VAN DER WAERDEN'S THEOREM ON ARITHMETIC PROGRESSIONS

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Abstract. We give a simplified version of a proof of van der Waerden's theorem that if a sufficiently long interval of integers is partitioned into a specified number of parts, one will contain an arithmetic progression of given length.

1. The Theorem

By a segment of $\mathbb{Z}$ I will mean an interval $\Delta = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ for finite a and b. By the length of Δ I will mean $b - a + 1$, the number of elements of Δ.

Theorem 1.1 (van der Waerden [2])**.** *Given positive integers k and ℓ there is an integer $n(k, \ell)$ with the following property. If a segment Δ of $\mathbb{Z}$ of length at least $n(k, \ell)$ is the union of k sets $\Delta = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_k$ then for some j, Δ_j contains an arithmetic progression of length ℓ.*

Van der Waerden's original proof was quite complicated. In [1] Khinchin presents a simpler proof due to M. A. Lukomskaya. I will give here a simplified version of her proof.

We can clearly assume that the Δ_j are disjoint. I will write

$$\Delta = \Delta_1 \sqcup \Delta_2 \sqcup \cdots \sqcup \Delta_k \tag{1}$$

to indicate this. If such a partition has been given I will write $a \sim b$ if a and b lie in the same Δ_j. The theorem asserts the existence of a function $f(i) = a + id$ such that $f(i)$ lies in Δ and $f(i) \sim f(0)$ for all $0 \leq i < \ell$.

We prove the theorem by induction on ℓ. It is trivial for $\ell = 1$ and also for $\ell = 2$ since we only need to make sure some Δ_j has length at least 2. We therefore assume the theorem is true for some $\ell \geq 2$ and prove it for $\ell + 1$.

By an m–fold arithmetic progression of length ℓ I will mean a function of the form

$$f(i_1, \ldots, i_m) = a + \sum_{1}^{m} i_\nu d_\nu \tag{2}$$

with all $d_\nu > 0$ and $0 \leq i_\nu < \ell$ for all $1 \leq \nu \leq m$. As an immediate consequence of the theorem we see that if Δ has length at least $n(k, \ell^m)$ and $\Delta = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_k$ then some Δ_j will contain an m–fold arithmetic progression $f(i_1, \ldots, i_m)$ of length ℓ such that f takes distinct values. We need only find an ordinary arithmetic progression $a + id$ with $0 \leq i < \ell^m$ in Δ_j and write $i = \sum_0^{m-1} i_\nu l^\nu$ where $0 \leq i_\nu < \ell$.

The idea of the proof is to construct an m–fold arithmetic progression $f(i_1, \ldots, i_m)$ of length ℓ in some Δ_j using only the induction hypothesis. This will be done in

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such a way that f can be extended to $i_\nu = \ell$ with values still in Δ. We then take a suitable subprogression.

2. The Proof

As observed in the last section it is sufficient to consider partitions $\Delta = \Delta_1 \sqcup \Delta_2 \sqcup \cdots \sqcup \Delta_k$ where the Δ_j are disjoint and we write $a \sim b$ if a and b lie in the same set Δ_j.

Lemma 2.1. *Suppose Theorem 1.1 holds for a given value of $\ell \geq 2$. Then for any $m > 0$ there is an integer $N(k, m, \ell)$ with the following property. If a segment Δ has length at least $N(k, m, \ell)$ and $\Delta = \Delta_1 \sqcup \Delta_2 \sqcup \cdots \sqcup \Delta_k$ then Δ contains an m–fold arithmetic progression of length $\ell+1$, $f(i_1, \dots, i_m) = a + \sum_1^m i_\nu d_\nu$, with all $d_\nu > 0$ and $0 \leq i_\nu \leq \ell$ such that if $i_1, \dots, i_s < \ell$ then $f(i_1, \dots, i_s, j_{s+1}, \dots, j_m) \sim f(0, \dots, 0, j_{s+1}, \dots, j_m)$ for all $j_{s+1}, \dots, j_m$.*

