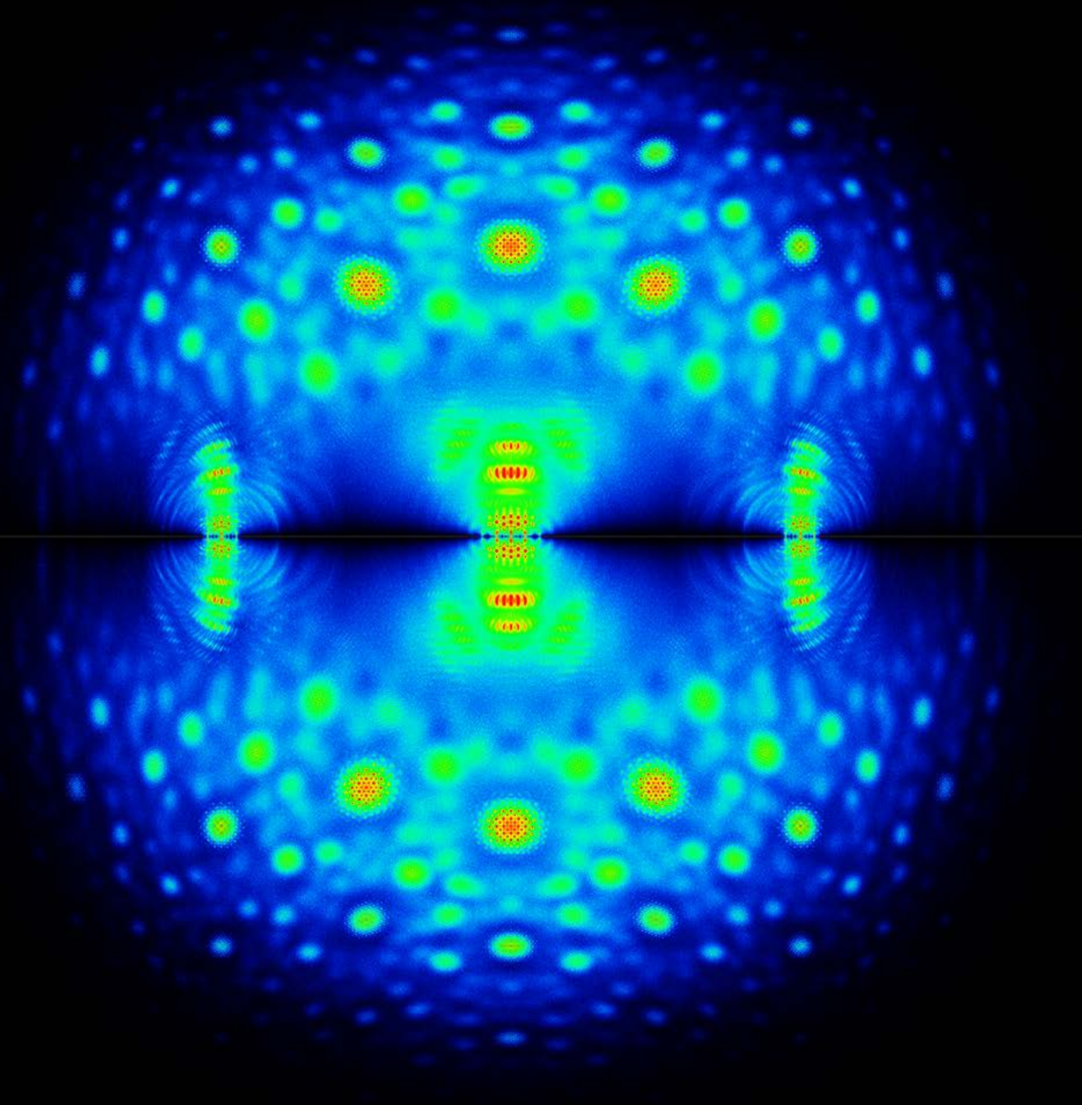
# Bohemian Eigenvalues



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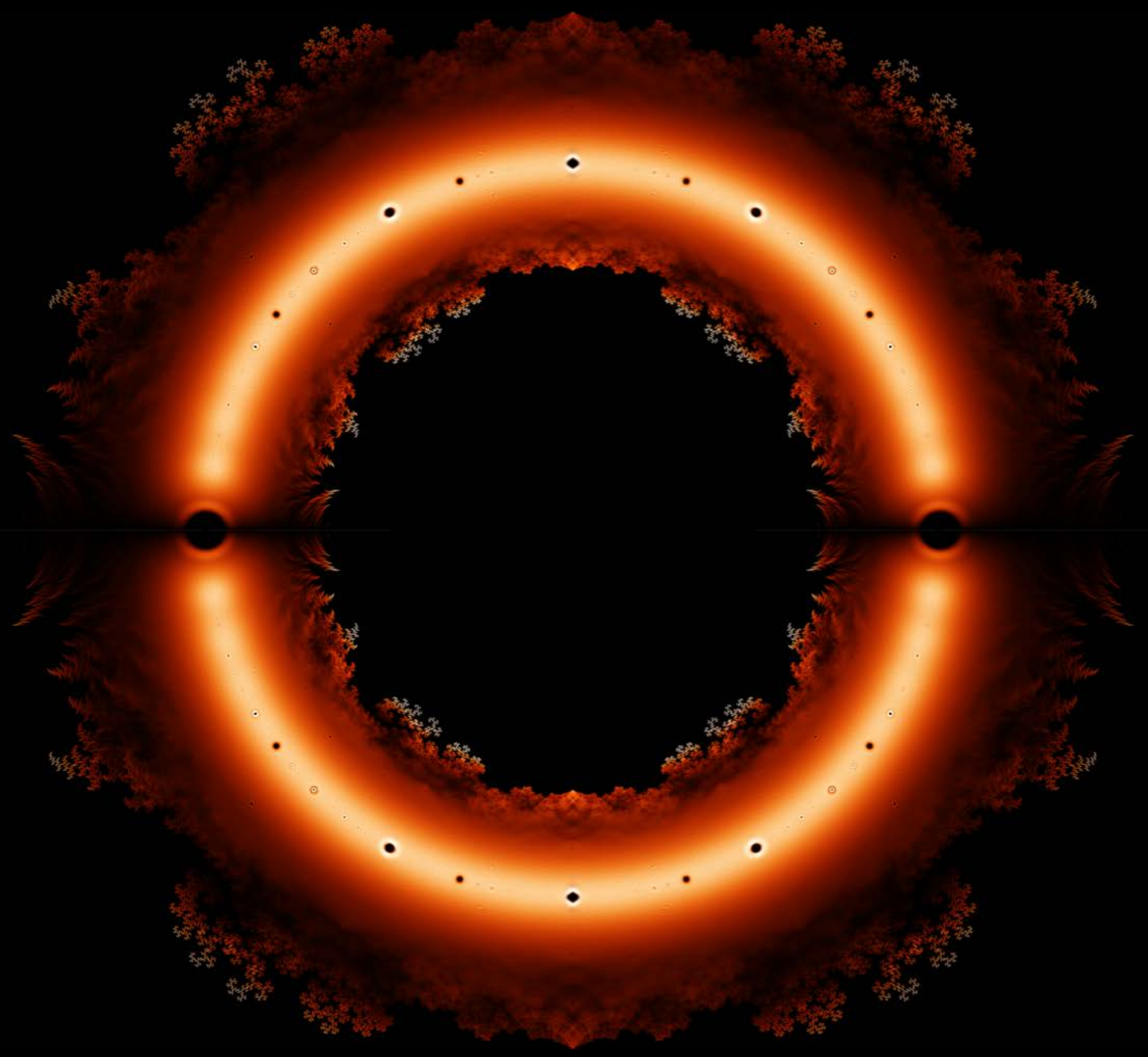
# History

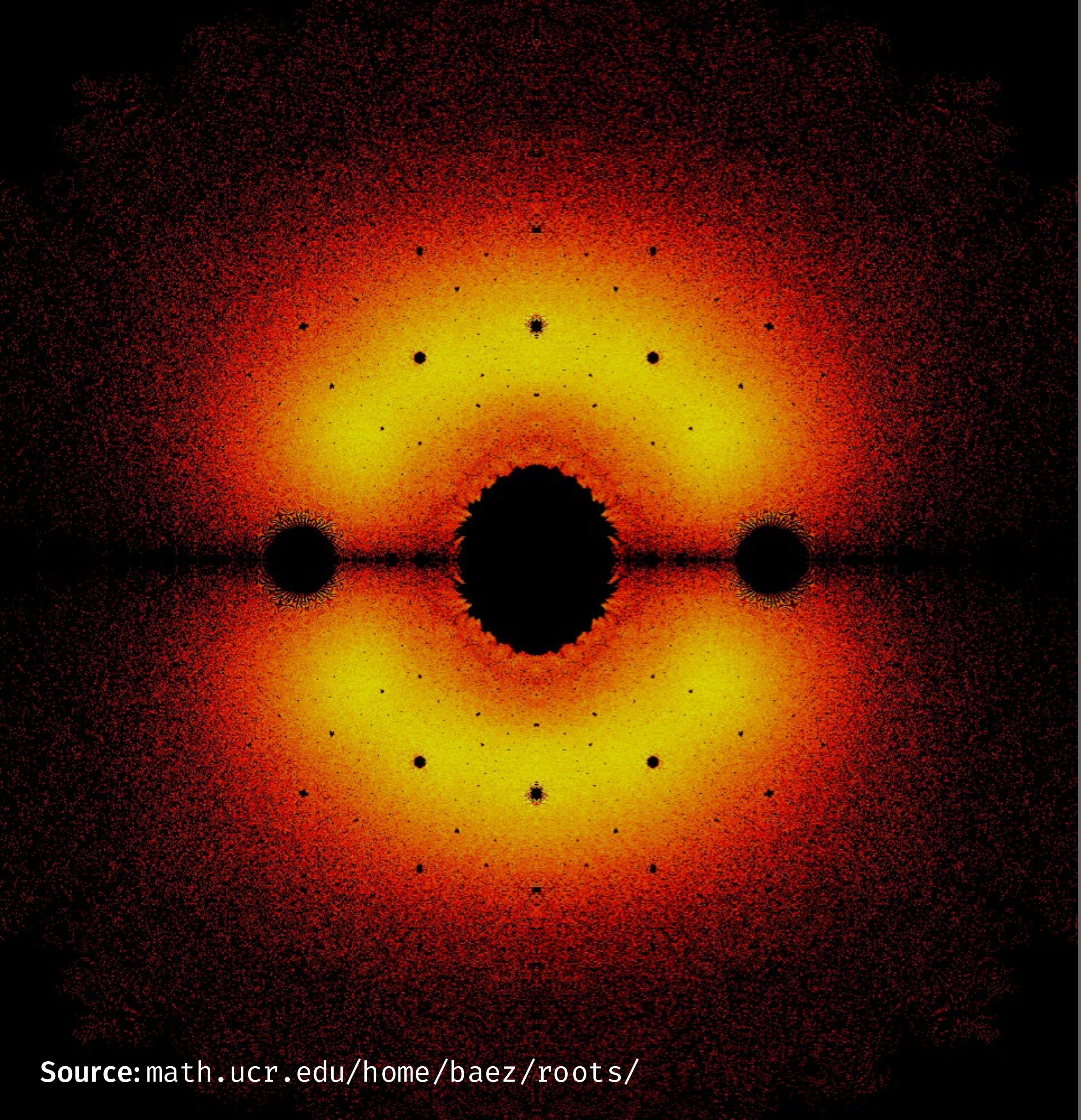
- Polynomials with coefficients of bounded height: Littlewood (50s), P. Borwein & L. Jörgenson 1995, D. Christensen more recently
- Corless 2004/2007 Lagrange basis pictures, and the realization that PBH $\subseteq$ Bohemian matrices because of companion matrices
- Lawrence & Corless 2011, Mandelbrot matrices show PBH $\subset$ Bohemian ($F = \{0, 1\}$ but coefficients of $p$ grow exponentially in degree)
- Random matrices: Wishart 1928, Wigner 1967, Tao & Vu 2009, 2015
- Graph theory (incidence matrices) since forever
- Finite difference matrices, again since forever



