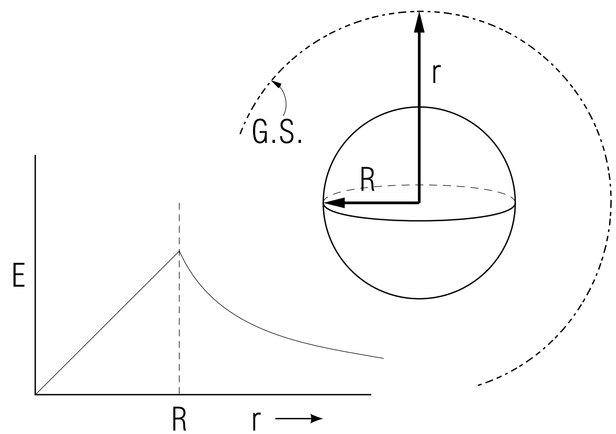
PHYSNET

MISN-0-132

GAUSS'S LAW FOR SPHERICAL SYMMETRY



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

GAUSS'S LAW FOR SPHERICAL SYMMETRY

by

Peter Signell

Title: **Gauss's Law for Spherical Symmetry**

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Input Skills:

1. Vocabulary: cylindrical symmetry, spherical symmetry (MISN-0-153).
2. For a charge distribution with spherical symmetry, indicate the direction of the electric field at any given point in space and draw the surface that includes all other points with the same magnitude of the electric field (MISN-0-153).

Output Skills (Knowledge):

K1. Vocabulary: Gauss's law, Gaussian surface, integration surface, surface integral, volume charge density.

K2. State the two rules for constructing a useful Gaussian surface.

K3. Derive Coulomb's law from Gauss's law, including all parts of the argument.

K4. State the basic physics difference between Gauss's law and Coulomb's law.

Output Skills (Problem Solving):

S1. Given a spherically symmetric charge distribution, use Gauss's law to determine the electric field at any specified point.

Post-Options:

1. "Gauss's Law Applied to Cylindrical and Planar Charge Distributions" (MISN-0-133).
2. "Electric Fields and Potentials Across Charge Layers and In Capacitors" (MISN-0-134).

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New authors, reviewers and field testers are welcome.

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1. Introduction

1a. The Importance of Gauss's Law. Gauss's law is generally thought of as being the integral form of one of the four great laws of electricity and magnetism. Gauss's law has exactly the same content as Coulomb's law, but, while Coulomb's law is formulated in terms of point charges, Gauss's law is formulated in terms of continuous charge distributions. Although either of these laws can be derived from the other, it is easier to derive Coulomb's law from Gauss's law than the other way around, and that makes us think of Gauss's law as the more basic form. Furthermore, it is easy to transform Gauss's law from its usual integral form to a differential form which is also widely used. In that form it is known as one of the four differential "Maxwell's equations" that govern all of electricity and magnetism.[1]

1b. Usefulness of the Law. Gauss's law is generally the option of choice for finding the electric field at various space points when the charges producing the electric field are symmetrically distributed. This situation often occurs in the design of new electronic components.

Even when the charges are not symmetrically distributed, Gauss's law can still be used to give a rough estimate for design exploration or for checking the result from a computer program.

For an example of the use of Gauss's law, look at the cross-sectional view of a coaxial cable shown in Fig. 1. Here the charge distributions have cylindrical symmetry. Using Gauss's law we can easily determine the electric field at any point inside or outside this cable, then use that knowledge to determine the way the cable affects signals that pass down it. By varying the cable parameters we can quickly optimize the design.[2]

For a wildly different example, Gauss's law provides a quick and exact proof of an important derivation in gravitation, a derivation that eluded Newton for nearly 20 years.[3]

[1]See "The Ampere-Maxwell Equation; The Displacement Current" (MISN-0-145) and "Maxwell's Equations" (MISN-0-146).

[2]The coaxial cable is treated in greater detail in "Gauss's Law Applied to Cylindrical and Planar Charge Distributions" (MISN-0-133).