Proof. We use induction on m. For $m = 1$ we can take $N(k, 1, \ell) = 2n(k, \ell)$. If the length of Δ is at least $2n(k, \ell)$ write $\Delta = \Delta' \sqcup \Delta''$ where Δ' and Δ'' are contiguous intervals of length at least $n(k, \ell)$. Choose an arithmetic progression $a + id$ with $d > 0$ and $0 \leq i < \ell$ in Δ'. Now a and $a + d$ lie in Δ' because $\ell \geq 2$ so $d \leq n(k, \ell)$. Since $a + (\ell - 1)d$ lies in Δ' it follows that $a + \ell d$ lies in Δ.

Suppose the lemma is true for a given m. Let $q = N(k, m, \ell)$ and define $N(k, m+1, \ell) = q + 2n(k^q, \ell)$. Suppose Δ has length at least $N(k, m+1, \ell)$. Write $\Delta = \Delta' \sqcup \Delta''$ where Δ' and Δ'' are contiguous intervals, Δ' has length $2n(k^q, \ell)$ and Δ'' has length at least q. Define an equivalence relation on Δ' by $x \approx y$ if $x + z \sim y + z$ for all z with $0 \leq z < q$. This relation has k^q possible equivalence classes. Since Δ' has length $2n(k^q, \ell)$ we can find an arithmetic progression $a + id$ with $d > 0$ and $0 \leq i \leq \ell$ in Δ' such that $a + id \approx a + jd$ if $0 \leq i, j < \ell$. Therefore $c + id \sim c$ for all $a \leq c < a + q$ and $0 \leq i < \ell$. Since $[a, a + q)$ lies in Δ and has length q, the induction hypothesis gives us an m–fold arithmetic progression $g(i_1, \dots, i_m) = b + \sum_1^m i_\nu d_\nu$ in $[a, a + q)$ such that for $i_1, \dots, i_s < \ell$ we have $g(i_1, \dots, i_s, j_{s+1}, \dots, j_m) \sim g(0, \dots, 0, j_{s+1}, \dots, j_m)$ for all $j_{s+1}, \dots, j_m$. Define $f(i_0, \dots, i_m) = i_0 d + g(i_1, \dots, i_m)$. If $i_0, \dots, i_s < \ell$ with $s > 0$ then $f(i_0, \dots, i_s, j_{s+1}, \dots, j_m) = i_0 d + g(i_1, \dots, i_s, j_{s+1}, \dots, j_m) \sim g(i_1, \dots, i_s, j_{s+1}, \dots, j_m) \sim g(0, \dots, 0, j_{s+1}, \dots, j_m) = f(0, \dots, 0, j_{s+1}, \dots, j_m)$ for all $j_{s+1}, \dots, j_m$ as required. □

To prove the theorem we now set $n(k, \ell + 1) = N(k, k, \ell)$. Given a partition $\Delta = \Delta_1 \sqcup \Delta_2 \sqcup \cdots \sqcup \Delta_k$ find f as in the lemma and set $a_r = f(0, \dots, 0, \ell, \dots, \ell)$ with r zeros where $0 \leq r \leq k$. There are $k + 1$ of these so two, say a_r and a_s, lie in the same Δ_j. Say $r < s$ and define $h(i) = f(0, \dots, 0, i, \dots, i, \ell, \dots, \ell)$ with r 0's, $s - r$ i's, and $k - s$ ℓ's. Then $h(i) \sim h(0)$ for $0 \leq i < \ell$ and $h(\ell) = a_r \sim a_s = h(0)$ so h is the required arithmetic progression.

References

1. A. Y. Khinchin, Three Pearls of Number Theory, Dover Publications, Mineola, NY 1998.
2. B. L. van der Waerden, Beweis einer Baudetsche Vermuting, Nieuw Arch. Wiskunde 15 (1927), 212–216.

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