Source: `math.ucr.edu/home/baez/roots/`

# Details

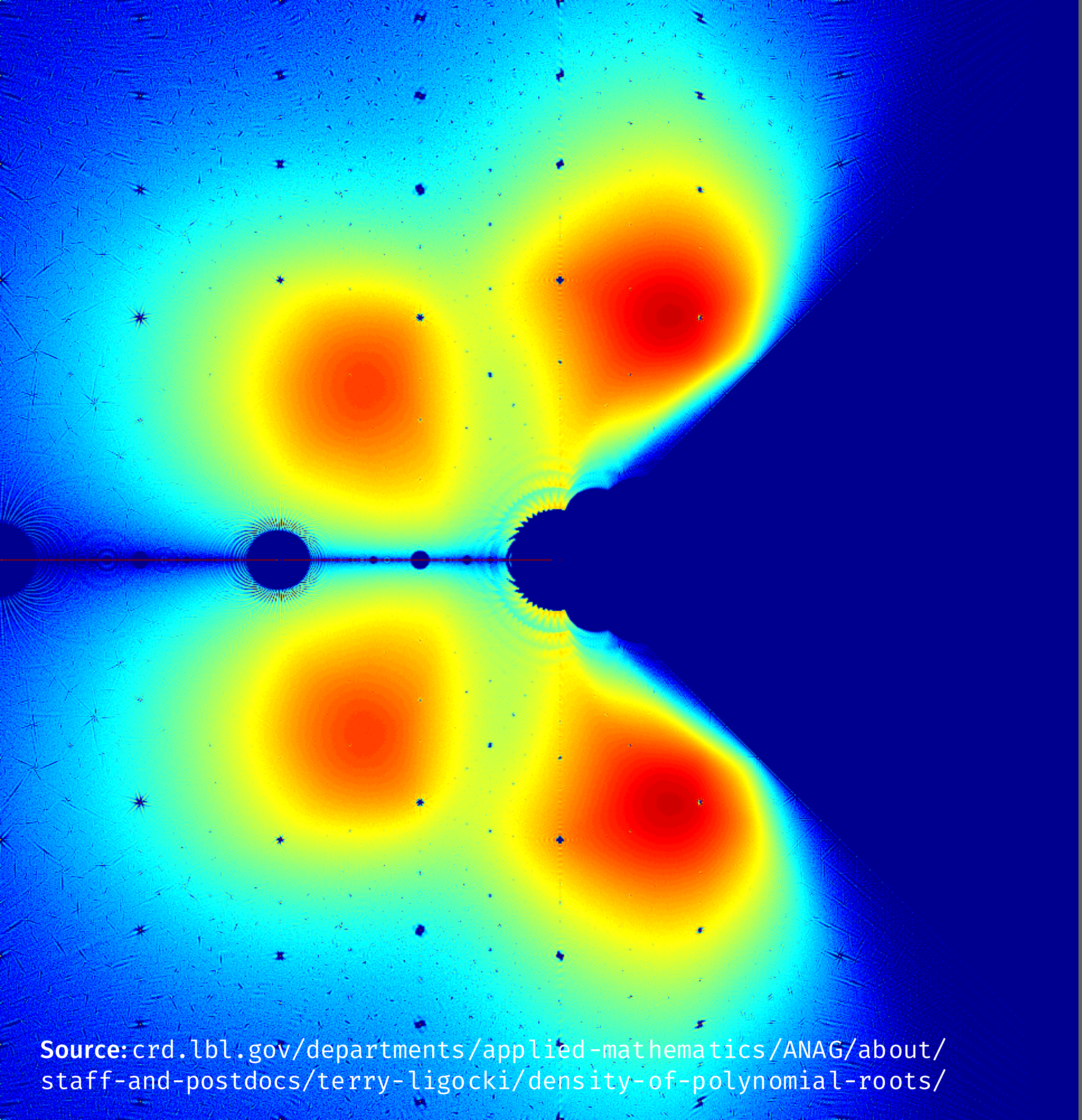
- Roots of all polynomials of degree $\leq 24$
- Coefficients in $\{-1, 1\}$
- Created by Sam Derbyshire
- 4 days to compute roots using Mathematica



Source: math.ucr.edu/home/baez/roots/

# Details

- Roots of all polynomials of degree $\leq 6$
- Integer coefficients between $-5$ and $5$



Source: crd.lbl.gov/departments/applied-mathematics/ANAG/about/staff-and-postdocs/terry-ligocki/density-of-polynomial-roots/

# Details

- Roots of all polynomials of degree $\leq 4$
- Coefficients in $\{0, 1, \ldots, 30\}$

# Eigenvalues of Random Matrices


- Edelman, A. (1988). Eigenvalues and condition numbers of random matrices. *SIAM Journal on Matrix Analysis and Applications*, 9(4), 543-560.
- Marčenko, V. A., & Pastur, L. A. (1967). Distribution of Eigenvalues for Some Sets of Random Matrices. *Mathematics of the USSR-Sbornik*, 1(4), 457.
- Tao, T., & Vu, V. (2011). Random Matrices: Universality of Local Eigenvalue Statistics. *Acta mathematica*, 206(1), 127-204.
- Diaconis, P., & Shahshahani, M. (1994). On the Eigenvalues of Random Matrices. *Journal of Applied Probability*, 49-62.
- Arnold, L. (1967). On the Asymptotic Distribution of the Eigenvalues of Random Matrices. *Journal of Mathematical Analysis and Applications*, 20(2), 262-268.

# Some Properties

| | | $\{-1, 0, 1\}$ | | $\{0, 1, 2\}$ | | $\{t, 1, 2\}$ | |
|---|---|---|---|---|---|---|---|
| $n$ | $\#M$ | $\#c$ | $\#c_m$ | $\#c$ | $\#c_m$ | $\#c$ | $\#c_m$ |
| 1 | 3 | 3 | 0 | 3 | 0 | 3 | 0 |
| 2 | 81 | 16 | 3 | 22 | 3 | 36 | 0 |
| 3 | 19,683 | 209 | 17 | 513 | 23 | 1782 | 16 |
| 4 | 43,046,721 | ? | ? | ? | ? | ? | ? |

# Some Properties

| | | $\{0,1\}$ | | $\{-1,1\}$ | | $\{t,1\}$ | |
|---|---|---|---|---|---|---|---|
| $n$ | $\#M$ | $\#c$ | $\#c_m$ | $\#c$ | $\#c_m$ | $\#c$ | $\#c_m$ |
| 1 | 2 | 2 | 0 | 2 | 0 | 2 | 0 |
| 2 | 16 | 6 | 2 | 6 | 1 | 9 | 0 |
| 3 | 512 | 32 | 8 | 28 | 6 | 68 | 6 |
| 4 | 65,536 | 333 | 50 | 203 | 48 | 1161 | 115 |
| 5 | 335,542,432 | ? | ? | ? | ? | ? | ? |

# Repeats

- For $F = \{0, 1\}$, $n = 3$ there are $2^{3^2} = 512$ matrices but only 32 distinct characteristic polynomials
- $\lambda(\lambda - 1)^2$ and $\lambda^2(\lambda - 1)$ occur 75 times each
- $(\lambda - 2)(\lambda + 1)^2$, $\lambda^2(\lambda - 3)$ **once** each
- Possible that some eigenvalues are more likely than others

# Why Are There Repeats?

- $A$ and $A^T$ have the same eigenvalues
- $A$ and $PAP^{-1}$ have the same eigenvalues
  - If $P$ is a permutation matrix and $A$ is Bohemian, then $PAP^{-1}$ is too. ($n!$ such symmetries)
- Number of coefficients of characteristic polynomials $= n$, not $n^2$. Therefore, $A \to \text{charpoly}(A)$ is a compression; matrices with same $\text{trace}(A)$, $\text{trace}(A^2), \ldots$ (Fadeev/Leverrier) have same characteristic polynomial. These are non-linear correlations

# Besides, We Hate Polynomials

- Harder to solve than eigenvalue problems
- For instance, a bug in `fsolve` for polynomials of degree 3 prevented one exploration (we reported it and they gave us a workaround)
- Eigenvalue computation always works (in our experience)
- Compressing to find the distinct characteristic polynomials seems to require solving systems of Diophantine equations
- So brute force it is...

## Entries in $\{-1, 0, 1\}$

$$\begin{bmatrix}-1 & -1\\-1 & -1\end{bmatrix} \begin{bmatrix}-1 & -1\\-1 & 0\end{bmatrix} \begin{bmatrix}-1 & -1\\-1 & 1\end{bmatrix} \begin{bmatrix}-1 & -1\\0 & -1\end{bmatrix} \begin{bmatrix}-1 & -1\\0 & 0\end{bmatrix} \begin{bmatrix}-1 & -1\\0 & 1\end{bmatrix} \begin{bmatrix}-1 & -1\\1 & -1\end{bmatrix} \begin{bmatrix}-1 & -1\\1 & 0\end{bmatrix}$$

$$\begin{bmatrix}-1 & -1\\1 & 1\end{bmatrix} \begin{bmatrix}-1 & 0\\-1 & -1\end{bmatrix} \begin{bmatrix}-1 & 0\\-1 & 0\end{bmatrix} \begin{bmatrix}-1 & 0\\-1 & 1\end{bmatrix} \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix} \begin{bmatrix}-1 & 0\\0 & 0\end{bmatrix} \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} \begin{bmatrix}-1 & 0\\1 & -1\end{bmatrix}$$

$$\begin{bmatrix}-1 & 0\\1 & 0\end{bmatrix} \begin{bmatrix}-1 & 0\\1 & 1\end{bmatrix} \begin{bmatrix}-1 & 1\\-1 & -1\end{bmatrix} \begin{bmatrix}-1 & 1\\-1 & 0\end{bmatrix} \begin{bmatrix}-1 & 1\\-1 & 1\end{bmatrix} \begin{bmatrix}-1 & 1\\0 & -1\end{bmatrix} \begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix} \begin{bmatrix}-1 & 1\\0 & 1\end{bmatrix}$$

$$\begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix} \begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix} \begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix} \begin{bmatrix}0 & -1\\-1 & -1\end{bmatrix} \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix} \begin{bmatrix}0 & -1\\-1 & 1\end{bmatrix} \begin{bmatrix}0 & -1\\0 & -1\end{bmatrix} \begin{bmatrix}0 & -1\\0 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & -1\\0 & 1\end{bmatrix} \begin{bmatrix}0 & -1\\1 & -1\end{bmatrix} \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} \begin{bmatrix}0 & -1\\1 & 1\end{bmatrix} \begin{bmatrix}0 & 0\\-1 & -1\end{bmatrix} \begin{bmatrix}0 & 0\\-1 & 0\end{bmatrix} \begin{bmatrix}0 & 0\\-1 & 1\end{bmatrix} \begin{bmatrix}0 & 0\\0 & -1\end{bmatrix}$$

