

**ON THE THEORY OF SUPERFLUIDITY\***By **N. BOGOLUBOV***Mathematical Institute, Academy of Sciences of the Ukrainian SSR  
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This paper presents an attempt of explaining the phenomenon of superfluidity on the basis of the theory of degeneracy of a non-perfect Bose-Einstein gas.

By using the method of the second quantization together with an approximation procedure we show that in the case of the small interaction between molecules the low excited states of the gas can be described as a perfect Bose-Einstein gas of certain "quasi-particles" representing the elementary excitations, which cannot be identified with the individual molecules.

The special form of the energy of a quasi-particle as a function of its momentum is shown to be connected with the superfluidity.

The object of this paper is an attempt to construct a consistent molecular theory explaining the phenomenon of superfluidity without assumptions concerning the structure of the energy spectrum.

The most natural starting point for such a theory seems to be the scheme of a non-perfect Bose-Einstein gas with a weak interaction between its particles.

It should be noted that similar attempts were done some time ago by Tisza and London to explain the phenomenon of superfluidity on the basis of the degeneracy of a perfect Bose-Einstein gas, but these attempts raised a counterblast of objections.

It has been pointed out, for example, that helium II has nothing to do with a perfect gas, because of the strong interaction between its molecules. However, this objection cannot be regarded as an essential one. Indeed, it is clear that a rigorous theoretical computation of the properties of a real liquid is hopelessly beyond the reach of a pure molecular theory based on usual "microscopic" equations of quantum mechanics. All we can require from a molecular theory of superfluidity, at least at the first stage of investigation, is to be able to account for the qualitative picture

of this phenomenon being based on a certain simplified scheme.

A really essential objection one can make against this idea is the following one. The particles of a degenerate perfect Bose-Einstein gas in the ground state cannot possess the property of superfluidity, since nothing prevents them from exchanging their momenta with excited particles colliding with them, and, therefore, from friction in their movement through the liquid.

In the present paper we try to overcome this difficulty and to show that under certain conditions the "degenerate condensate" of a "nearly perfect" Bose-Einstein gas can move without any friction with respect to the elementary excitations, with an arbitrary, sufficiently small velocity. It is to be pointed out that the necessity of considering the collective elementary excitations rather than individual molecules was suggested by L. Landau in his well known paper "Theory of Superfluidity of Helium II" where he, by postulating their existence in form of phonons and rotons, was enabled to explain the property of the superfluidity.

In our theory the existence and the properties of the elementary excitations follow directly from the basic equations describing the Bose-Einstein condensation of non perfect gases.

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1. Let us consider a system of  $N$  identical monoatomic molecules enclosed in a certain macroscopic volume  $V$  and subjected to Bose statistics.

Suppose the Hamiltonian of our system to be, as usually, of the form:

$$H = \sum_{(1 \leq i \leq N)} T(p_i) + \sum_{(1 \leq i < j \leq N)} \Phi(|q_i - q_j|),$$

where

$$T(p_i) = \frac{|p_i|^2}{2m} = \sum_{(1 \leq \alpha \leq 3)} \frac{(p_i^\alpha)^2}{2m}$$

represents the kinetic energy of the  $i$ th molecule and  $\Phi(|q_i - q_j|)$ —the mutual potential energy of the pair  $(i, j)$ .

Applying then the method of second quantization, let us present the basic equation in the form:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + \int \Phi(|q - q'|) \Psi^+(q') \Psi(q') dq' \cdot \Psi, \quad (1)$$

whereby

$$\Psi = \sum_f a_f \varphi_f(q); \quad \Psi^+ = \sum_f a_f^+ \varphi_f^+(q).$$

Here  $a_f$ ,  $a_f^+$  are conjugated operators with commutation rules of the well-known type:

$$a_f a_{f'} - a_{f'} a_f = 0; \\ a_f a_{f'}^+ - a_{f'}^+ a_f = \Delta_{f, f'} = \begin{cases} 1, & f = f' \\ 0, & f \neq f' \end{cases}$$

and  $\{\varphi_f(q)\}$  is a complete orthonormal set of functions:

$$\int \varphi_f^+(q) \varphi_{f'}(q) dq = \Delta_{f, f'}.$$

For the sake of simplicity, we shall further employ the system of eigenfunctions of the momentum operator of a single particle:

$$\varphi_f(q) = \frac{1}{V^{1/2}} e^{i \frac{(f \cdot q)}{\hbar}}; \quad (f \cdot q) = \sum_{(1 \leq \alpha \leq 3)} f^\alpha q^\alpha,$$

the operator  $N_f = a_f^+ a_f$  then representing the number of molecules with momentum  $f$ . For finite  $V$  vector  $f$  is obviously quantized. For example, under usual boundary conditions of the periodicity type,

$$f^\alpha = \frac{2\pi n^\alpha \hbar}{l},$$

where  $(n^1, n^2, n^3)$  are integers, and  $l$  denotes the side of a cube of the volume  $V$ .