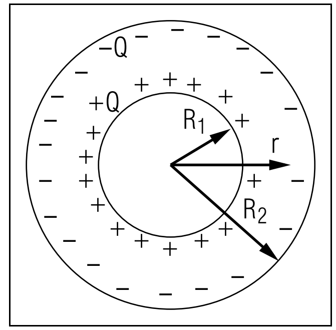


Figure 1. Cross section of a coaxial cable, showing the charges at some instant.

2. Gauss's Law

2a. Introduction. Our approach to Gauss's law will be to: (1) develop some ideas about the Gaussian surface that occurs in the law; (2) review the rules for finding the component of the electric field that is normal to the Gaussian surface, since this is what occurs in the law; (3) give a precise statement of the law; and then (4) give precise rules for choosing the Gaussian surface.

2b. Gaussian Surfaces. Before we proceed to the statement of Gauss's law, we introduce some notions about the Gaussian surface occurring in the mathematical statement of the law; further study of the law will then convert those notions into precise statements.

The surface that occurs in Gauss's law, called a Gaussian surface, is a closed imaginary surface that passes through the space point at which we want to know the electric field and which has a shape determined by the symmetry of the charge distribution. In all of the examples we will be dealing with, at least part of the Gaussian Surface will be the Electric Field Equi-Magnitude Surface (EMS) that goes through the point at which we wish to know the field.[4] The only difference is that additional surface areas must be added to the EMS, if necessary, in order to make it into a closed surface (one that totally encloses some volume). Here are typical Gaussian surfaces: (1) a sphere (see Fig. 2); (2) a cylinder with flat

[3]See "The Gravitational Field Outside a Homogeneous Spherical Mass" (MISN-0-109).

[4]For the necessary discussion of Equi-Magnitude Surfaces, see "Electric Fields From Symmetric Charge Distributions" (MISN-0-153).

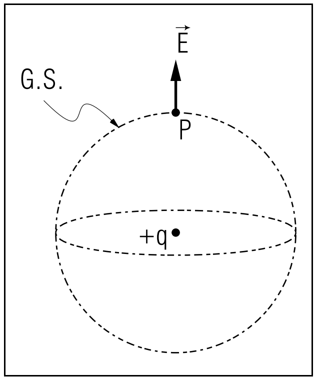


Figure 2. A Spherical Gaussian Surface.

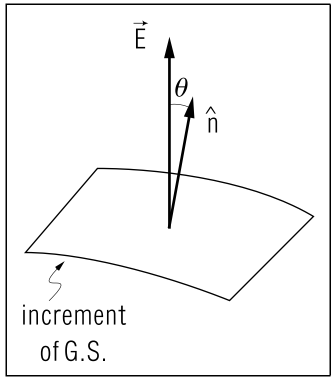


Figure 3. Illustration of an electric field vector not normal to the Gaussian surface.

circular ends that are normal to the axis of the cylinder; and (3) two flat sheets that are parallel and identical in shape, with the volume between the parallel sheets closed by a surface normal to the planes of the sheets.

2c. Normal Component of $\vec{E}$. Gauss's law uses only the component of the electric field that is normal (perpendicular) to the Gaussian surface. This component is denoted E_n. Where the electric field is not normal to the Gaussian surface we must find its normal component (see Fig. 3):

$$E_n = \vec{E} \cdot \hat{n} = E \cos\theta \,,$$

where $\hat{n}$ is the outward-directed unit vector normal to the Gaussian surface at the point under consideration.

2d. Statement of Gauss's Law. Gauss's law states that the integral of the normal component of the electric field over any closed surface is proportional to the net charge contained inside that surface. Thus Gauss's law contains the following quantities:

1. $\oint_S E_n \, dS \equiv$ the integral of the normal component of the electric field, E_n, over any closed surface "S";[5] and

[5] The circle on the integral sign reminds you that the surface must be a *closed* one, meaning that it must completely enclose some region of space. When you actually use Gauss's law, the suface integrals will reduce to just a constant times the surface areas of spheres, cylinders, or rectangles.

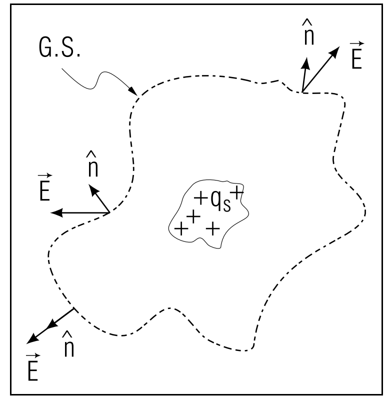


Figure 4. A fanciful Gaussian surface for illustrating Gauss's law.