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} \begin{bmatrix}0 & 0\\1 & -1\end{bmatrix} \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix} \begin{bmatrix}0 & 1\\-1 & -1\end{bmatrix} \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} \begin{bmatrix}0 & 1\\-1 & 1\end{bmatrix}$$

$$\begin{bmatrix}0 & 1\\0 & -1\end{bmatrix} \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} \begin{bmatrix}0 & 1\\1 & -1\end{bmatrix} \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix} \begin{bmatrix}1 & -1\\-1 & -1\end{bmatrix} \begin{bmatrix}1 & -1\\-1 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} \begin{bmatrix}1 & -1\\0 & -1\end{bmatrix} \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix} \begin{bmatrix}1 & -1\\1 & -1\end{bmatrix} \begin{bmatrix}1 & -1\\1 & 0\end{bmatrix} \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} \begin{bmatrix}1 & 0\\-1 & -1\end{bmatrix}$$

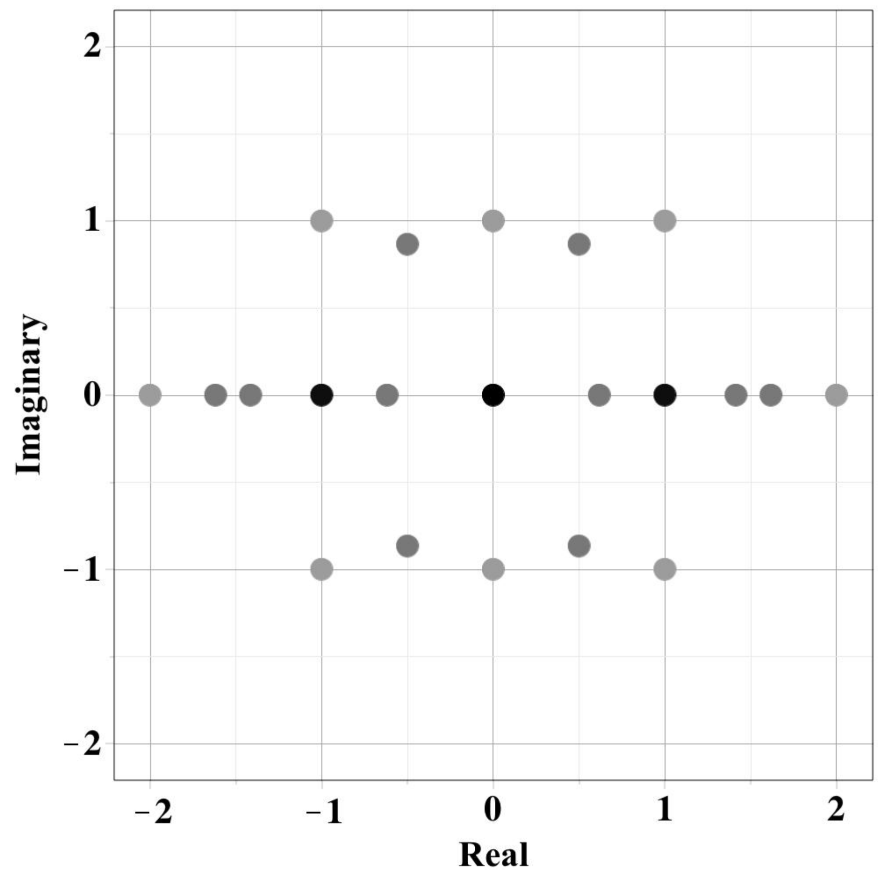
$$\begin{bmatrix}1 & 0\\-1 & 0\end{bmatrix} \begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix} \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \begin{bmatrix}1 & 0\\1 & -1\end{bmatrix} \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\-1 & -1\end{bmatrix} \begin{bmatrix}1 & 1\\-1 & 0\end{bmatrix} \begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix} \begin{bmatrix}1 & 1\\0 & -1\end{bmatrix} \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$$

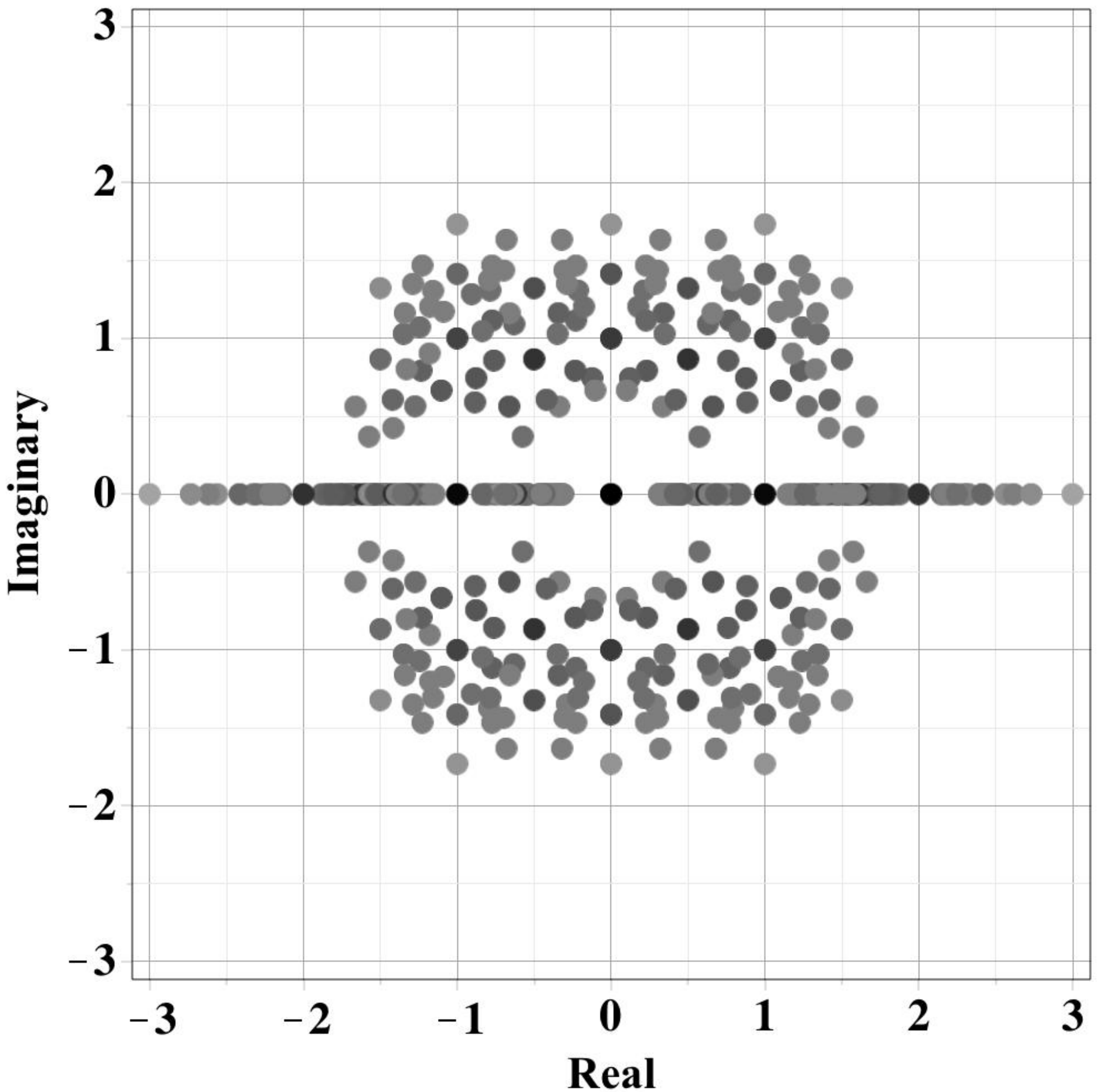
$$\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

# $2 \times 2$ Matrices, {-1, 0, 1}

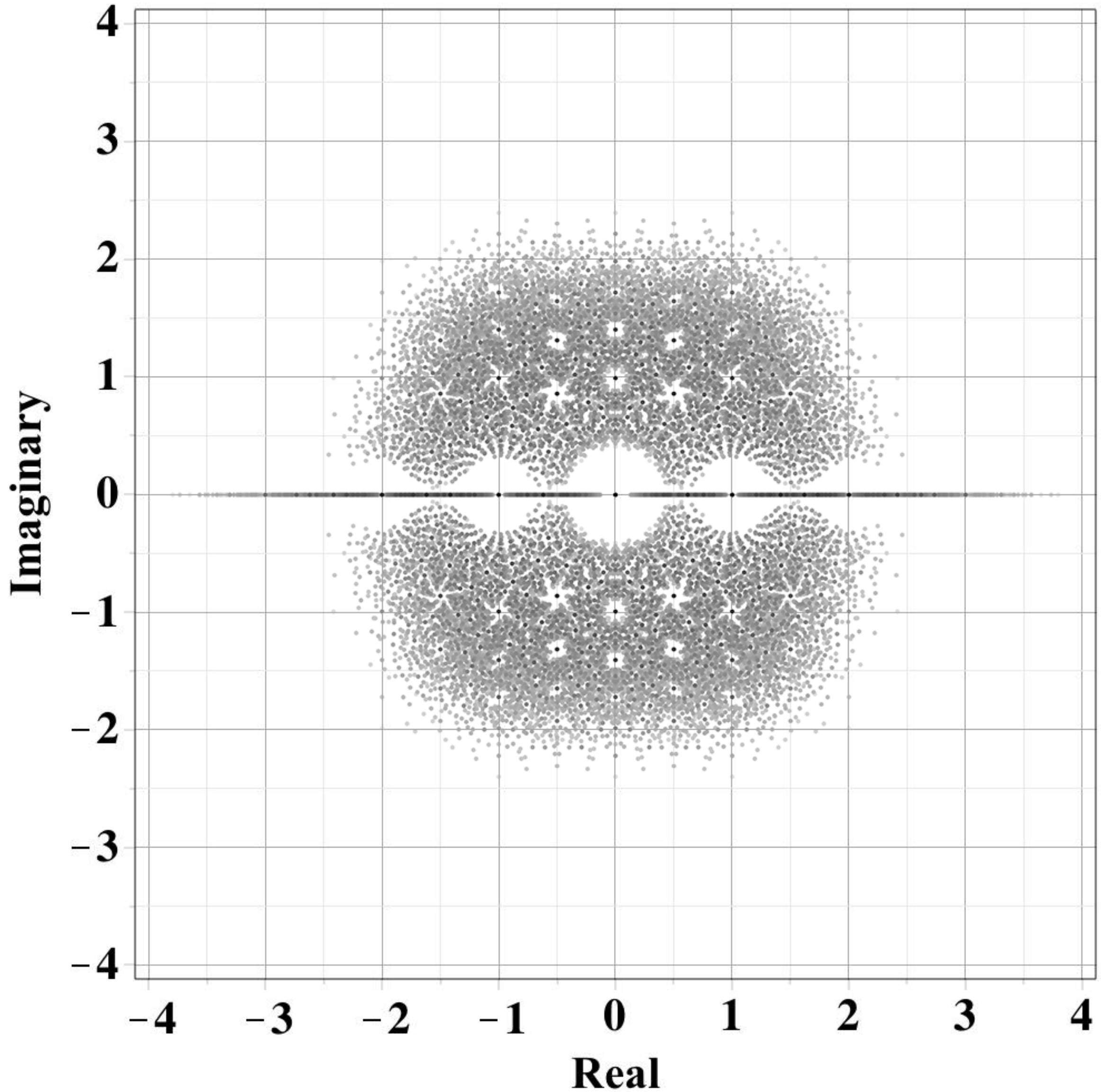
| Eigenvalue | Count |
|---|---|
| $0$ | 42 |
| $1$ | 32 |
| $-1$ | 32 |
| $\sqrt{2}$ | 4 |
| $-\sqrt{2}$ | 4 |
| $\frac{1+\sqrt{5}}{2}$ | 4 |
| $\frac{1-\sqrt{5}}{2}$ | 4 |
| $\frac{-1+\sqrt{5}}{2}$ | 4 |
| $\frac{-1-\sqrt{5}}{2}$ | 4 |
| $\frac{1+i\sqrt{3}}{2}$ | 4 |
| $\frac{1-i\sqrt{3}}{2}$ | 4 |
| $\frac{-1+i\sqrt{3}}{2}$ | 4 |
| $\frac{-1-i\sqrt{3}}{2}$ | 4 |
| $2$ | 2 |
| $-2$ | 2 |
| $i$ | 2 |
| $-i$ | 2 |
| $1+i$ | 2 |
| $1-i$ | 2 |
| $-1+i$ | 2 |
| $-1-i$ | 2 |



