} \in A^{<\omega} \ \psi(\vec{x}))),$$

where ϕ, ψ are formulas which contain only numerical quantifiers and do not refer to A, but may refer to a fixed analytic subset of ω^ω as a predicate. The formula ψ is called the *resolvent* of the sentence Φ. A *resolvent* is a formula which is the resolvent of some tame sentence. A *witness* to a tame sentence of the above form is a set $A \subseteq \omega^\omega$ for which

$$(\forall \vec{x} \in (\omega^\omega)^{<\omega} \ \exists \vec{y} \in A^{<\omega} \ \phi(\vec{x}, \vec{y})) \wedge (\forall \vec{x} \in A^{<\omega} \ \psi(\vec{x}))$$

holds.

Example 1.2. The tame consequences of the Axiom of Choice considered in [3] included the following:

1. there is an infinite maximal almost disjoint family of subsets of ω. The resolvent formula is "$x_0 \cap x_1$ is finite";

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2. there is a Hamel basis for the space of real numbers;

3. there is an ω_1 sequence of distinct reals;

In this paper we add the following example : there is a non-Lebesgue measurable set. Here the witnessing set A can be taken to code a set $A' \subseteq 2 \times \omega^\omega$, with the resolvent saying that no set of the form $2 \times \{x\}$ is contained in A' while the other subformula of the tame expression asserts that each set $2 \times \{x\}$ ($x \in \omega^\omega$) intersects A', and that each attempt to witness Lebesgue measurability for the set $\{x \in \omega^\omega : (1, x) \in A'\}$ fails. Although we have no use for it, one could similarly express the existence of a set of real without the Baire property as a tame Σ^2_1 sentence with the same resolvent.

A tame Σ^2_1 sentence with resolvent ψ is associated with a natural partial order P_ψ of countable approximations. Given a resolvent ψ, a *ψ-set* is a set $a \subseteq \omega^\omega$ such that $\forall \vec{x} \in a^{<\omega}\psi(\vec{x})$ holds. We let P_ψ be the partial order of countable ψ-sets, ordered by reverse inclusion. Then P_ψ is σ-closed and adds a ψ-set $A \subseteq \omega^\omega$ as a union of the generic filter. For many naturally arising tame sentences Φ it is the case that P_ψ forces the generic set A to be a witness for Φ. We say that $A \subseteq \omega^\omega$ is a *generic witness* for Φ if it is obtained from a filter on P_ψ which is generic over $L(\mathbb{R})$.

In [3], we proved a variety of consistency results regarding non-implications between tame consequences of the Axiom of Choice by considering models of the form $L(\mathbb{R})[A]$, where A was a generic witness for a tame Σ^2_1 sentence. In this paper we consider one additional model (with a generic Hamel basis) and prove some additional facts about a model from [3] (with a generic improved MAD family). In the latter case, we use the fact that the existence of sets of reals without the standard regularity properties can be expressed as a tame Σ^2_1 sentence.

2 Independence

We briefly review the theorem from which all of our independence results derive.

Definition 2.1. Let Φ_0, Φ_1 be tame Σ^2_1 sentences with respective resolvents ψ_0, ψ_1. Let A_0 and A_1 be subsets of ω^ω. We say that A_1 is *(Φ_0, Φ_1)-independent* of A_0 if there exists an infinite cardinal κ such that for every poset Q collapsing κ to $\aleph_0$, and for all Q-names τ_0, τ_1 for witnesses to Φ_0 and Φ_1 respectively extending A_0 and A_1 (that is, agreeing on $(\omega^\omega)^V$ with A_0 and A_1 respectively) there exist $n \in \omega$ and (in some generic extension) V-filters $G_i \subseteq Q$ ($i \in n$) such that

$$\forall \vec{x} \in \bigcup_i \tau_0/G_i \ \psi_0(\vec{x})$$

holds and

$$\forall \vec{x} \in \bigcup_{i \in n} \tau_1/G_i \ \psi_1(\vec{x})$$

fails.

We say that witnesses for Φ_1 are Φ_0-independent of A_0 if every witness A_1 for Φ_1 is (Φ_0, Φ_1)-independent of A_0. Similarly, we say that witnesses for Φ_1 are *independent of* witnesses for Φ_0 if every witness A_1 for Φ_1 is (Φ_0, Φ_1)-independent of every witness A_0 for Φ_0.