However, since we are going to deal with thermodynamic volume properties of the system, we have to bear in mind a limiting process, such that when  $N \rightarrow \infty$ , the boundary of the container tends to infinity,  $V \rightarrow \infty$ , but the volume per molecule  $v = V/N$  remains constant. Therefore, we shall pass over in the final results to a continuous spectrum, replacing the sums  $\sum_f F(f)$  by integrals

$$\frac{V}{(2\pi\hbar)^3} \int F(f) df.$$

Equations (1) are the exact equations of the problem of  $N$  bodies. It becomes, therefore, a necessity in order to push forward the investigation of the motion of the considered system, to apply some approximation method based on a supposition that the energy of interaction is sufficiently small. According to this supposition, the potential function  $\Phi(r)$  is assumed to be proportional to a certain small parameter  $\varepsilon$ .

Later on we shall see what dimensionless quantity can be chosen for  $\varepsilon$ . For the present it suffices to notice that the above supposition, strictly speaking, corresponds to a neglect of the finiteness of molecular radius, since we do not take into account the intensive increase of  $\Phi(r)$  for small  $r$ , which causes the impenetrability of molecules. We shall see, however, that the results, which will be obtained, can be generalized so as to include the case of a finite molecular radius.

Now let us turn to the formulation of an approximate method. If there be no interaction at all, *i. e.* if  $\varepsilon$  be exactly zero, we could put at zero temperature:  $N_0 = N$ ,  $N_f = 0$  ( $f \neq 0$ ).

But in the case under consideration, when  $\varepsilon$  is small and the gas is in a weakly excited state, these relations are valid approximately, which means that the momenta of the overwhelming majority of molecules approach zero. Of course, the fact, that zero momentum is the limiting one for particles in the ground state, is due to a specific choice of the coordinate system: namely, we choose the system with respect to which our "condensate" is at rest.

Our approximation method, based on the above considerations, runs as follows.

1) Since  $N_0 = a_0^+ a_0$  is very large as compared with unity, the expression:

$$a_0 a_0^+ - a_0^+ a_0 = 1,$$

is small as compared with  $a_0^+$ ,  $a_0$  themselves, which enables us to treat them as ordinary numbers\* neglecting their non-commutability.

2) Let us put

$$\Psi = \frac{a_0}{\sqrt{V}} + \vartheta; \quad \vartheta = \frac{1}{\sqrt{V}} \sum_{(f \neq 0)} a_f e^{i \frac{(f \cdot q)}{\hbar}}$$

and consider  $\vartheta$  as a "correction term of the first order". Neglecting all the terms in equation (1), involving the second and higher powers of  $\vartheta$ , this being permissible since the excitations are supposed to be weak, we obtain the following basic approximate equations:

$$\begin{aligned} i\hbar \frac{\partial \vartheta}{\partial t} &= -\frac{\hbar^2}{2m} \Delta \vartheta + \frac{N_0}{V} \Phi_0 \vartheta + \\ &+ \frac{N_0}{V} \int \Phi(|q - q'|) \vartheta(q') dq' + \\ &+ \frac{a_0^2}{V} \int \Phi(|q - q'|) \vartheta^*(q') dq', \end{aligned} \quad (2)$$

$$i\hbar \frac{\partial a_0}{\partial t} = \frac{N_0}{V} \Phi_0 a_0,$$

where

$$\Phi_0 = \int \Phi(|q|) dq.$$

In order to pass from the operator wave function  $\vartheta$  to operator amplitudes let us apply Fourier's expansion;

$$\Phi(|q - q'|) = \sum_f \frac{1}{V} e^{i \frac{(f \cdot q - q')}{\hbar}} \nu(f). \quad (3)$$

From the radial symmetry of the potential function it follows that the amplitudes of this expansion:

$$\nu(f) = \int \Phi(|q|) e^{-i \frac{(f \cdot q)}{\hbar}} dq,$$

depend upon the length  $|f|$  of vector  $f$  only. The substitution of (3) into equation (2) gives:

$$\begin{aligned} i\hbar \frac{\partial a_f}{\partial t} &= \left\{ T(f) + E_0 + \frac{N_0}{V} \nu(f) \right\} a_f + \\ &+ \frac{a_0^2}{V} \nu(f) a_f^*; \end{aligned}$$

\* A similar procedure was used by Dirac in his book "The Principles of Quantum Mechanics" (second edition), cf. the end of § 63: Waves and Bose-Einstein particles.