# 3 × 3 Matrices, {-1, 0, 1}

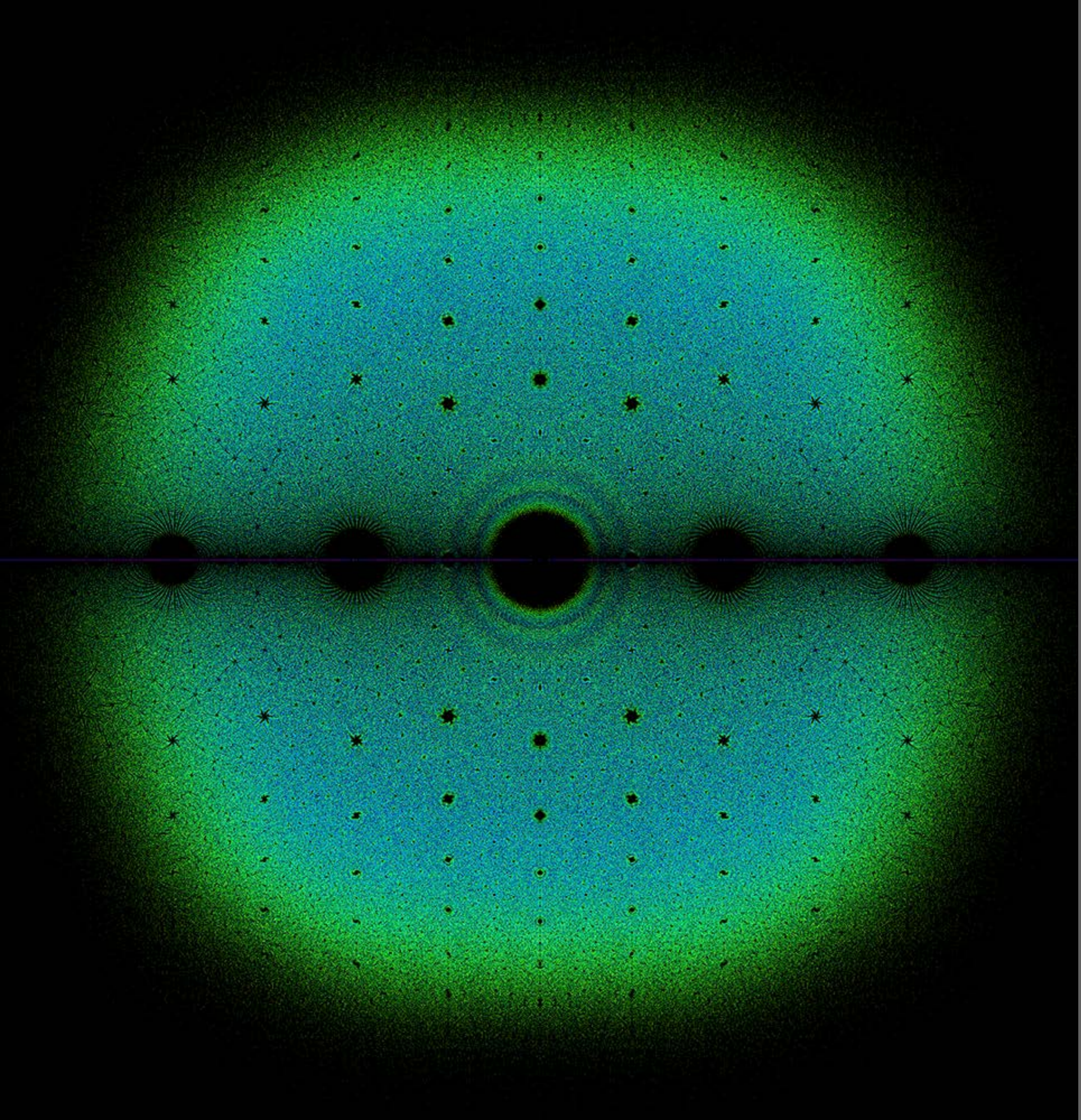


# 4 × 4 Matrices, {-1, 0, 1}



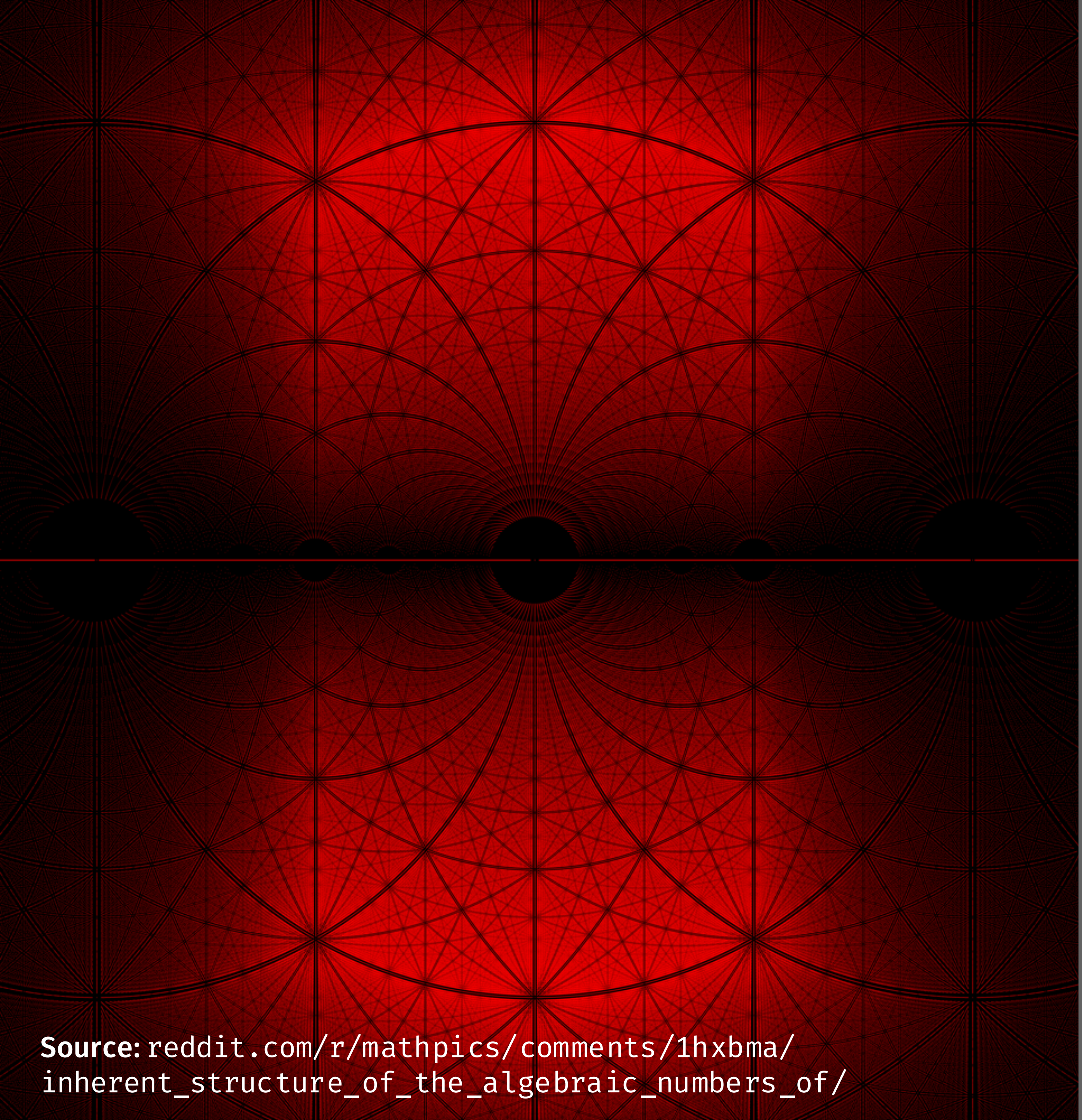
# Sampling

- Get the most probable characteristic polynomimals
- Most probable eigenvalues
- Can draw entries from plausible distributions
- Most of the images we've created were done this way



# Details

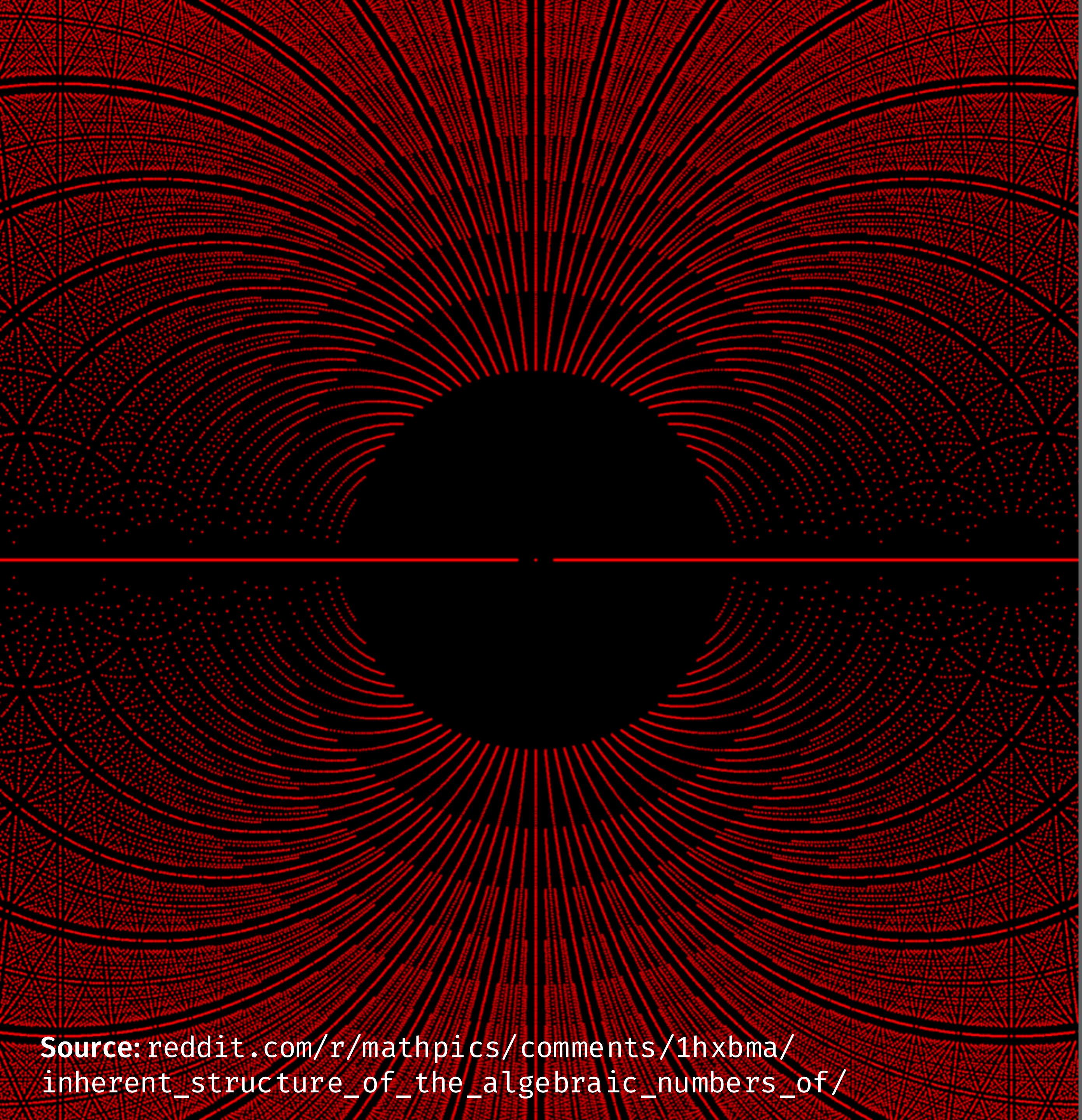
- $5 \times 5$ matrices
- Entries sampled uniformly from $\{-1, 0, 1\}$
- Random sample of 3 billion matrices
- Represents 0.35% of class
- Approximately 20 hours to produce this image (on a 2011 MacBook Pro, 2.3GHz Intel Core i7, 16GB RAM)