We write LC for the hypothesis that there exist proper class many Woodin cardinals.

Theorem 2.2. (ZFC + LC) *Suppose that Φ_0, Φ_1 are tame Σ^2_1 sentences with respective resolvents ψ_0, ψ_1. Let $A_0 \subseteq \omega^\omega$ be a P_{ψ_0}-generic witness to Φ_0. If, in $V[A_0]$, witnesses for Φ_1 are Φ_0-independent of A_0, then $L(\mathbb{R})[A_0] \models \neg\Phi_1$.*

3 Hamel bases

In this section we consider the model produced by adding a generic Hamel basis. In this case a ψ-set is a set of irrational numbers which is linearly independent over $\mathbb{Q}$. The following answers a question raised by Schindler, Wu and Yu, and later answered by them in [5].

Theorem 3.1. (ZFC+LC) *Let A be a generic Hamel basis. In the model $L(\mathbb{R})[A]$, there is no injection from ω_1 into $\mathcal{P}(\omega)$, and no infinite MAD family.*

Theorem 3.1 is a straightforward application of Theorem 2.2, in the special (easy) case of mutual genericity. One has to check that injections from ω_1 into $\mathcal{P}(\omega)$ and infinite MAD families are independent of Hamel bases. This follows from the following lemmas, the first two of which are Claim 3.4 and 3.5 of [3].

Lemma 3.2. *If $A \subseteq \mathcal{P}(\omega)$ is an infinite MAD family, Q is any poset collapsing $2^{\mathfrak{c}}$, τ is a Q-name for a MAD family extending A, and $G_i \subseteq Q$ for $i \in 2$ are mutually generic filters over V, the set $\tau/G_0 \cup \tau/G_1$ is not an AD family.*

Lemma 3.3. *If Q_0, Q_1 are posets collapsing $2^{\mathfrak{c}}$ and τ_0, τ_1 are Q_0, Q_1-names for injections from ω_1 to 2^ω, then there are conditions $q_0 \in Q_0$ and $q_1 \in Q_1$ such that for any pair $G_0 \subseteq Q_0$, $G_1 \subseteq Q_1$ of filters separately generic over V and containing the conditions q_0, q_1 respectively, the set $\tau/G_0 \cup \tau/G_1$ is not a function.*

Lemma 3.4. *Suppose that A is a Hamel basis, P is a partial order, (G, H) is V-generic for $P \times P$, B is a Hamel basis in $V[G]$ extending A and C is a Hamel basis in $V[H]$ extending A. Then $B \cup C$ is linearly independent.*

Proof. If not, then (implicitly using the fact that $\mathbb{R} \cap V[G] \cap V[H] = \mathbb{R} \cap V$) there exist $a_1, ..., a_n$ in $A, b_1, ..., b_m$ in $B - A$, $c_1, ..., c_p$ in $C - A$ and rationals $r_1, ..., r_n, s_1, ..., s_m, t_1, ..., t_p$ such that

$$a_1 \cdot r_1 + \cdots + a_n \cdot r_n + b_1 \cdot s_1 + \cdots + b_m \cdot s_m + c_1 \cdot t_1 + \cdots + c_p \cdot t_p = 0.$$

This means that $c_1 \cdot t_1 + \cdots + c_p \cdot t_p$ is in V, which contradicts the assumption that A is a Hamel basis and C is linearly independent. $\square$

4 Improved MAD families again

In [3] we considered the following type of MAD family.

Definition 4.1. An *improved AD family* is a pair $\langle A, B\rangle$ such that

1. A is an infinite AD family in $\mathcal{P}(\omega)$;
2. B is a set consisting of pairs $\langle s, a\rangle$ such that s is a partition of ω into finite sets and $a \subseteq A$ is a countable set;
3. for every pair $\langle s, a\rangle \in B$ and every finite set $b \subseteq A \setminus a$, there are infinitely many sets $c \in s$ such that $\bigcup b \cap c = 0$.

An improved AD family $\langle A, B\rangle$ is *maximal* if A is a MAD family and for every partition s there is a with $\langle s, a\rangle \in B$.