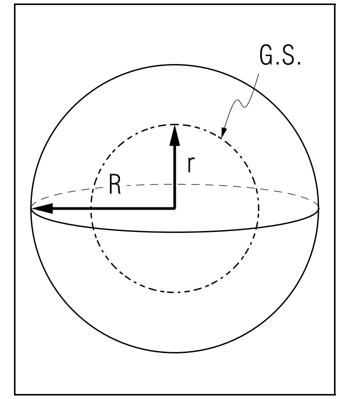


Figure 5. The Gaussian surface (G.S.) for applying Gauss's law to a spherically symmetric charge distribution that extends to radius R.

2. $q_S \equiv$ the total (net) electric charge contained within that same surface S.

The mathematical statement of Gauss's law is:

$$\oint_S E_n \, dS = 4\pi k_e \, q_S \,. \tag{1}$$

This law holds for any surface whatever, even one as wild as that shown in Fig. 4.

2e. Choosing the Gaussian Surface . There are two rules for constructing a useful Gaussian surface:

1. the surface must pass through the point at which you wish to know a particular component of the electric field, and the surface must be normal to that component at the point in question;

2. the surface must be a completely closed one and, at every point on the surface, the normal component of the electric field should either have the same value as at the point in question or be zero.

Using the above two rules, the electric field due to a symmetric charge distribution can generally be determined in a few lines in a few seconds.

Advice: People learning Gauss's law for the first time often use a proper imaginary Gaussian Surface on the left side of Eq. (1) but then incorrectly compute the charge within a real surface, different from the Gaussian Surface, for the right side of that equation. You *must* compute the charge within the imaginary Gaussian Surface for the right side of the equation, and it must be the same Gaussian Surface you used to evaluate the left side of the equation.

3. Spherically Symmetric Charge

For charge distributions having spherical symmetry, we choose our Gaussian surface to be a sphere of radius r centered on the center of the charge distribution (see Fig. 5). Then $\vec{E}$ is always normal to the surface: $E_n = \pm|\vec{E}|$, the sign depending on whether the electric field points outward or inward. Also, because the charge distribution looks the same from any point on the spherical Gaussian surface, the magnitude of E is constant over that surface. That is, E_n can be a function of radius but it will not be a function of position for fixed radius: $E_n = E(r)$. This constancy over a spherical surface S enables us to easily perform the integration:

$$\oint_S E_n\,dS = E(r)\oint_S dS = E(r)\,4\pi r^2\,.$$

Then by Gauss's law, Eq. (1), the electric field is:

$$4\pi r^2 E(r) = 4\pi k_e q_S(r)\,, \qquad \text{(spherical symmetry)}\,, \tag{2}$$

where $q_S(r)$ is the net amount of charge inside the spherical surface of radius r.

4. A Point Charge: Coulomb's Law

The application of Gauss's law to the case of a single point charge at the origin of some coordinate system produces the electric field appropriate to Coulomb's law. The single point charge is an especially simple case because any spherical surface centered on the charge will enclose all of the charge. Writing the value of the point charge as Q, Eq. (2) immediately becomes:

$$E(r) = k_e\frac{Q}{r^2}\,.$$

Inserting the radial direction of the field gives us:

$$\vec{E}(r) = k_e\frac{Q}{r^2}\,\hat{r}\,. \qquad \textit{Help: [S-1]}\,. \tag{3}$$

If we now place a charge q at the space-point $\vec{r}$, it experiences a force $\vec{F} = q\vec{E}(r)$ due to the charge Q at the origin. Using Eq. (3), we obtain an expression for the force between the two charges:

$$\vec{F} = k_e\frac{qQ}{r^2}\,\hat{r}\,,$$

which is Coulomb's law.

5. A Sphere of Uniform Charge

5a. Overview. Given a sphere with electric charge distributed uniformly throughout its volume, we can use Gauss's law to easily find the electric field at any point outside or even inside the sphere. All we need do is evaluate the net amount of charge inside the Gaussian surface at the radius of the point, then insert that charge into Eq. (2).

