Van der Waerden's *Modern Algebra*

Saunders Mac Lane

B. L. van der Waerden's early studies in the Netherlands of algebraic geometry led him to think about useful definitions of intersection multiplicity for curves and surfaces. He heard that Emmy Noether at the University of Göttingen used newer ideas about ideals to provide precise definitions of such multiplicities. So he went to Göttingen to listen to her inspired but sometimes confusing lectures on ideal theory and presently gave a very clear course of lectures on ideal theory. He then visited the University of Hamburg, where Professor Emil Artin (in 1921) was giving his impassionately insightful lectures on modern algebra. For a brief period there developed a plan that Artin and van der Waerden would collaborate to prepare an algebra text, but Artin did not get around to writing up his planned chapters. Then van der Waerden proceeded alone to write and publish (with Springer) his two-volume 1931 text *Modern Algebra*, with the caption "using lectures by E. Artin and E. Noether".

This beautiful and eloquent text served to transform the graduate teaching of algebra, not only in Germany, but elsewhere in Europe and the United States. It formulated clearly and succinctly the conceptual and structural insights which Noether had expressed so forcefully. This was combined with the elegance and understanding with which Artin had lectured. The first volume included his neat and clean presentation of the Galois theory, a presentation which rapidly replaced the earlier, often obscure treatments. The volume also covered formally real fields and valuation theory. The second volume covered ideal theory, algebraic integers, linear algebra, and representation theory. The whole was inspired by a facility for conceptual clarity and was written in simple, understandable German. Upon its publication it was soon clear that this was the way in which algebra must now be presented. Its simple but austere style set the pattern for mathematical texts in other subjects, from Banach spaces to topological group theory. When I first taught modern algebra as a beginning instructor at Harvard University in 1934, I of course used van der Waerden as my text.

The presentation then of newer ideas from Dedekind, Noether, and Artin should not blind us to the decisive contribution made by van der Waerden. For comparison, recall a two-volume text in algebra by Otto Haupt, then a professor at Erlangen. Haupt was well acquainted with Emmy Noether's new ideas, and he presented them very carefully in his two volumes (published in 1928). I chanced to have studied the volumes then and found them helpful but heavy-handed—indeed, pedantic. It was van der Waerden who understood the real thrust of abstract algebra and who presented it abstractly but without pedantry. His two volumes remained

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the text of choice—in German, or in the second edition published during the war by the alien property custodian, or in English translation, or in a later German edition in which the title dropped the claim to "modernity".

Van der Waerden himself wrote a number of other books and carried on an active research program in algebraic geometry, publishing in the 1930s and 40s many papers "Zur Algebraische Geometrie" (#1 to #20) in the *Mathematische Annalen*, as well as a text (1939) *Introduction to Algebraic Geometry*. This text provided, inter alia, a much needed systematic and precise formulation of the notion of intersection multiplicity. At the time, I bought a copy, but soon realized that this was only the start of the needed reform of algebraic geometry.

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The Italian school of algebraic geometry had made effective and imaginative use of intersection multiplicity, but it was clear that many proofs were not rigorous and that indeed the concepts needed a careful analysis and redevelopment. This need must have been clear to many mathematicians; for example, it became clear to me when in 1937 I learned how R. J. Walker had developed in his thesis a careful treatment of the "infinitely near points" used for the intersection multiplicity of algebraic curves. In 1939 and 1942 Oscar Zariski had published in the *Annals of Mathematics* his careful resolution of the singularities of an algebraic surface. In 1946 Andre Weil's *Foundations of Algebraic Geometry* (AMS Colloquium Publications, vol. 29) had given an extensive analysis of intersection multiplicity (and this volume had aspects which suggested to me the possible uses of category theory, as was done in the later work of Grothendieck).

But Italian-style algebraic geometry still flourished in Belgium, where a young Belgian, L. Derwidué, studied the singularities of algebraic varieties of higher dimension. At that time, van der Waerden had a summer home in the Netherlands. Derwidué visited him and explained his method of resolving singularities in all dimensions. Soon van der Waerden accepted and published in the Mathematische Annalen a paper by Derwidué, "*La problème de la réduction des singularités d'une variété algébrique*" (vol. 123, 302–330, 1951). At the time of publication, I happened to read the title in the volume in the Widener Library at Harvard; I at once searched out and found Oscar Zariski. "Oscar, they have solved your problem of the resolution in all dimensions." Professor Zariski at once wrote the editor of the *Mathematical Reviews*, asking to review this paper. His resulting review (MR, vol. 13, p. 67 ff) demolishes the purported resolution with analysis and striking counterexamples.

It required Zariski's expert understanding of the resolution problem to correct van der Waerden's mistake in accepting the Derwidué paper for publication. This is just one dramatic moment in the long and elaborate process in which algebraic geometry was gradually and totally transformed by the successful efforts of van der Waerden, Lefschetz, Walker, Zariski, Weil, Chevalley, Serre, Hironaka, Grothendieck, and many other mathematicians. One might hope that soon experts will examine and explore the many stages involved in this complex process, including the role of van der Waerden.

The historical situation of abstract algebra is a simpler one. I again emphasize the decisive contribution made by his two volumes on modern algebra. They dramatically changed the way algebra is now taught by providing a decisive example of a clear and perspicuous presentation. It is, in my view, the most influential text in algebra of the twentieth century.

Such assessments are uncertain and are perhaps to be avoided, as in van der Waerden's own judicious work in the history of mathematics. For example, his 1985 book *A History of Algebra from Al Kharisme to Emmy Noether* indeed gives a brief and objective description of Noether and her lectures on crossed products, on hypercomplex numbers, on division algebras, and on general representation theory (of groups and algebras). There is mention of Noether's use of van der Waerden's notes on some of her lectures. His 1985 book notes her profound influences on the development of modern algebra (p. 241) and quotes Herman Weyl's memorial address on page 217: "She could just utter a far-seeing remark like this, 'Norm residue symbol is nothing else than cyclic algebra' in her prophetic lapidary manner, out of her mighty imagination." We are fortunate that her imagination has been made accessible by van der Waerden.