Improved MAD families are naturally added by a poset of countable improved AD families ordered by coordinatewise inclusion. The following is a combination of Corollaries 5.6 and 5.8 of [3], where E_0 is the relation of mod-finite equivalence on $\mathcal{P}(\omega)$.

Theorem 4.2. (ZFC+LC) *Let A be a generic improved maximal almost disjoint family. In the model $L(\mathbb{R})[A]$,*

1. *there are no ω_1 sequences of reals;*
2. *there are no nonatomic measures on ω;*
3. *the quotient space of E_0 cannot be linearly ordered;*
4. *there are no total selectors for E_0.*

The following theorem is a weak variant of the main theorem from [2], whose large cardinal hypothesis is a single strongly inaccessible cardinal (as in their paper, our result extends to Q-measurability for bounded forcings Q). The model produced in that paper seems to be very similar to ours : roughly, their model is to $L(\mathbb{R})[A]$ below as a Solovay model ([6]) is to the inner model $L(\mathbb{R})$ in the presence of a proper class of Woodin cardinals.

Theorem 4.3. (ZFC+LC) *Let A be a generic improved maximal almost disjoint family. In the model $L(\mathbb{R})[A]$, every set of reals is Lebesgue measurable.*

Again, the proof of Theorem 4.3 is an application of Theorem 2.2. One has to check that nonmeasurable sets of reals are independent of improved MAD families. This follows from the following facts, the first of which is Theorem 3.5 of [3]. Instead of taking mutual generics as above, we pass to a random forcing extension before building a suitable pair of generic filters.

An extension V' of V is *bounding* if each element of $\omega^\omega \cap V'$ is dominated by an element of $\omega^\omega \cap V$. A forcing extension via random forcing bounding (see [1]).

Theorem 4.4 ([3]). *Suppose that*

- $\langle A, B\rangle$ *is an improved MAD family,*
- $n \in \omega$,
- $V[G_i]$ $(i \in n)$ *are bounded forcing extensions of* V *contained in some common extension* $V[G]$,
- $P_i \in V[G_i]$ $(i \in n)$ *are posets*
- *for each* $i \in n$, $\langle \dot{A}_i, \dot{B}_i\rangle \in V[G_i]$ *is a* P_i*-name for an improved MAD family extending* $\langle A, B\rangle$.

Then, in some forcing extension, there are filters $H_i \subseteq P_i$ $(i \in n)$, *each generic over the respective* $V[G_i]$, *such that* $\langle \bigcup_{i\in n} \dot{A}_i/H_i, \bigcup_{i\in n} \dot{B}_i/H_i\rangle$ *is an improved AD family.*

Remark 4.5. The proof of Theorem 4.4 in [3] constructs filters H_i $(i \in n)$ that for each pair $i < i' \in n$, $\dot{A}_i/H_i \cap \dot{A}_{i'}/H_{i'} = A$.

The following observation completes the proof of Theorem 4.3, using Theorem 2.2. We let $\mathbb{R}$ denote random forcing, and let $\dot{r}$ denote the canonical name for the generic real added by $\mathbb{R}$, that is, the unique real number in the forcing extension which is in every (reinterpreted) ground model Borel set corresponding to a condition in the generic filter. More generally, in the statement of Proposition 4.6, for any iteration of the form $\mathbb{R} * \dot{Q}$, we let $\dot{r}$ the canonical name for random real added by the first step of the iteration. A real number r is random over V (i.e., is $\dot{r}_G$ for some V-generic filter $G \subseteq \mathbb{R}$) if and only if r is not in any reinterpreted Borel null set from V (see [1]).

Partial orders P and Q are said to be *forcing-equivalent* if every forcing extension by either of P and Q is also a forcing extension by the other. When P and Q are forcing equivalent, a P-name τ for a generic filter for Q giving rise to the same extension induces a translation of each P-name σ to a Q-name σ' such that, if $G \subseteq P$ is V-generic, then σ_G is the realization of σ' by τ_G. In the statement of Proposition 4.6 we suppress the mention of τ and write σ^0 and σ^1 for the induced versions of σ'.