In the next two sections we will determine the electric field inside and outside a sphere of charge, but first we will give a quick qualitative run-through. Imagine starting with a very small Gaussian surface, one with radius $r \ll R$ where R is the radius of the sphere of charge. If we then increase r, the amount of charge enclosed by the Gaussian surface at r will increase like r^3 until R is reached. As r increases beyond R, the amount of charge enclosed by the surface will stay constant since we are now outside the sphere. However, the Gaussian surface's area [in the left side of Eq. (2)] will continually increase like r^2. Combining these radial dependencies, we find that the electric field should increase linearly with radius inside R and decrease as the inverse square of the radius outside R (see Fig. 6).

5b. Field Inside the Charge Distribution. We will here use Gauss's law to compute the electric field at a radius r that is less than the radius R of the surface of a sphere of uniformly distributed charge totaling Q (see Fig. 5). Since the charge distribution is spherically symmetric we can immediately pass from Eq. (1) to Eq. (2). For our case, the charge enclosed by the imaginary integration surface of radius r is: $q_S = Q\,(r/R)^3$ (*Help: [S-6]*). Then Eq. (2) becomes:

$$4\pi r^2 E(r) = 4\pi k_e q_S = 4\pi k_e Q\,r^3/R^3 \qquad (\text{for } r \le R)\,.$$

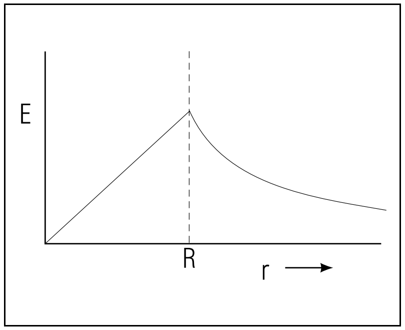


Figure 6. Radial dependence of the electric field due to a homogeneously charged sphere of radius R.

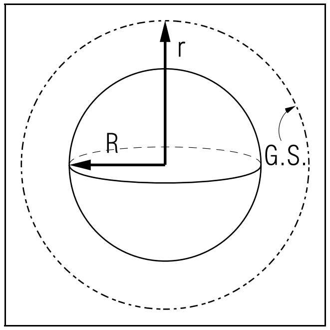


Figure 7. Gaussian Surface (G.S.) for applying Gauss's law outside a spherical charge distribution.

Since E's direction is radial,

$$\vec{E}(r) = k_e \frac{Qr}{R^3}\,\hat{r} \qquad (\text{for } r \leq R)\,.$$

This says that the field is zero at the center and increases linearly as we go out toward the edge (see Fig. 6).

5c. Field Outside the Distribution. In the region beyond a spherically symmetric distribution of charge, any integration surface (see Fig. 7) encloses the entire charge and thus Eq. (2) reduces to Coulomb's law. If the total charge is Q, then the charge enclosed by the surface in Eq. (2) is $q_S = Q$ (for $r \geq R$). Then putting in the direction gives us:

$$\vec{E}(r) = k_e \frac{Q}{r^2}\,\hat{r} \qquad (\text{for } r \geq R)\,.$$

This field is identical to the one that would be produced by a point charge Q located at the center of our spherical charge distribution (see Fig. 6). Note also that the expressions for the field inside and outside the sphere give the same answer at the surface of the charge sphere ($r = R$). They'd better!

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

Glossary

- **Gauss's law**: a statement that the integral of the normal component of the electric field over any closed surface is a known constant times the net charge inside that surface. The law is the integral form of one of Maxwell's equations.
- **Gaussian surface**: a surface enclosing a charge distribution, chosen so that: (1) the surface passes through the point at which you wish to know a particular component of the electric field, and the surface is normal to that component at the point in question; and (2) the surface is a completely closed one and, at every point on the surface, the normal component of the electric field either has the same value as at the point in question or is zero.
- **integration surface**: a surface over which some function is integrated.
- **surface integral**: the integral of some function over a surface S, written: $I = \int_S F(\vec{r})\,dS$, or $I = \int_S \vec{F}(\vec{r}) \cdot d\vec{S}$, depending on the phenomenon being described. If the surface is bounded by a line L, then the integral can be written $I = \int_{SL} \vec{F}(\vec{r})\,d\vec{S}$ where $\int_{SL}$ should be read as "the integral over the surface S that is bounded by the line L." The function $\vec{F}(\vec{r})$ varies according to where one is on the surface S. The infinitesimal element of area $d\vec{S}$ has a direction defined as the local normal to the surface: $d\vec{S} = \hat{n} dS$.
- **closed-surface integral**: the integral of some function over a closed surface S, called the integration surface; written as $I = \oint_S F(\vec{r})\,dS$, where the circle on the integral sign denotes a *closed* surface, $F(\vec{r})$ is some function of coordinate space, and dS is an infinitesimal element of area on the surface S.
- **volume charge density**: the amount of charge per unit volume, usually expressed in coulombs per cubic meter.