$$E_0 = \frac{N_0}{V} \Phi_0,$$

whence on setting

$$a_f = e^{i \frac{E_0}{\hbar} t} b_f; \quad a_0 = e^{i \frac{E_0}{\hbar} t} b \quad (4)$$

we get:

$$\begin{aligned} i\hbar \frac{\partial b_f}{\partial t} &= \left\{ T(f) + \frac{N_0}{V} \nu(f) \right\} b_f + \frac{b^2}{V} \nu(f) b_f^*, \\ -i\hbar \frac{\partial b_f^*}{\partial t} &= \frac{(b^*)^2}{V} \nu(f) b_f + \left\{ T(f) + \frac{N_0}{V} \nu(f) \right\} b_f^*. \end{aligned} \quad (5)$$

Solving this system of two differential equations with constant coefficients, we find that the operators  $b_f$ ,  $b_f^*$  depend upon the time by means of a linear combination of exponentials possessing the form

$$e^{\pm i \frac{E(f)}{\hbar} t},$$

where\*

$$E(f) = \sqrt{2T(f) \frac{N_0}{V} \nu(f) + T^2(f)}. \quad (6)$$

Now let us observe that the inequality:

$$\nu(0) = \int \Phi(|q|) dq > 0, \quad (7)$$

implies the positiveness of the expression under the sign of the radical (6) since  $\epsilon$  is considered to be sufficiently small; thus  $b_f$ ,  $b_f^*$  prove to be periodical functions of time. On the contrary, if  $\nu(0) < 0$ , this expression is negative for small momenta and, therefore,  $E(f)$  receives complex values. As a consequence,  $b_f$ ,  $b_f^*$  will involve a real exponential increasing with time, whence it follows that the states with small  $N_f = b_f^* b_f$  are unstable.

In order to be sure in the stability of the excited states, let us restrict the class of possible types of interaction forces, supposing inequality (7) to be satisfied for all types we shall consider. It is interesting to note that the inequality (7) just represents the condition of thermodynamic stability of a gas at absolute zero.

\* We may remark that if we form the corresponding frequency  $\omega = \hbar E(f)$  and put:  $\hbar \rightarrow 0$ ,  $f/\hbar = k = \text{const}$  we obtain the classical formula for the dispersion of the frequency deduced by A. Vlasov, cf. Journal of Physics, 9, 25 (1945).

Indeed, at absolute zero the free energy coincides with the mean energy. The main term of the latter has the form

$$E = \frac{N^2}{2V} \int \Phi(|q|) dq,$$

since the correction terms (for instance, the mean kinetic energy) are proportional to higher powers of  $\varepsilon$ .

Therefore, the pressure  $P$  is expressed by the following formula:

$$P = -\frac{\partial E}{\partial V} = \frac{N^2}{2V^2} \int \Phi(|q|) dq = \frac{\rho^2}{2m^2} \nu(0),$$

where  $\rho = Nm/V$  represents the density of the gas.

This proves that the inequality (7) is equivalent to the condition of thermodynamic stability:

$$\frac{\partial P}{\partial \rho} > 0.$$

Finally, let us note that we can write instead of (6), with the same degree of accuracy:

$$\begin{aligned} E(f) &= \sqrt{2T(f) \frac{N}{V} \nu(f) + T^2(f)} = \\ &= \sqrt{\frac{|f|^2 \nu(f)}{mv} + \frac{|f|^4}{4m^2}}, \end{aligned} \quad (6')$$

since we take into account the main terms only; it follows from (6') for small momenta:

$$\begin{aligned} E(f) &= \sqrt{\frac{\nu(0)}{mv}} |f| (1 + \dots) = \\ &= \sqrt{\frac{\partial P}{\partial \rho}} |f| (1 + \dots), \end{aligned}$$

where by dots are denoted the terms vanishing together with  $f$ .

In what follows we shall attribute the positive sign to each square root we shall deal with. We thus have for small momenta:

$$E(f) = c |f| (1 + \dots), \quad (8)$$

where  $c$  denotes the velocity of sound at absolute zero.

On the contrary, for sufficiently large momenta  $E(f)$  can be expanded in powers of  $\varepsilon$ :

$$E(f) = \frac{|f|^2}{2m} + \frac{\nu(f)}{v} + \dots \quad (9)$$

Since  $\nu(f)$  tends to zero with increasing  $|f|$ ,  $E(f)$  is seen to approach the kinetic energy

$T(f)$  of a single molecule for sufficiently large momenta.