Proposition 4.6. *Suppose that* P *is a forcing which makes* $(2^{\aleph_0})^V$ *countable, and that* σ *is a* P*-name for a non-Lebesgue measurable set of reals. Then* P *is forcing-equivalent to two iterations* $\mathbb{R} * \dot{Q}_0$ *and* $\mathbb{R} * \dot{Q}_1$ *such that for some* $p \in \mathbb{R}$ *and some conditions* $(p, \dot{q}_0)$ *in* $\mathbb{R} * \dot{Q}_0$ *and* $(p, \dot{q}_1)$ *in* $\mathbb{R} * \dot{Q}_1$, $(p, \dot{q}_0) \Vdash \dot{r} \in \sigma^0$ *and* $(p, \dot{q}_1) \Vdash \dot{r} \notin \sigma^1$.

Proof. Let $G \subseteq P$ be a V-generic filter. Since forcing with P makes $(2^{\aleph_0})^V$ countable, in $V[G]$ the set of reals which are random over V is conull. Call this set X. Since $X \neq \emptyset$, P is forcing-equivalent to an iteration of the form $\mathbb{R} * \dot{Q}$.

Suppose first that there exist

- an iteration $\mathbb{R} * \dot{Q}$ forcing-equivalent to P,

- a condition $p \in \mathbb{R}$ and
- conditions $(p, \dot{q}_0), (q, \dot{q}_1) \in \mathbb{R} * \dot{Q}$

such that, letting σ' be a version of σ corresponding to $\mathbb{R} * \dot{Q}$, $(p, \dot{q}_0) \Vdash \dot{r} \in \sigma'$ and $(p, \dot{q}_1) \Vdash \dot{r} \notin \sigma'$. Then of course we are done.

If this is not the case, then for every iteration of the form $\mathbb{R} * \dot{Q}$ which is forcing-equivalent to P, there is a Borel set $b \in V$ such that, letting p be the condition in $\mathbb{R}$ corresponding to b, and p' be the condition corresponding to the complement of b, and again letting σ' be a corresponding version of σ, $(p, 1_{\dot{Q}}) \Vdash \dot{r} \in \sigma'$ and $(p', 1_{\dot{Q}}) \Vdash \dot{r} \notin \sigma'$ (that is, the statement $\dot{r} \in \sigma'$ is decided by the generic for $\mathbb{R}$; for the sake of notational convenience we pretend that the "condition" corresponding to a null set forces every statement). For each $r \in X$, then, we may choose

- an iteration $\mathbb{R} * \dot{Q}_r$ which is forcing equivalent to P,
- a V-generic filter (H_r, K_r) for $\mathbb{R} * \dot{Q}_r$ such that $\dot{r}_{H_r} = r$ and $V[H_r, K_r] = V[G]$ and
- a Borel set $b_r \in V$ such that the $\mathbb{R}$-condition corresponding to b_r forces in $\mathbb{R} * \dot{Q}_r$ that $\dot{r}$ is in a fixed set σ^r corresponding to σ, and the the $\mathbb{R}$-condition corresponding to the complement of b_r forces in $\mathbb{R} * \dot{Q}_r$ that $\dot{r}$ is not in σ^r.

For each $r \in X$, the conditions just listed imply that $r \in \sigma_G$ if and only if $r \in b_r$ (as reinterpreted in $V[G]$). If there exist $r, r' \in X$ such that $b_r \triangle b_{r'}$ is nonnull, then we can finish by taking p to be the condition in $\mathbb{R}$ corresponding to any nonnull member of $\{b_r \setminus b_{r'}, b_{r'} \setminus b_r\}$, and using the iterations $\mathbb{R} * \dot{Q}_r$ and $\mathbb{R} * \dot{Q}_{r'}$. If there do not exist such r, r', then for each $r \in X$ the symmetric difference of X (and therefore the symmetric difference of σ_G) with the reinterpretation of b_r is Lebesgue null, giving a contradiction. □

We give two proofs that $\mathbb{R}$ can't be injected into the generic improved MAD family A in the model $L(\mathbb{R})[A]$ above. This answers a question of Ali Enyat asked on Math Overflow.[1] The first adapts the machinery from [3]. To do this, we have to generalize Definition 1.1, defining a *weakly tame* Σ^2_1 formula to be a unary formula in a variable v with the syntactic form of a tame Σ^2_1 sentence in which v is allowed to appear only in the non-resolvent clause ϕ from Definition 1.1. We generalize Definition 2.1 as follows.