**Source:** `reddit.com/r/mathpics/comments/1hxbma/inherent_structure_of_the_algebraic_numbers_of/`

# Details

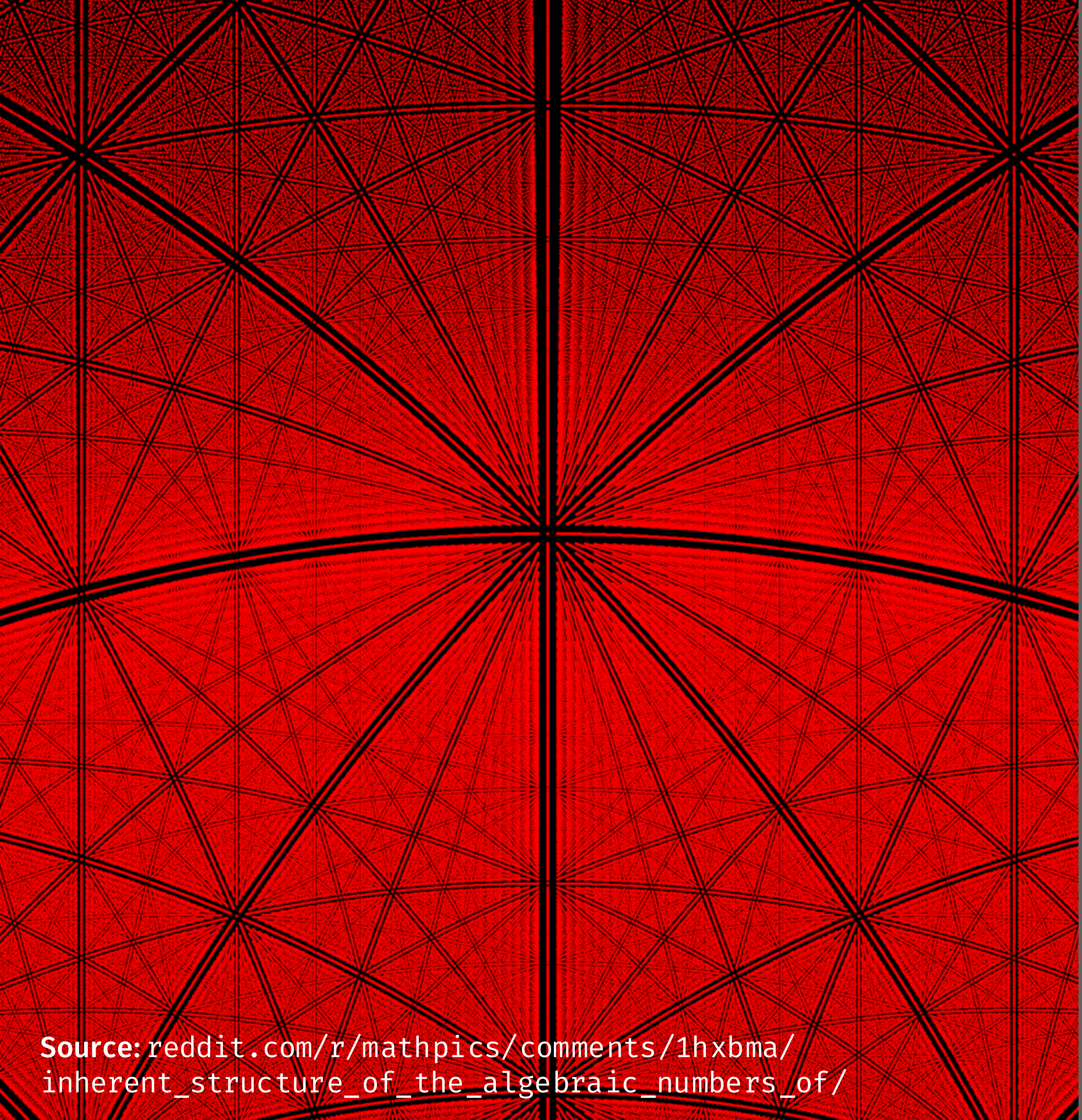
- Algebraic numbers
- Solutions to quadratic polynomials with coefficients no greater than 100 in magnitude
- Viewed on $\pm 1.2 \pm 1.2i$



**Source:** reddit.com/r/mathpics/comments/1hxbma/inherent_structure_of_the_algebraic_numbers_of/

# Details

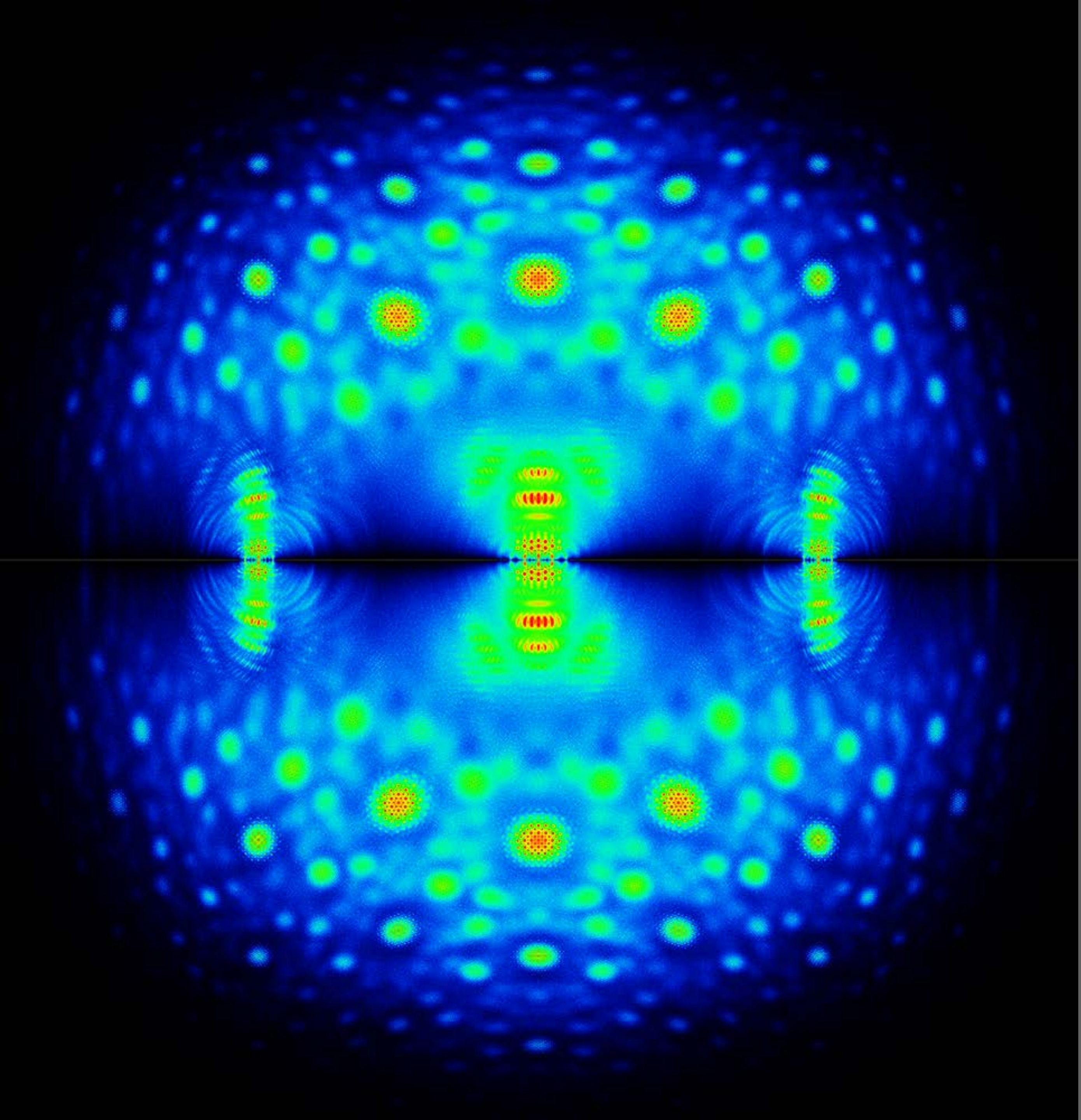
- Algebraic numbers
- Solutions to quadratic polynomials with coefficients no greater than 100 in magnitude
- Close up of the origin



