Taylor Expansion and Derivative Formulas for Matrix Logarithms

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I give the derivation of formulas for the Taylor expansion and derivative of a matrix logarithm. They may well be in the literature, but I have not found derivations by using standard search tools, so thought it useful to document them as a memo.

I. INTRODUCTION

A query from "Backpacker" on Physics Forum says "A paper I'm reading states that: for positive hermitian matrices A and B, the Taylor expansion of $log(A+tB)$ at $t=0$ is

$$log(A+tB) = log(A) + t\int_0^\infty \frac{1}{B+zI}A\frac{1}{B+zI}dz + O(t^2) \quad . \tag{1}$$

However, there is no source or proof given, and I cannot seem to find a derivation of this identity anywhere!" (No citation for the paper containing this formula was given in the query.)

Derivations of this and related formulas are given in the following sections, with no attempt at mathematical rigor.

II. TAYLOR EXPANSION OF THE MATRIX LOG

Let x and y be noncommuting matrices or operators. Then the expansion

$$\frac{1}{x+y} = \frac{1}{x} - \frac{1}{x}y\frac{1}{x} + \frac{1}{x}y\frac{1}{x}y\frac{1}{x} - ... \tag{2}$$

is easily verified by multiplying through from the left (or from the right) by $x+y$. Replacing x by $x+a1$ and integrating the left hand side with respect to a from 0 to an upper limit U gives

$$\begin{aligned}\int_0^U da\frac{1}{x+y+a1} &= \log(x+y+U1) - \log(x+y) \quad , \\ \int_0^U da\frac{1}{x+a1} &= \log(x+U1) - \log(x) \quad ,\end{aligned} \tag{3}$$

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so that subtracting and substituting Eq. (2) gives

$$\begin{aligned}\log(x+y)-\log(x)-\log(x+y+U)+\log(x+U) &= \int_0^U da\Big(\frac{1}{x+a1}-\frac{1}{x+y+a1}\Big)\\ &= \int_0^U da\Big(\frac{1}{x+a1}y\frac{1}{x+a1}-\frac{1}{x+a1}y\frac{1}{x+a1}y\frac{1}{x+a1}+...\Big) \quad .\end{aligned} \tag{4}$$

Since

$$\log(x+y+U)-\log(x+U)=\log\big(1+(x+y)/U\big)-\log(1+x/U) \tag{5}$$

vanishes as $U\to\infty$, taking this limit gives the Taylor expansion formula

$$\log(x+y)-\log(x)=\int_0^\infty da\Big(\frac{1}{x+a1}y\frac{1}{x+a1}-\frac{1}{x+a1}y\frac{1}{x+a1}y\frac{1}{x+a1}+...\Big) \quad . \tag{6}$$

The first term in this expansion is the equation given in the query.

III. DERIVATIVE OF THE MATRIX LOG

Letting $y=dx(t)$ and dividing by dt, one gets

$$\frac{d}{dt}\log\big(x(t)\big)=\int_0^\infty da\frac{1}{x(t)+a1}\frac{dx(t)}{dt}\frac{1}{x(t)+a1} \quad , \tag{7}$$

even when $x(t)$ and $dx(t)/dt$ do not commute. Making the substitution $a=b/(1-b)$, a related formula can be given as an integral $\int_0^1 db$,

$$\frac{d}{dt}\log\big(x(t)\big)=\int_0^1 db\frac{1}{(1-b)x(t)+b1}\frac{dx(t)}{dt}\frac{1}{(1-b)x(t)+b1} \quad . \tag{8}$$

These formulas are analogs of the well-known formula for the derivative of the matrix exponential,

$$\frac{d}{dt}\exp\big(x(t)\big)=\int_0^1 da\exp\big(ax(t)\big)\frac{dx(t)}{dt}\exp\big((1-a)x(t)\big) \quad , \tag{9}$$

which is given in the Wikipedia article on the matrix exponential.