Returning now to equations (5), let us introduce new mutually conjugated operators  $\xi_f, \xi_f^*$  instead of  $b_f, b_f^*$  by means of the following relations

$$\xi_f = \frac{b_f - L_f b_{-f}^*}{\sqrt{1 - |L_f|^2}}; \quad \xi_f^* = \frac{b_{-f}^* - L_f^* b_f}{\sqrt{1 - |L_f|^2}}, \quad (10)$$

where  $L_f$  are numbers determined by the equalities:

$$L_f = \frac{V b^2}{N_0^2 \nu(f)} \left\{ E(f) - T(f) - \frac{N_0}{V} \nu(f) \right\},$$

so that

$$\begin{aligned} |L_f|^2 &= \left( \frac{(N_0/V) \nu(f)}{E(f) + T(f) + (N_0/V) \nu(f)} \right)^2; \\ 1 - |L_f|^2 &= \frac{2E(f)}{E(f) + T(f) + (N_0/V) \nu(f)}. \end{aligned} \quad (11)$$

Reversing (10) we obtain:

$$b_f = \frac{\xi_f + L_f \xi_{-f}^*}{\sqrt{1 - |L_f|^2}}; \quad b_f^* = \frac{\xi_{-f}^* + L_f^* \xi_f}{\sqrt{1 - |L_f|^2}}. \quad (12)$$

The substitution of these expressions into equations (5) gives:

$$i\hbar \frac{\partial \xi_f}{\partial t} = E(f) \xi_f; \quad -i\hbar \frac{\partial \xi_f^*}{\partial t} = E(f) \xi_f^*. \quad (13)$$

It can be immediately verified that the new operators satisfy the same commutation relations as the operators  $a_f, a_f^*$ :

$$\xi_f \xi_{f'} - \xi_{f'} \xi_f = 0; \quad \xi_f \xi_{f'}^* - \xi_{f'}^* \xi_f = \Delta_{f, f'}. \quad (14)$$

This alone suffices to conclude that the excited states of the given assemblage of molecules can be treated as a perfect gas composed of "elementary excitations"—"quasi-particles" with energy depending on the momentum by means of the relation:  $E = E(f)$ . These quasi-particles are described by the operators  $\xi_f, \xi_f^*$  in the same way as molecules were described by the operators  $a_f, a_f^*$ , and, therefore, they are also subjected to Bose statistics. The operator

$$n_f = \xi_f^* \xi_f$$

represents the number of quasi-particles with the momentum  $f$ .

The above conclusion becomes quite clear if we consider the total energy

$$H = H_{\text{kin}} + H_{\text{pot}}$$

where

$$\begin{aligned} H_{\text{kin}} &= -\frac{\hbar^2}{2m} \int \Psi^*(q) \Delta \Psi(q) dq, \\ H_{\text{pot}} &= \frac{1}{2} \int \Phi(|q - q'|) \Psi^*(q) \Psi^*(q') \Psi(q) \Psi(q') dq dq' = \\ &= \frac{1}{2V} \sum_f \nu(f) \int e^{i \frac{(f \cdot q - q')}{\hbar}} \Psi^*(q) \Psi^*(q') \Psi(q) \Psi(q') dq dq'. \end{aligned}$$

For the kinetic energy we obtain

$$H_{\text{kin}} = \sum_f T(f) a_f^\dagger a_f = \sum_f T(f) b_f^\dagger b_f.$$

In order to compute the potential energy with the assumed degree of accuracy, we shall disregard in the expression:

$$\Psi^*(q) \Psi^*(q') \Psi(q) \Psi(q') = \left( \frac{a_0^*}{\sqrt{V}} + \vartheta^*(q) \right) \left( \frac{a_0^*}{\sqrt{V}} + \vartheta^*(q') \right) \left( \frac{a_0}{\sqrt{V}} + \vartheta(q) \right) \left( \frac{a_0}{\sqrt{V}} + \vartheta(q') \right),$$

all terms of the third and higher order with respect to  $\vartheta$ ,  $\vartheta^*$ . This gives

$$H_{\text{pot}} = \Phi_0 \left\{ \frac{1}{2} \frac{N_0^2}{V} + \frac{N_0}{V} \sum_{f \neq 0} b_f^\dagger b_f \right\} + \frac{b^2}{2V} \sum_{f \neq 0} \nu(f) b_f^\dagger b_{-f}^\dagger + \frac{(b^+)^2}{2V} \sum_{f \neq 0} \nu(f) b_f b_{-f} + \frac{N_0}{V} \sum_{f \neq 0} \nu(f) b_f^\dagger b_f.$$

Noticing here that

$$\sum_{f \neq 0} b_f^\dagger b_f = \sum_{f \neq 0} N_f = N - N_0,$$

we can write with the same degree of accuracy:

$$\frac{1}{2} \frac{N_0^2}{V} + \frac{N_0}{V} \sum_{f \neq 0} b_f^\dagger b_f = \frac{1}{2} \frac{N^2}{V}.$$