**Source:** `reddit.com/r/mathpics/comments/1hxbma/inherent_structure_of_the_algebraic_numbers_of/`

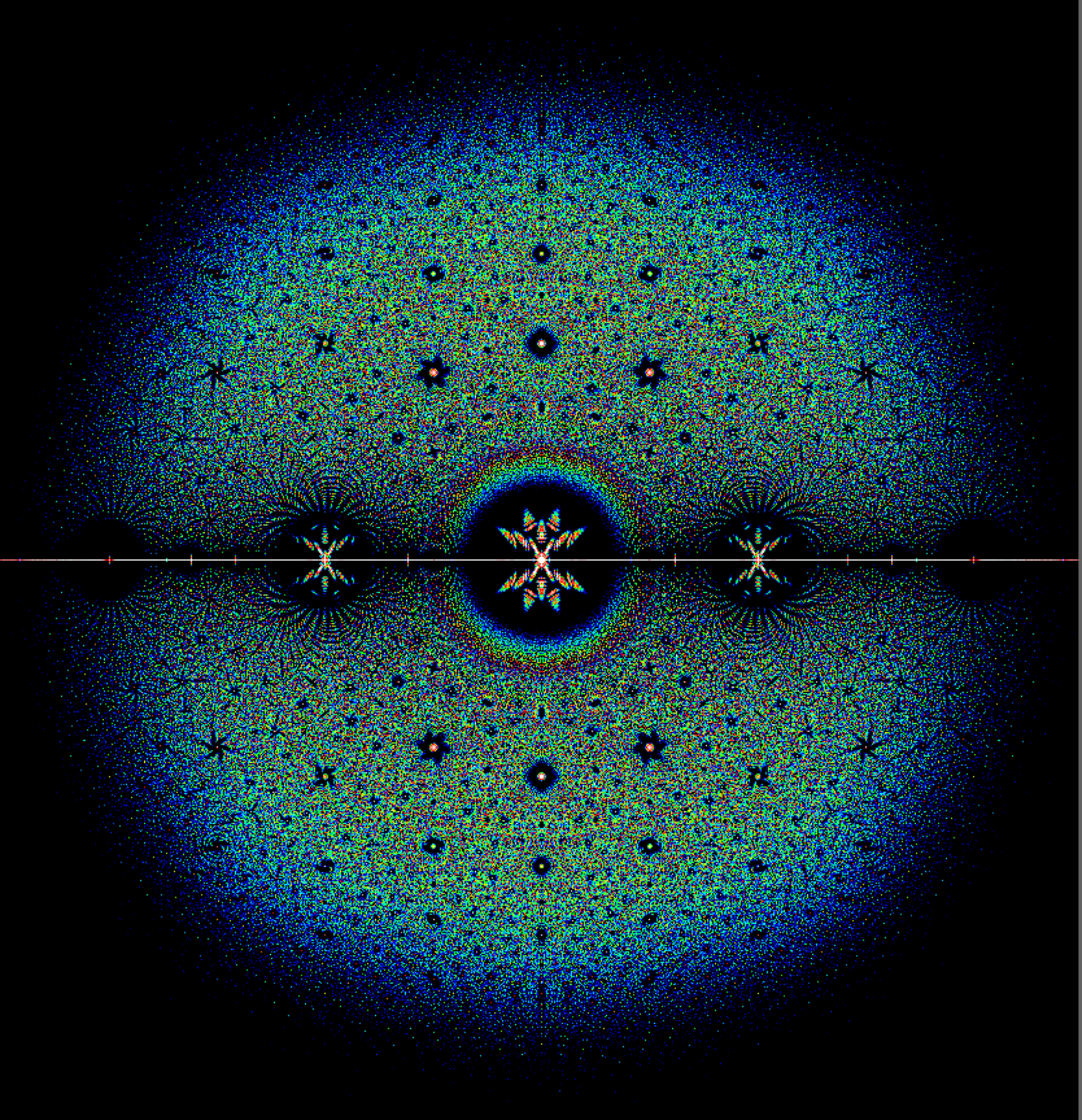
# Details

- Algebraic numbers
- Solutions to quadratic polynomials with coefficients no greater than 100 in magnitude
- Close up of *i*



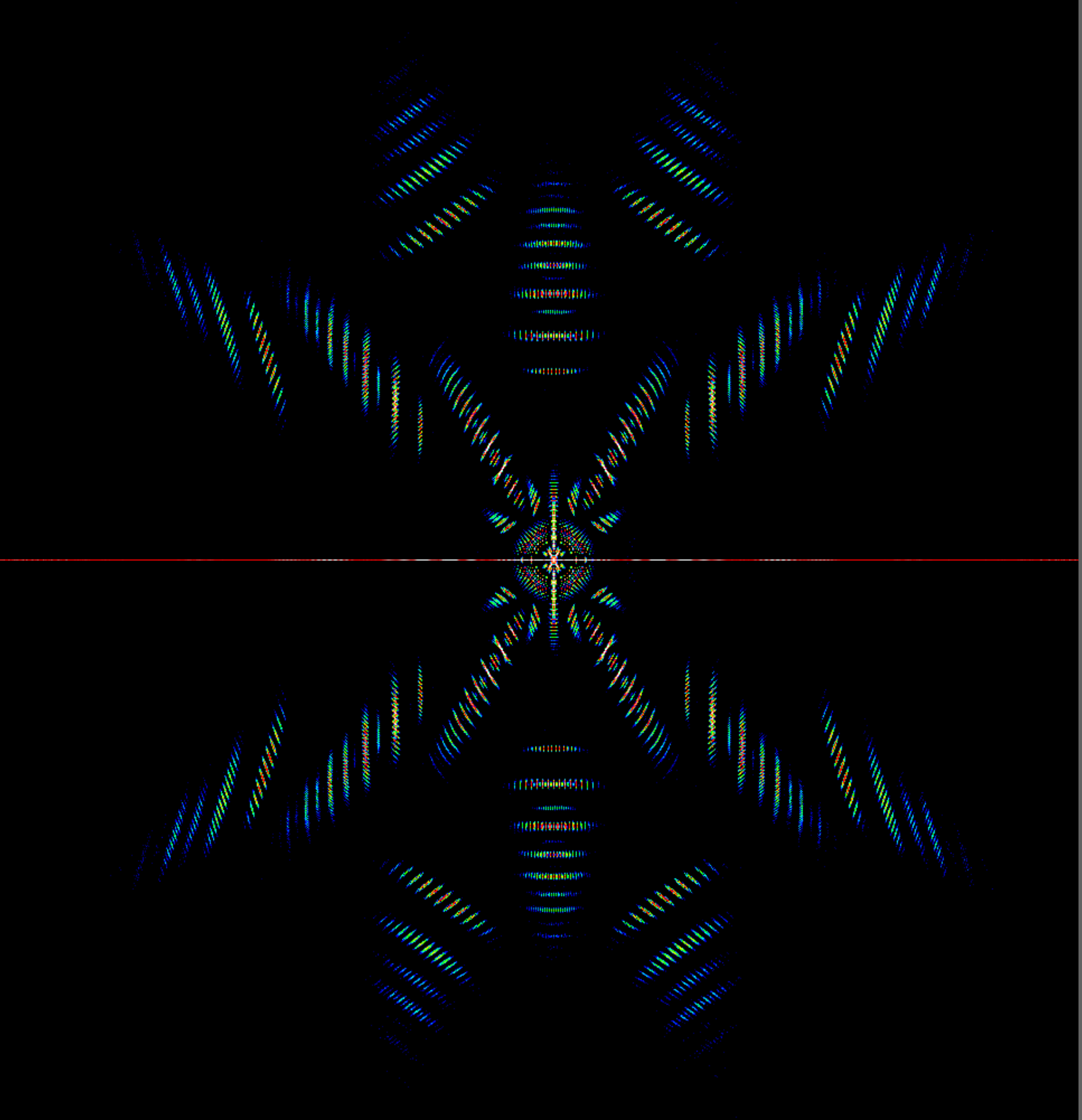
# Details

- $5 \times 5$ matrices
- Entries sampled uniformly from $\{-20, -1, 0, 1, 20\}$
- Random sample of 73 million matrices
- Colored by density of eigenvalues



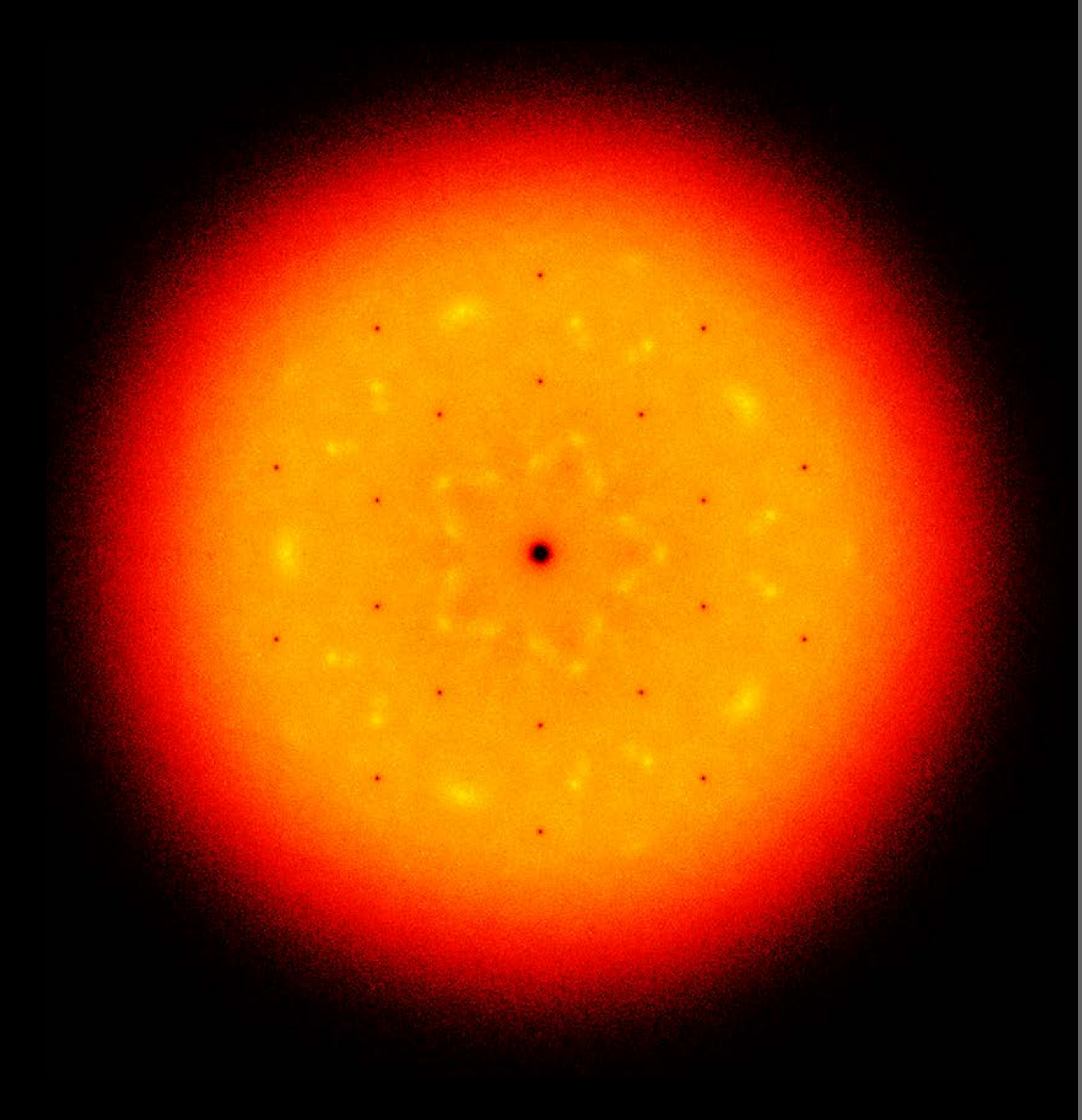
# Details

- $5 \times 5$ matrices
- Entries sampled uniformly from $\{-1, -1/10000, 0, 1/10000, 1\}$
- Random sample of 50 million matrices