PROBLEM SUPPLEMENT

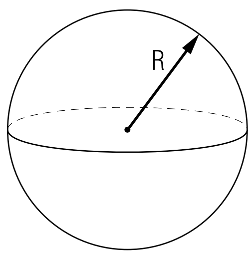
Note: Problems 3, 4, and 5 also occur in this module's *Model Exam*.

$k_e = 8.99 \times 10^9\,\mathrm{N\,m^2\,C^{-2}}$

1. Determine the electric field for a constant charge density throughout a spherical volume of radius R, as shown. Evaluate $\vec{E}$ at:

   a. $r = 2.00\,\mathrm{cm}$

   b. $r = 4.00\,\mathrm{cm}$



   where $R = 3.00\,\mathrm{cm}$ and the volume charge density is $2.00 \times 10^{-6}\,\mathrm{C/m^3}$. *Help: [S-7]*

2. Given a hollow spherical shell of charge:

   Region III: $0 < r < R_1$: no charge;

   Region II: $R_1 < r < R_2$: uniform charge distribution, total charge Q;

   Region I: $R_2 < r < \infty$: no charge.

   a. Draw a cross sectional view, labeling radii and regions.

   b. Determine $\vec{E}$ in Region I *Help: [S-3]*, Region II *Help: [S-5]* and Region III *Help: [S-2]* in terms of the given quantities.

   c. Check that $\vec{E}_I$, $\vec{E}_{II}$, and $\vec{E}_{III}$ agree at their common boundaries. *Help: [S-4]*

   d. Sketch $E(r)$ for this case and for $R_1 \to 0$.

3. Given a spherical shell that has a constant charge density within the shell (between the two surfaces of the shell). Assume there are no charges inside the inner surface of the shell or outside its outer surface. Use Gauss's law to determine the electric field inside, outside, and within the spherical shell. Give your answers in terms of the shell's total charge Q, inner radius a, outer radius b, and the radius r at which the electric field is to be evaluated.

4. At a distance of $2.00\,\mathrm{m}$ from the surface of a sphere of charge, what force would each of these particles feel if it was at rest with respect to the sphere?

   a. an electron

   b. a neutron (zero charge)

   The sphere's charge density is $-16\,\mathrm{C/m^3}$ and the radius of the sphere is $5.00\,\mathrm{cm}$.

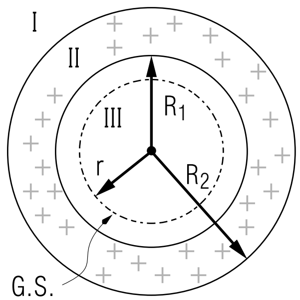
5. What is the ratio of the magnitude of the gravitational force to the magnitude of the electrostatic force on an electron due to a solid sphere with a volume charge density of $8.0 \times 10^2\,\mathrm{C/m^3}$ and a radius of $2.0\,\mathrm{cm}$? The distance between the electron and the center of the sphere is $2.0\,\mathrm{m}$. (Take the gravitational force as $F = 1.52 \times 10^{-41}\,\mathrm{N}$.)