Hence

$$H = \frac{N^2}{2V} \Phi_0 + \frac{b^2}{2V} \sum_{f \neq 0} \nu(f) b_f^\dagger b_{-f}^\dagger + \frac{(b^+)^2}{2V} \sum_{f \neq 0} \nu(f) b_f b_{-f} + \frac{N_0}{V} \sum_{f \neq 0} \nu(f) b_f^\dagger b_f + \sum_f T(f) b_f^\dagger b_f.$$

Replacing here the operators  $b_f$ ,  $b_f^\dagger$  by the operators  $\xi_f$ ,  $\xi_f^\dagger$  with the aid of relations (12), we obtain finally

$$H = H_0 + \sum_{f \neq 0} E(f) n_f; \quad n_f = \xi_f^\dagger \xi_f, \quad (15)$$

where

$$\begin{aligned} H_0 &= \frac{1}{2} \frac{N^2}{V} \Phi_0 + \sum_{f \neq 0} \frac{E(f) - T(f) - (N_0/V) \nu(f)}{2} = \\ &= \frac{1}{2} \frac{N^2}{V} \Phi_0 + \frac{V}{2(2\pi\hbar)^3} \int \left\{ E(f) - T(f) - \frac{N_0}{V} \nu(f) \right\} df. \end{aligned} \quad (16)$$

Thus we see that the total energy of the considered non-ideal gas consists of the energy of the ground state and the individual energies of each of the quasi-particles. It means that the quasi-particles do not interact with each other and thus form a perfect Bose-Einstein gas. The absence of interaction between the quasi-particles is evidently caused by the admitted approximation, namely by neglecting the terms of the third and higher order with

respect to  $\xi_f, \xi_f^*$  involved in the expression for energy. Therefore, the above results are valid for weakly excited states only.

On having taken into account the disregarded third-order terms in the expression for energy or the second-order terms in equations (13) which are to be considered as a small perturbation, we could discover a weak interaction between the quasi-particles. This interaction enables the assemblage of quasi-particles to attain the state of statistical equilibrium. Proceeding to the study of this state, let us first observe that the total momentum of quasi-particles:  $\sum_f f n_f$ , is conserved. To prove this consider the components of the total momentum. From

$$\sum_{(1 \leq i \leq N)} p_i^\alpha = \int \Psi^*(q) \left\{ -i\hbar \frac{\partial \Psi(q)}{\partial q^\alpha} \right\} dq = \sum_f f^\alpha a_f^\dagger a_f = \sum_f f^\alpha b_f^\dagger b_f$$

it follows, according to the transformation relations (12), that

$$\sum_{(1 \leq i \leq N)} p_i^\alpha = \sum_f f^\alpha \frac{(\xi_f^* + L_f^* \xi_{-f})(\xi_f + L_f \xi_{-f}^*)}{1 - |L_f|^2}.$$

The invariance of  $L_f, L_f^*$  with respect to the replacement of  $f$  by  $-f$  implies:

$$\begin{aligned} \sum_f f^\alpha \frac{L_f^* \xi_{-f} \xi_f}{1 - |L_f|^2} &= \sum_f f^\alpha \frac{L_f \xi_f^* \xi_{-f}^*}{1 - |L_f|^2} = 0, \\ \sum_f f^\alpha \frac{|L_f|^2 \xi_{-f} \xi_f^*}{1 - |L_f|^2} &= \sum_f f^\alpha \frac{|L_f|^2 (\xi_{-f} \xi_{-f}^* - 1)}{1 - |L_f|^2} = - \sum_f f^\alpha \frac{|L_f|^2 \xi_f^* \xi_f}{1 - |L_f|^2}. \end{aligned}$$

Hence we conclude that

$$\sum_{(1 \leq i \leq N)} p_i^\alpha = \sum_f f^\alpha n_f,$$

*i. e.* that the total momentum of the assemblage of molecules is equal to that of the assemblage of quasi-particles. Since the former is conserved, the sum  $\sum_f f n_f$  proves to be conserved too.

It is also easy to see that the total number of quasi-particles  $\sum_f n_f$  is not invariant; quasi-particles can appear and disappear.

For this reason we obtain in the usual way the following formula for average occupation numbers  $\bar{n}_f$  ( $f \neq 0$ ) in the state of statistical equilibrium:

$$\bar{n}_f = \left\{ A \exp \frac{E(f) - (f \cdot u)}{\Theta} - 1 \right\}^{-1}; \quad A = 1, \quad (17)$$

where  $\Theta$  is the temperature modulus, while  $u$  denotes an arbitrary vector. However, the length of this vector must have an upper

limit. In fact, since all average occupation numbers have to be positive, the inequality

$$E(f) > (f \cdot u),$$

and, therefore, the inequality

$$E(f) > |f| \cdot |u|$$

is satisfied for all  $f \neq 0$ .