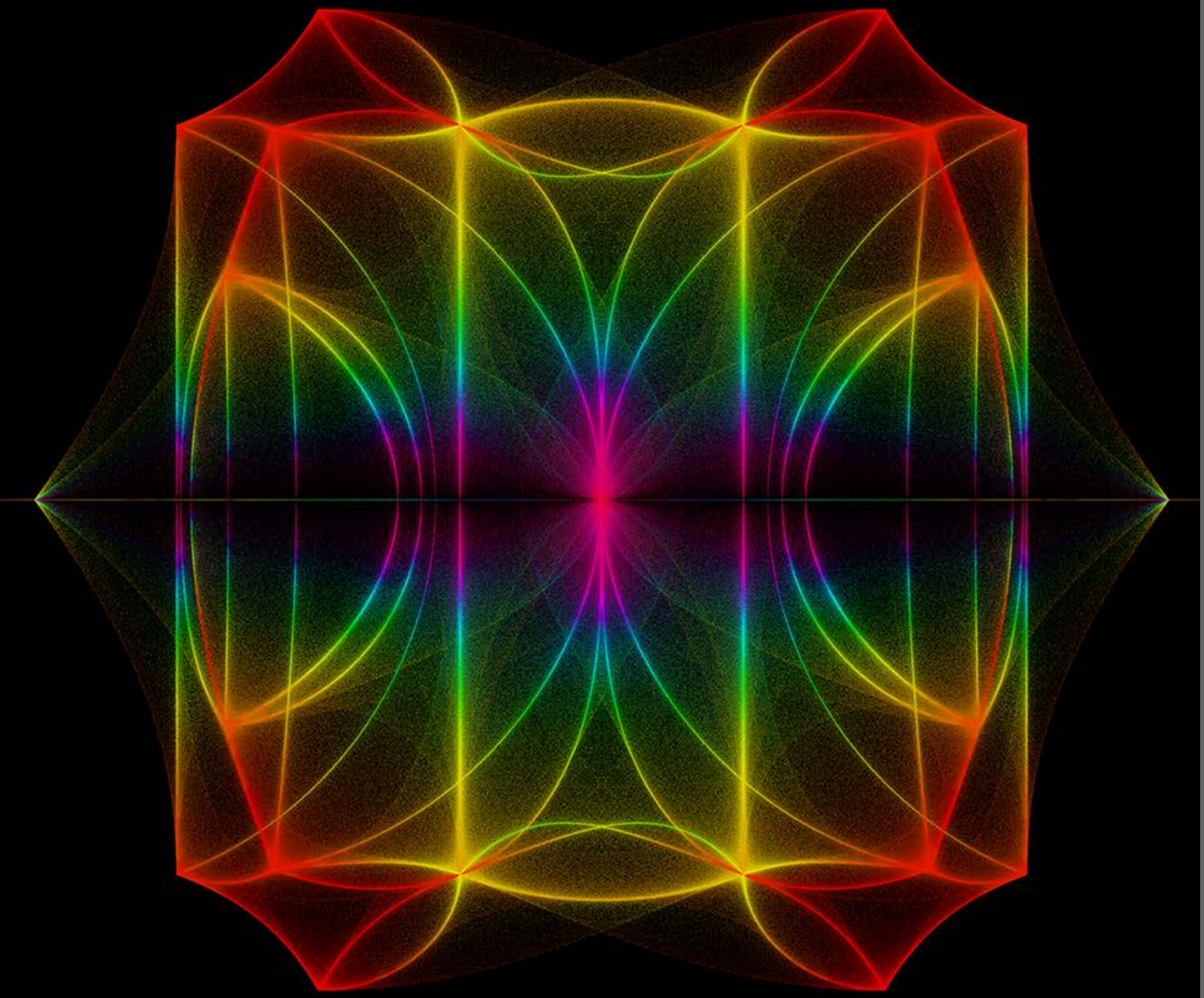
# Details

- $5 \times 5$ matrices
- Entries sampled uniformly from $\{-1, -1/10000, 0, 1/10000, 1\}$
- Random sample of 50 million matrices



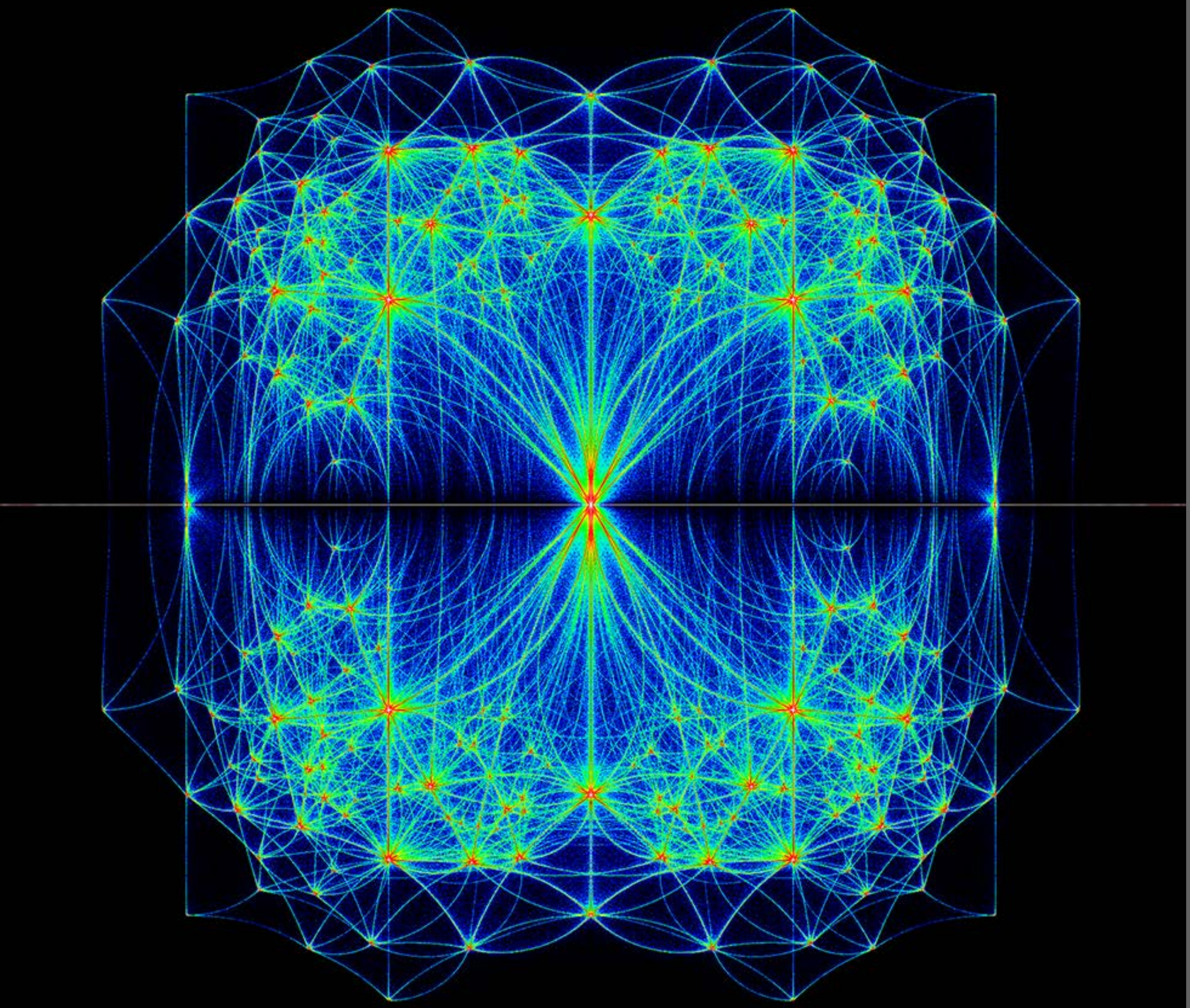
# Details

- $5 \times 5$ complex symmetric matrices
- Entries sampled uniformly from $\{e^{2\pi ij/5} \mid 0 \leq j < 5\}$
- Random sample of 20 million matrices
- Colored by density of eigenvalues



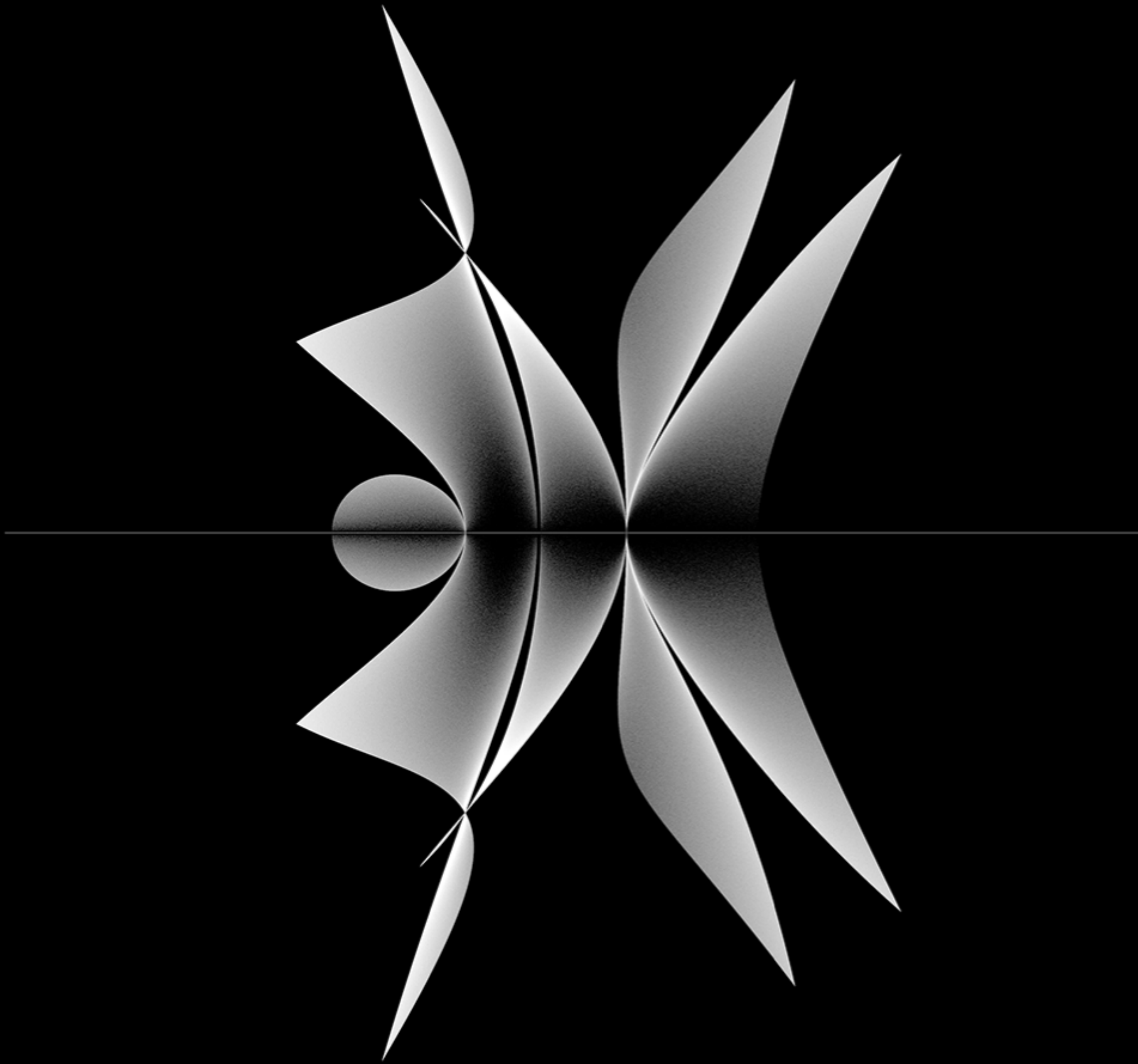
# Details

- $3 \times 3$ matrices
- Each entry is sampled from $2X - 1$ where $X \sim \mathrm{Beta}(0.01, 0.01)$
- Random sample of 10 million matrices
- Colored by eigenvalue condition number



# Details

- $4 \times 4$ matrices
- Each entry is sampled from $2X - 1$ where $X \sim \text{Beta}(0.01, 0.01)$
- Random sample of 1 million matrices
- Colored by density of eigenvalues