Brief Answers:

1. a. $E = 1.51 \times 10^3\,\mathrm{N/C}$ *Help: [S-8]*

   b. $E = 1.27 \times 10^3\,\mathrm{N/C}$ *Help: [S-9]*

2. a.



   b. $\vec{E}_I = k_e Q \hat{r}/r^2$;

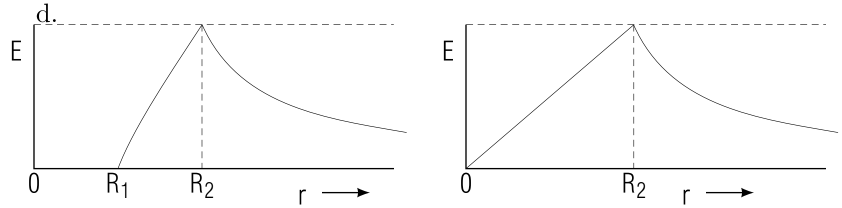
   $\vec{E}_{II} = k_e Q\,\hat{r}\left(r^3 - R_1^3\right)/\left[r^2\left(R_2^3 - R_1^3\right)\right]$;

   $\vec{E}_{III} = 0$.

   c. $\vec{E}_{III}(R_1) = \vec{E}_{II}(R_1) = 0$;

   $\vec{E}_{II}(R_2) = \vec{E}_I(R_2) = k_e Q \hat{r}/R_2^2$.

   d.



3. Outside: $\vec{E} = k_e \dfrac{Q}{r^2}\,\hat{r}$

   Within: $\vec{E} = k_e \dfrac{Q\left(r^3 - a^3\right)}{r^2\left(b^3 - a^3\right)}\,\hat{r}$

   Inside: $\vec{E} = 0$

4. a. $F = 2.8 \times 10^{-12}\,\mathrm{N}$, away from the sphere.

   (The answer is NOT $2.8 \times 10^{-15}\,\mathrm{N}$.)

   b. $\vec{F} = 0$

5. $F_G/F_E = 1.57 \times 10^{-30}$

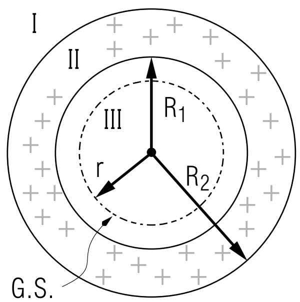
   (The answer is NOT 1.57×10^{-27}.)

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from TX-4, TX-5c)*

The definition of $\hat{r}$ is: $\hat{r} = \vec{r}/r$. Then $\vec{r} = r\hat{r}$ and both $\vec{r}$ and r have the dimensions of length.

S-2 *(from PS-problem 2b)*

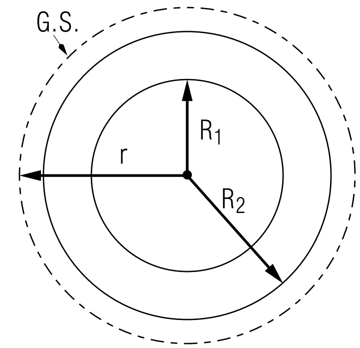


Examine the sketch and notice that there is no charge in Region III ($r \leq R_1$). Gauss's law refers only to the charge contained inside the integration surface. If we pick a spherical Gaussian surface in Region III, so its radius is $r \leq R_1$, it will contain no charge and Gauss's law leads to this form for Eq. (2):

$$4\pi r^2 E_{III}(r) = 0.$$

We can immediately conclude that $E_{III}(r) = 0$.

S-3 *(from PS-problem 2b)*



Outside the sphere, in Region I ($r \geq R_2$), our Gaussian surface encloses the entire sphere (see sketch) and so Gauss's law leads to this form for Eq. (2):

$$4\pi r^2 E_I(r) = 4\pi k_e q_S = 4\pi k_e Q\,,$$

where Q is the total charge of the hollow sphere. Then:

$$E_I(r) = k_e \frac{Q}{r^2}.$$

Putting in the radial direction of the field, $\vec{E}_I(r) = k_e \dfrac{Q}{r^2}\,\hat{r}$.

S-4 *(from PS-problem 2c)*

Simply evaluate $\vec{E}_I(R_2)$ and $\vec{E}_{II}(R_2)$ and show that they are the same. Then evaluate $\vec{E}_{II}(R_1)$ and $\vec{E}_{III}(R_1)$ and show that they are the same (see the Brief Answers).