But from the above properties of  $E(f)$  it follows that the ratio

$$\frac{E(f)}{|f|}$$

is a continuous positive function of  $|f|$ . This function is equal to  $c > 0$  at  $|f| = 0$  and is increasing as  $|f|^2/2m$ , for  $|f| \rightarrow \infty$ ; therefore, the minimum value of the considered ratio is essentially positive. Hence the condition for the positiveness of  $\bar{n}_f$  is equivalent to inequality

$$|u| \leq \min \frac{E(f)}{|f|}. \quad (18)$$

If the decrease of  $E(f)$  for small momenta be proportional not to the momentum itself, but to its square (as it is the case for the kinetic energy of a molecule), the right-hand side of the obtained inequality would

be equal to zero, this value being the only possible one for  $u$ . But in the case under consideration the vector  $u$  can be chosen arbitrarily, provided that its length is sufficiently small.

Now let us note that from the momentum distribution over the gas, composed of quasi-particles, given by formula (17), it follows that this gas is moving as a whole with the velocity  $u$ . At first, we have chosen such a coordinate system with respect to which the condensate (*i. e.* the assemblage of molecules in the ground state) is at rest. Inversely, by transition to a coordinate system with respect to which the quasi-particle gas, as a whole, is at rest, we can discover the motion of the condensate with the velocity  $u$ .

This relative motion goes on stationarily in the state of statistical equilibrium without any external forces. Hence we see that it is not accompanied by friction and thus represents the property of superfluidity\*.

As we have seen, the energy of a quasi-particle is asymptotically equal to  $c|f|$  for small momenta,  $c$  being the velocity of sound. Therefore, a quasi-particle for small momenta is just a phonon. When the momentum is increasing, the kinetic energy becomes large as compared with the binding energy of a molecule, and the energy of a quasi-particle tends continuously toward the individual energy of a molecule  $T(f)$ .

Thus we see that no division of quasi-particles into two different types, phonons and rotons, can even be spoken of.

2. Now let us consider the distribution of momenta over an assemblage of molecules, in the state of statistical equilibrium. We introduce a function  $W(f)$  defined in such a way that  $NW(f)df$  represents the average number of molecules whose momenta belong to an elementary volume  $df$  of the momentum space. This function is seen to be normalized by means of

$$\int W(f)df = 1. \quad (19)$$

\* By using the argument pointed out by Landau in §4 of his paper "Theory of Superfluidity of Helium II" it may be noted that the existence of superfluidity is directly evident from the above mentioned properties of the function  $E(f)$ , following from the inequality:  $v(0) > 0$ . This inequality can thus be considered as the condition of superfluidity.

Suppose, further,  $F(f)$  to be an arbitrary continuous function of the momentum. For the mean value of a dynamical variable

$$\sum_{(1 \leq i \leq N)} F(p_i),$$

we then obtain obviously:

$$N \int F(f)W(f)df. \quad (20)$$

On the other hand, we have for the same mean value:

$$\sum_f F(f)\bar{N}_f = \frac{V}{(2\pi\hbar)^3} \int F(f)\bar{N}_f df. \quad (21)$$

Thus the comparison of expressions (20), (21) gives:

$$NW(f) = \frac{V}{(2\pi\hbar)^3} \bar{N}_f = N \frac{v}{(2\pi\hbar)^3} \bar{b}_f^* \bar{b}_f,$$

whence we obtain, by expressing  $b_f, b_f^*$  through  $\xi_f, \xi_f^*$ :

$$\begin{aligned} W(f) &= \\ &= \frac{v}{(2\pi\hbar)^3} (1 - |L_f|^2)^{-1} (\xi_f^* + L_f^* \xi_{-f}) (\xi_f + L_f \xi_f^*) = \\ &= \frac{v}{(2\pi\hbar)^3} \frac{\bar{n}_f + |L_f|^2 (\bar{n}_{-f} + 1)}{1 - |L_f|^2}, \end{aligned} \quad (22)$$

where, in virtue of (17)

$$\bar{n}_f = \left\{ \exp \left( \frac{E(f) - (f \cdot u)}{\theta} \right) - 1 \right\}^{-1}. \quad (23)$$

Expression (22), obtained for the distribution function, holds evidently only for  $f \neq 0$ . Therefore, the complete expression for the momentum distribution function is seen to be, in virtue of the normalization condition (19):

$$W(f) = C \delta(f) + \frac{v}{(2\pi\hbar)^3} \frac{\bar{n}_f + |L_f|^2 (\bar{n}_{-f} + 1)}{1 - |L_f|^2}, \quad (24)$$

where  $\delta(f)$  is Dirac's  $\delta$ -function, and the number  $C$  is determined by the equation:

$$C = 1 - \frac{v}{(2\pi\hbar)^3} \int \frac{\bar{n}_f + |L_f|^2 (\bar{n}_{-f} + 1)}{1 - |L_f|^2} df. \quad (25)$$

$C$  is obviously equal to  $N_0/N$  since  $CN$  represents the average number of molecules with zero momentum.