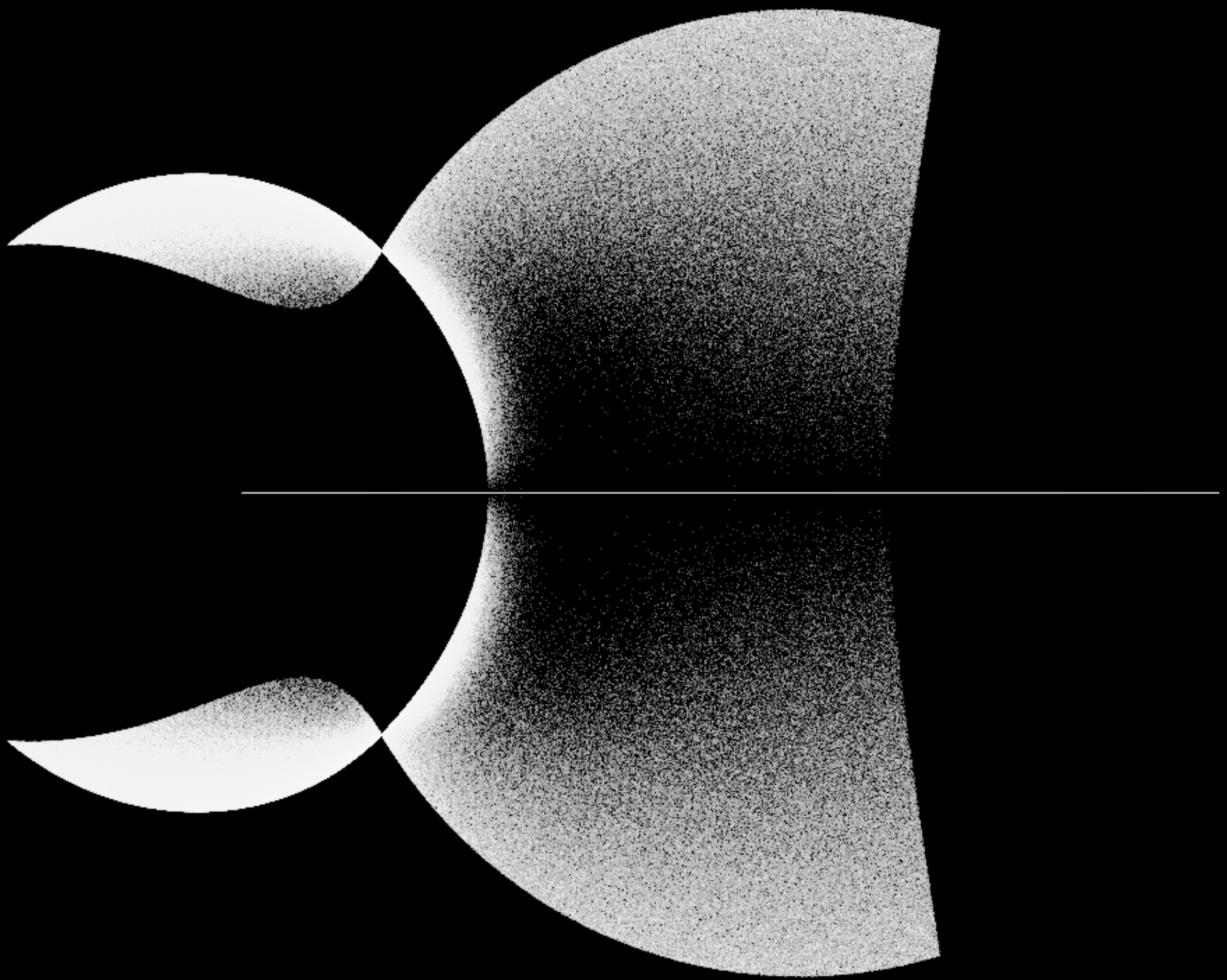


# Details

- Eigenvalues of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & A \\ -1 & -1 & 1 & 0 \\ B & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

  where $A$ and $B$ are continuous uniform random variables on $(-5, 5)$
- Random sample of 30 million matrices

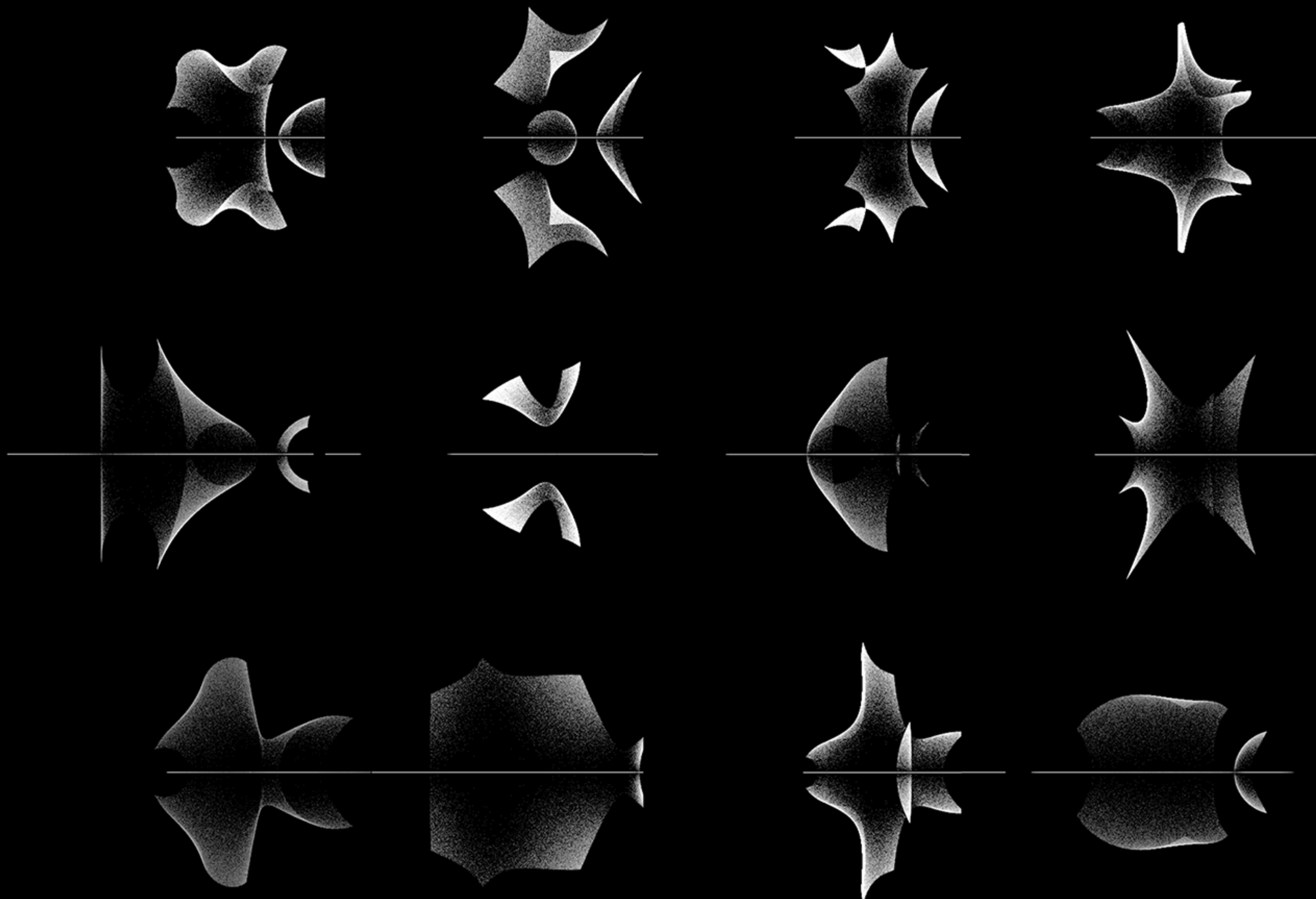


# Details

- Eigenvalues of the matrix

$$\begin{bmatrix} -1 & 1 & -1 \\ -1 & A & 1 \\ 0 & B & 1 \end{bmatrix}$$

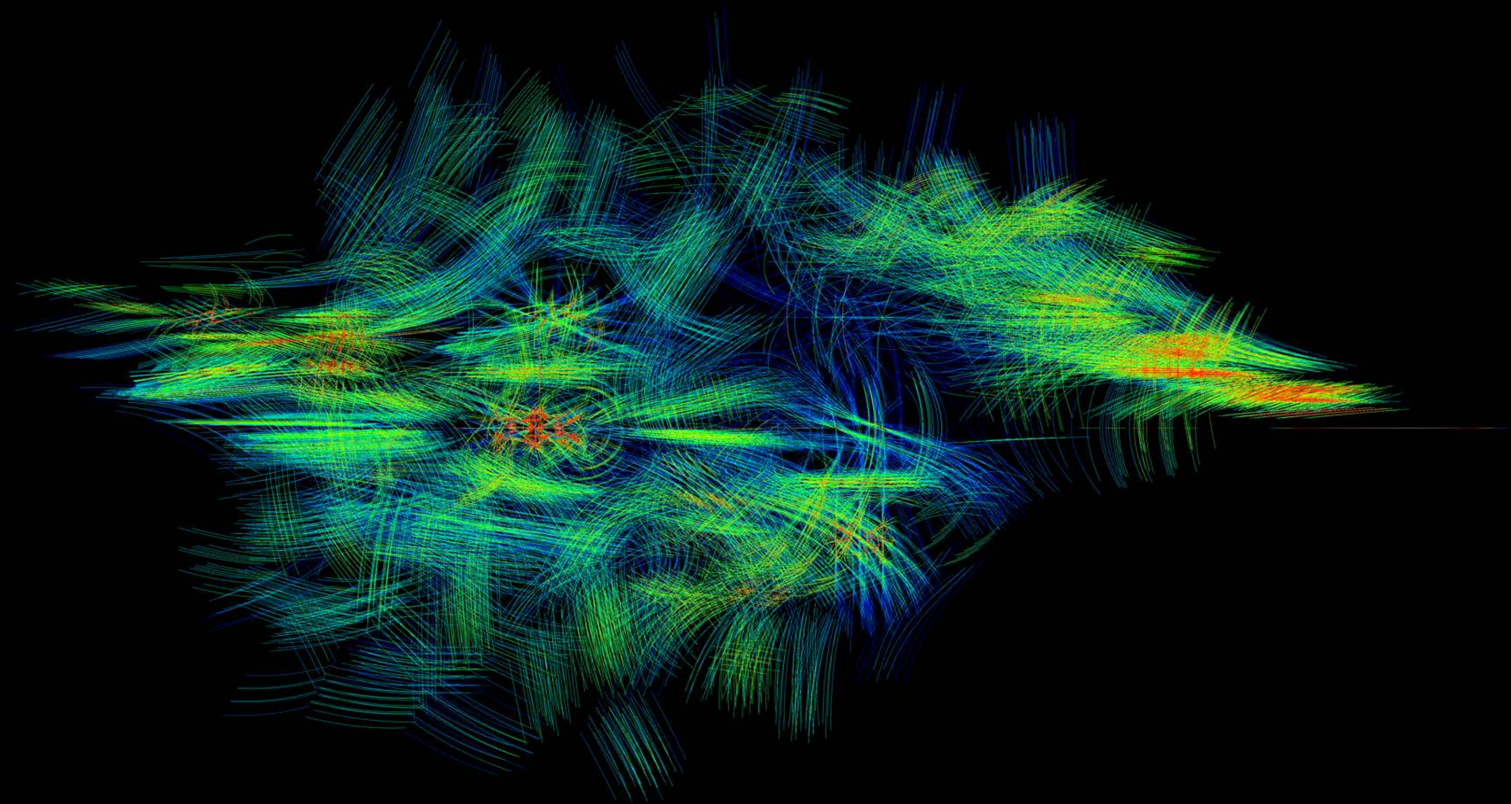
  where $A$ and $B$ are continuous uniform random values on $(-2, 2)$
- Random sample of 1 million matrices



# Future Work

- Analysis of exclusion zones (holes) related to approximation by algebraic numbers
- Analysis of exclusion zones and spectral lines (complex symmetric case)
- Diffraction patterns: wave-like behaviour from particle-like zeros: connection to algebraic approximation

# Thank You!



**Source code with examples available at:**
`github.com/steventhornton/BHIME-Project`