Definition 4.7. Let Φ_0 be a tame Σ^2_1 sentence with resolvent ψ_0, and let Φ_1 be weakly tame Σ^2_1 formula with resolvent ψ_1. Let A_0 and A_1 be subsets of ω^ω. We say that A_1 is (Φ_0, Φ_1)-*independent* of A_0 if there exists an infinite cardinal κ such that for every poset Q collapsing κ to $\aleph_0$, and for all Q-names τ_0, τ_1 such that

- τ_0 is a Q-name for a witness to Φ_0 extending A_0 and

[1] https://mathoverflow.net/questions/72047/lebesgue-measurability-and-weak-ch

- τ_1 is a Q-name for a witness to $\Phi_1(\tau_0)$ extending A_1

for witnesses to Φ_0 and Φ_1 respectively there exist $n \in \omega$ and (in some generic extension) V-filters $G_i \subseteq Q$ ($i \in n$) such that

$$\forall \vec{x} \in \bigcup_i \tau_0/G_i \ \psi_0(\vec{x})$$

holds and

$$\forall \vec{x} \in \bigcup_{i \in n} \tau_1/G_i \ \psi_1(\vec{x})$$

fails.

We say that witnesses for Φ_1 are Φ_0-independent of A_0 if every witness A_1 for Φ_1 is (Φ_0, Φ_1)-independent of A_0. Similarly, we say that witnesses for Φ_1 are *independent of* witnesses for Φ_0 if every witness A_1 for Φ_1 is (Φ_0, Φ_1)-independent of every witness A_0 for Φ_0.

Since the variable in a weakly tame Σ^2_1 formula does not appear in the resolvent, the corresponding version of Theorem 2.2 has the same proof.

Theorem 4.8. (ZFC + LC) *Suppose that Φ_0 is a tame Σ^2_1 sentence with resolvent ψ_0, and Φ_1 is a weakly tame Σ^2_1 formula. Let $A_0 \subseteq \omega^\omega$ be a P_{ψ_0}-generic witness to Φ_0. If, in $V[A_0]$, witnesses for Φ_1 are Φ_0-independent of A_0, then $L(\mathbb{R})[A_0] \models \neg\Phi_1(A_0)$.*

The existence of an injection from $\mathbb{R}$ into a set of reals B can naturally be expressed as a weakly tame Σ^2_1 assertion about B, where the resolvent says that the witnessing set A describes a partial function (it might be more natural to say that B contains the range of the function, but we avoid doing that so that we can reuse the proof of Theorem 2.2). Theorems 4.4 and 4.8, Proposition 4.6 and Remark 4.5 give the following theorem. The corresponding version of the theorem appears in [2], again from the assumption of a single strongly inaccessible cardinal.

Theorem 4.9. (ZFC+LC) *Let $\langle A, B\rangle$ be a generic improved maximal almost disjoint family. In the model $L(\mathbb{R})[A, B]$, there is no injection from $\mathbb{R}$ into A.*

As always, the theorem is established once we show that injections from $\mathbb{R}$ into A are independent of A, whenever A is a generic improved MAD family. The following lemma gives this.

Lemma 4.10. *Suppose that*

- *$\langle A, B\rangle$ is an improved MAD family,*
- *P is a partial making $\mathbb{R}^V$ countable,*
- *$\dot{A}$ and $\dot{B}$ are P-names such that $\langle \dot{A}, \dot{B}\rangle$ is forces to be an improved MAD family extending A and*

- $\dot{f}$ *is a* P*-name which is forces to be an injection from* $\mathbb{R}$ *into* $\dot{A}$*.*

Then P *is forcing-equivalent to an iteration* $\mathbb{R} * \dot{Q}$ *such that in some forcing extension there exist* V*-generic filters* (G, H_0) *for* (G, H_1) *for* $\mathbb{R} * \dot{Q}$ *such that*

$$\langle \dot{A}'_{G,H_0} \cup \dot{A}'_{G,H_1}, \dot{B}'_{G,H_0} \cup \dot{B}'_{G,H_1} \rangle$$

is an improved AD family, but $\dot{f}'_{G,H_0} \cup \dot{f}'_{G,H_1}$ *is not a function, where* $\dot{A}'$*,* $\dot{B}'$ *and* $\dot{f}'$ *are induced versions of* $\dot{A}$*,* $\dot{B}$ *and* $\dot{f}$*.*