It follows from (14) that

$$\frac{|L_f|^2}{1-|L_f|^2} = \frac{\left(\frac{N_0}{V} \nu(f)\right)^2}{2E(f) \left\{ E(f) + T(f) + \frac{N_0}{V} \nu(f) \right\}}; \quad (26)$$

$$\frac{1}{1-|L_f|^2} = \frac{E(f) + T(f) + \frac{N_0}{V} \nu(f)}{2E(f)}.$$

Hence the momentum distribution function at absolute zero is of the form:

$$W(f) = C\delta(f) + \frac{v}{(2\pi\hbar)^3} \frac{\left(\frac{N_0}{V} \nu(f)\right)^2}{2E(f) \left\{ E(f) + T(f) + \frac{N_0}{V} \nu(f) \right\}},$$

whereby

$$1 - C = \frac{v}{(2\pi\hbar)^3} \int \frac{\left(\frac{N_0}{V} \nu(f)\right)^2}{2E(f) \left\{ E(f) + T(f) + \frac{N_0}{V} \nu(f) \right\}} df. \quad (27)$$

We thus see that even at  $\Theta = 0$  only a fraction of molecules possesses momenta which are exactly zero, the rest being continuously distributed over the whole momentum spectrum.

As we have pointed out, our approximation method holds only when  $(N - N_0)/N = 1 - C \ll 1$ , and, therefore, the interaction of molecules must be sufficiently small in order to secure the smallness of the integral (27).

We can now make clear the meaning of the assumed smallness of the interaction.

Let us put  $\Phi(r) = \Phi_m F(r/r_0)$ , where  $F(\rho)$  is a function assuming together with its derivatives the values of the order of unity for  $\rho \sim 1$  and rapidly approaching zero as  $\rho \rightarrow \infty$ . Then

$$\nu(f) = \Phi_m r_0^3 \omega\left(\frac{|f|r_0}{\hbar}\right),$$

$\omega(x)$  being a function assuming the values  $\sim 1$  for  $x \sim 1$ , rapidly approaching zero as  $x \rightarrow \infty$ .

Transforming (27) to the dimensionless variables and reducing the three-dimensional integral to the one-dimensional one, we get:

$$\frac{N - N_0}{N} = \frac{v}{r_0^3} \eta \frac{1}{(2\pi)^2} \int_0^\infty \frac{\eta \omega^2(x) x dx}{\alpha(x) \{x\alpha(x) + x^2 + \eta \omega(x)\}}, \quad (28)$$

where

$$\alpha(x) = \sqrt{x^2 + 2\eta \omega(x)};$$

$$\eta = \frac{(r_0^3 N_0 / V) \Phi_m}{\hbar^2 / 2mr_0^2} \sim \frac{(r_0^3 / v) \Phi_m}{\hbar^2 / 2m_0^2}.$$

Now it is easy to see that for small values of  $\eta$  the integral in (28) is of the order of  $\sqrt{\eta}$ , and hence the condition of the validity of our method may be represented by the inequalities:

$$\eta \ll 1; \quad \frac{v}{r_0^3} \eta^{3/2} \ll 1,$$

i. e.

$$\frac{r_0^3}{v} \Phi_m \ll \frac{\hbar^2}{2mr_0^2}; \quad \sqrt[3]{\frac{r_0^3}{v} \Phi_m} \ll \frac{\hbar^2}{2mr_0^2}. \quad (29)$$

For  $\Theta > 0$  an analogous consideration of the general formula (24) leads us to a supplementary condition of the weakness of the excitation which requires the temperature to be small as compared with the  $\lambda$ -point temperature.

It is to be pointed out that the inequalities (29) automatically exclude the possibility of accounting for the short-range repulsion forces, since this would require to admit the strong increment of  $\Phi(r)$  in the vicinity of  $r = 0$ .

It seems nevertheless possible to modify the obtained results in the way to get them extended to the more real case of a low density gas of molecules possessing a finite radius.

To this aim we observe that the potential function  $\Phi(r)$  appears in our final formula only in the form of the expression:

$$\nu(f) = \int \Phi(|q|) e^{i\frac{(f \cdot q)}{\hbar}} dq, \quad (30)$$

proportional to the amplitude of Born's collision probability for binary collision.