S-5 *(from PS-problem 2b)*

We first define the relationship between volume charge density ρ and total charge Q for the hollow sphere, then assume ρ is constant:

$Q = \int \rho\, dV = \int_{R_1}^{R_2} \rho 4\pi r^2\, dr = \rho(4/3)\pi(R_2^3 - R_1^3)$

so

$$\rho = \frac{Q}{(4/3)\pi(R_2^3 - R_1^3)}.$$

Now a spherical Gaussian surface of radius r inside region II, where $R_1 \le r \le R_2$, will enclose a net amount of charge:

$$q_S = \int_{R_1}^{r} \rho 4\pi r'^2\, dr' = \rho(4/3)\pi(r^3 - R_1^3),$$

$$= \frac{Q}{(4/3)\pi(R_2^3 - R_1^3)} \times (4/3)\pi(r^3 - R_1^3) = Q\frac{r^3 - R_1^3}{R_2^3 - R_1^3}.$$

So Eq. (2) gives us:

$$4\pi r^2 E_{II}(r) = 4\pi k_e q_S = 4\pi k_e Q\frac{r^3 - R_1^3}{R_2^3 - R_1^3},$$

or

$$E_{II}(r) = k_e\frac{Q(r^3 - R_1^3)}{r^2(R_2^3 - R_1^3)}.$$

Putting in the radial direction of the field, $E_{II}(r) = k_e\dfrac{Q(r^3 - R_1^3)}{r^2(R_2^3 - R_1^3)}\,\hat{r}$.

S-6 *(from TX-5b)*

$$q_S = \frac{V_r}{V_R}Q = \frac{(4/3)\pi r^3}{(4/3)\pi R^3}Q$$

or

$$q_S = \int_0^r \rho 4\pi r'^2\, dr' = \rho(4/3)\pi r^3 = \frac{Q}{(4/3)\pi R^3}(4/3)\pi r^3$$

S-7 *(from PS-problem 1)*

The volume charge density for a uniformly distributed charge is $\rho = Q/V$, where Q is the total charge and V is the volume over which it is distributed. If the distribution is spherical with radius R, then $V = 4\pi R^3/3$.

S-8 *(from PS-problem 1a)*

$$E = \frac{(8.99 \times 10^9\,\mathrm{N\,m^2\,C^{-2}})(2.00 \times 10^{-6}\,\mathrm{C\,m^{-3}})(4\pi)(0.02\,\mathrm{m})}{3}$$
$$= 1.51 \times 10^3\,\mathrm{N/C}$$

S-9 *(from PS-problem 1b)*

$$E = \frac{(8.99 \times 10^9\,\mathrm{N\,m^2\,C^{-2}})(2.00 \times 10^{-6}\,\mathrm{C\,m^{-3}})(4\pi)(0.03\,\mathrm{m})^3}{3(0.04\,\mathrm{m})^2}$$
$$= 1.27 \times 10^3\,\mathrm{N/C}$$

MODEL EXAM

$k_e = 8.99 \times 10^9\,\mathrm{N\,m^2\,C^{-2}}$

electron charge $= -1.60 \times 10^{-19}\,\mathrm{C}$

1. See Exam Skills K1-K4 in this module's *ID Sheet*.
2. Use Gauss's law to determine the electric field inside, outside, and within a spherical shell of constant charge density. Give your answers in terms of the shell's total charge Q, inner radius a, outer radius b, and the radius r at which the electric field is to be evaluated.
3. At a distance of 2.00 m from the surface of a sphere of charge, what force would each of these particles feel if it was at rest with respect to the sphere?

   a. an electron

   b. a neutron (zero charge)

   The sphere's charge density is $-16\,\mathrm{C/m^3}$ and the radius of the sphere is 5.00 cm.
4. What is the ratio of the magnitude of the gravitational force to the magnitude of the electrostatic force on an electron due to a solid sphere with a volume charge density of $8.0 \times 10^2\,\mathrm{C/m^3}$ and a radius of 2.0 cm? The distance between the electron and the center of the sphere is 2.0 m. (Take the gravitational force as $F = 1.52 \times 10^{-41}\,\mathrm{N}$.)

Brief Answers:

1. See this module's *text*.
2. See Problem 3 in this module's *Problem Supplement*.
3. See Problem 4 in this module's *Problem Supplement*.
4. See Problem 5 in this module's *Problem Supplement*.