Proof. Since forcing with P makes $\mathbb{R}^V$ countable, P is forcing-equivalent to an iteration of the form $\mathbb{R} * \dot{Q}$ for which there exists a condition $(p, \dot{q})$ forcing that $\dot{f}'(\dot{r})$ will not be in A. Let $G \subseteq \mathbb{R}$ be a V-generic filter containing p. Applying Theorem 4.4 and Remark 4.5, we can find $V[G]$-generic filters H_0, H_1 for $\dot{Q}_G$ such that $\dot{A}'_{G,H_0} \setminus A$ and $\dot{A}'_{G,H_1} \setminus A$ are disjoint. Then the values of $\dot{f}'_{G,H_0}$ and $\dot{f}'_{G,H_1}$ at $\dot{r}_G$ are distinct. □

Our second proof is simply the proof of Theorem 2.2 adapted to the case under consideration.

Theorem 4.11. (ZFC) *Assume that there exist proper class many Woodin cardinals. Suppose that* Φ *is the tame* Σ^2_1 *sentence asserting the existence of an improved MAD family, with respective resolvent* ψ*. Let* $(A, B) \subseteq \omega^\omega$ *be a* P_ψ*-generic witness to* Φ*. Then in* $L(\mathbb{R})[A, B]$*, there is no injection from* $\mathbb{R}$ *into* A*.*

Proof. Work in the model $V[A, B]$. Suppose towards a contradiction that the model $L(\mathbb{R})[A, B]$ does contain an injection f from $\mathbb{R}$ into A. In such a case, there must be a name $\dot{f} \in L(\mathbb{R})$ such that $f = \dot{f}_{(A,B)}$. The name $\dot{f}$ is coded by a set $C \subseteq \omega^\omega$ in $L(\mathbb{R})$, and some P_ψ-condition contained coordinatewise in (A, B) forces that $\dot{f}$ is an injection from $\mathbb{R}$ into the first coordinate of the generic improved MAD family.

Let δ be a Woodin cardinal and let $\mathbb{Q}_{<\delta}$ be the countably based stationary tower at δ which, collapses κ to $\aleph_0$ (see [4], for instance). Let τ_A, τ_B and τ_f be $\mathbb{Q}_{<\delta}$-names for $j(A)$, $j(B)$ and $j(f)$ respectively, where j is the generic elementary embedding derived from forcing with $\mathbb{Q}_{<\delta}$. By Lemma 4.10, in some generic extension $V[A, B][G]$, there exist $V[A, B]$-generic filters $H_i \subseteq \mathbb{Q}_{<\delta}$ $(i \in 2)$ such that

$$\langle \tau_{A,H_0} \cup \tau_{A,H_1}, \tau_{B,H_0} \cup \tau_{B,H_1} \rangle$$

is an improved AD family, but $\tau_{f,H_0} \cup \tau_{f,H_1}$ is not a function.

By results (due to Woodin) in Chapter 3 of [4] (especially Exercise 3.3.18), there exists in $V[A, B]$ a tree T on $\omega \times \gamma$, for some ordinal γ, such that

- $\mathrm{p}[T] = C$;
- $j(T) = T$ whenever j is an elementary embedding derived from forcing with $\mathbb{Q}_{<\delta}$;

- the model $\langle L(\mathbb{R}), \in \mathrm{p}[T]\rangle$ of $V[A,B]$ is elementarily equivalent to the same structure computed in $V[A,B][G]$.

It follows that, in $V[A,B][G]$,

$$\langle \tau_{A,H_0} \cup \tau_{A,H_1}, \tau_{B,H_0} \cup \tau_{B,H_1} \rangle$$

is a ψ-set forcing in P_ψ that, for each $i \in 2$, τ_{f,H_i} is a subset of the realization of the P_ψ-name coded by $\mathrm{p}[T]$. However, this contradicts the choice of the filters H_i $(i \in 2)$. □

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