Hence, as the interaction of molecules for the low density gas reveals itself principally by means of these binary collisions, it seems that expression (30) is to be replaced\* by the corresponding expression proportional to the amplitude of the exact probability of the binary collisions, calculated for the limiting case of zero density, *i. e.* we have to put:

$$\nu(f) = \int \Phi(|q|) \varphi(q, f) d \quad (31)$$

where  $\varphi(q, f)$  is the solution of the Schrödinger equation for the relative movement of an isolated pair of molecules:

$$-\frac{\hbar^2}{m} \Delta \varphi + \{\Phi(|q|) - E\} \varphi = 0,$$

going over into  $e^{i(f \cdot q)/\hbar}$  at infinity. The replacement of (31) instead of (30) in the formula for  $E(f)$  will lead us to the results, referring to low density gases.

This being admitted we see, *e. g.*, that the condition of the superfluidity  $\nu(0) > 0$  may be written in the form:

$$\int \Phi(|q|) \varphi(|q|) dq > 0, \quad (32)$$

where  $\varphi(|q|)$  is the radially symmetric solution of the equation:

$$-\frac{\hbar^2}{m} \Delta \varphi + \Phi(|q|) \varphi = 0$$

going over into unity at infinity.

In order to connect, as before, inequality (32) with the condition of the thermodynamic stability let us compute the principal term in the expansion of the free energy in powers of density at the absolute zero of temperature. The free energy at absolute zero being equal to the mean energy, we have the following expression for this energy per one molecule:

$$\mathcal{E} = \bar{T} + \frac{1}{2v} \int \Phi(|q|) G(|q|) dq, \quad (33)$$

where  $\bar{T}$  is the mean kinetic energy of one molecule and  $G(r)$  is the molecular distribution function normalized in the way that  $G(r) \rightarrow 1$  as  $r \rightarrow \infty$ . On the other hand, by using the virial theorem we see that the pressure  $P$  can be determined by the formula

$$Pv = \frac{2}{3} \bar{T} - \frac{1}{6v} \int \Phi'(|q|) |q| G(|q|) dq. \quad (34)$$

Let us now remark that for  $\Theta = 0$  the principal term in the expansion of the molecular distribution function in powers of density is obviously equal to  $\varphi^2(|q|)$ . Therefore, by neglecting in (33), (34) the terms of the second order in density one gets:

$$\mathcal{E} = \bar{T} + \frac{1}{2v} \int \Phi(|q|) \varphi^2(|q|) dq;$$

$$Pv = \frac{2}{3} \bar{T} - \frac{1}{6v} \int \Phi'(|q|) |q| \varphi^2(|q|) dq.$$

Hence, taking into account that

$$Pv = -v \frac{\partial \mathcal{E}}{\partial v},$$

we obtain the equation for the evaluation of the principal term in the expression for  $\bar{T}$ . In this way one gets:

$$\mathcal{E} = \frac{1}{2v} \int \Phi(|q|) \varphi(|q|) dq = \frac{\nu(0)}{2v}, \quad P = \frac{\nu(0)}{2v^2},$$

and thus the condition of superfluidity in the considered case of low density gas is also equivalent to the usual condition of the thermodynamic stability at absolute zero:

$$\frac{\partial P}{\partial v} < 0.$$

It can also be seen that the energy of a quasi-particle goes over again into  $c|f|$  for small  $f$ .

Consider now, for instance, the model of hard impenetrable spheres of the radius  $r_0/2$  and put:

$$\Phi(r) = +\infty, \quad r < r_0;$$

$$\Phi(r) = 0, \quad r > r_0.$$

\* I am indebted to L. D. Landau for this important remark.

Then by simple calculation we get:

$$\nu(0) = 2\pi \frac{\hbar^2 r_0}{m}.$$

$$\nu(0) = 2\pi \frac{\hbar^2 r_0}{m} + 2\pi \int_{r_0}^{\infty} r^2 \Phi(r) dr.$$

If a weak attraction between molecules is admitted here in the way that

$$\Phi(r) = +\infty, \quad r < r_0;$$

$$\Phi(r) = \varepsilon \Phi_0(r) < 0, \quad r > r_0.$$

$\varepsilon$  being a small parameter, we obtain, up to the terms of the order of  $\varepsilon^2$ :

Thus, the superfluidity in the considered gas model is conditioned by the play of repulsion and attraction forces, the first "encouraging" and the latter "hindering" it.

Let us note in conclusion that the extension of the present theory to the case of the real liquid seems possible if we be permitted to use such semi-phenomenological conceptions as that of the free energy of slightly non-equilibrium states.