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The scientific work of Arnold Walfisz

by

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Professor A. Walfisz published scientific papers, expository papers and books on various topics of the theory of numbers, altogether a 100 in number. We shall begin with his work on the lattice points theory, Walfisz's favourite subject, to which he has contributed about 30 papers and a large book.

Let $A_k(x)$ stand for the number of lattice points of the sphere $\xi_1^2+\ldots+\xi_k^2 \leqslant x$, where $k \geqslant 2$. Van der Corput has shown that for arbitrary $\varepsilon > 0$

$$A_2(x)-\pi x = O(x^{\theta+\varepsilon}),$$

where θ is a constant less than $1/3$. It is proved in [3], on expanding $A_2(x)-\pi x$ in a Fourier series, that $\theta = 37/112$.

As is well-known, Hardy published in 1918 — without a detailed proof — the important formula for the number of representations of positive integer n as a sum of k squares of integers

$$r_k(n) = \pi^{k/2}\Gamma^{-1}\left(\frac{k}{2}\right)n^{k/2-1}\sum_{q=1}^{\infty}\sum_{\substack{h \bmod q\\(h,q)=1}}\left(\frac{S(h,q)}{q}\right)^k \exp\left(-\frac{2\pi i n h}{q}\right)+O(n^{k/4}),$$

where $S(h,q)$ are Gaussian sums. Walfisz was the first to have proved this formula with all particulars (in [5], using Hardy-Littlewood's method). He also generalized it to the case of the representation of n by positive quadratic forms in k variables with integer coefficients. From this formula, he derives the best possible estimate

$$P(x) = O(x^{k/2-1}),$$

where $P(x)$ denotes the difference of the number of lattice points of a k-dimensional rational ellipsoid $Q \leqslant x$ and its volume. This estimate holds if $k \geqslant 8$, but in case of a sphere if $k \geqslant 5$. Paper [5] proved to be not only a starting point of further investigations of Walfisz, but also stimulated a great deal of work by other scholars: Ch. H. Müntz, E. Landau, H. Petersson, V. Jarník, H. D. Kloosterman.

In [7] it is proved, by using the method of [3], that for an arbitrary $\varepsilon > 0$

$$A_3(x) = \tfrac{4}{3}\pi x^{3/2} + O(x^{43/58+\varepsilon}).$$

In [12], Walfisz showed by means of his estimate for a summatorial function connected with the sum of divisors, that

$$A_4(x) = \frac{\pi^2}{2}x^2 + O\left(\frac{x\log x}{\log\log x}\right).$$

This estimate sharpens a previous one, due to Landau. Further, it is shown that

$$A_4(x) - \frac{\pi^2}{2}x^2 = \Omega(x\log\log x)\ (^1).$$

The same paper contains also an improvement of the above-mentioned estimate of [3], to the effect that

$$A_2(x) = \pi x + O(x^{163/494}).$$

In [13], one finds a number of Ω-results for $P(x)$.

Paper [18] is the first by any writer in this field to deal with the simplest irrational ellipsoids $Q \leqslant x$, namely those given by $Q(x_1, \ldots, x_k) = \alpha x_1^2 + Q_1(x_2, \ldots, x_k)$, where α is a positive irrationality. It is proved that for $k \geqslant 10$ and an arbitrary α,

$$P(x) = o(x^{k/2-1}),$$

while for $k \geqslant 10$ and almost all α's

$$P(x) = O(x^{k/2-6/5}\log^{1/4}x).$$

Paper [23] deals with k-dimensional irrational ellipsoids $Q \leqslant x$ of diagonal type, and it is proved there that for $k = 5$

$$P(x) = O(x^{3/2}).$$

In [33], on using the method of H. Weyl of estimation of exponential sums and the theory of modular forms of Hecke for four-dimensional integer ellipsoids, Walfisz derives the estimate

$$P(x) = O\left(\frac{x\log x}{\log\log x}\right).$$

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(1) Here and in the following by $f(x) = \Omega(s)$, $f(x) = \Omega_+(s)$ and $f(x) = \Omega_-(s)$, for $s > 0$, we understand that there is a constant K such that respectively for an infinity of values of x, $|f(x)| > Ks$, $f(x) > Ks$ and $f(x) < -Ks$.

This is an improvement on the previously known estimate of Kloosterman which held only for diagonal four-dimensional ellipsoids.

In [43], using again the theory of modular forms of Hecke, it is shown for four-dimensional rational ellipsoids that

$$\int_0^x P^2(u)\,du = \varkappa x^2 + O(x^{5/2}\log^2 x),$$

where $\varkappa = \varkappa_Q$ is a certain positive constant. Further, the constant $\varkappa$ for the forms $Q = x_1^2+x_2^2+x_3^2+x_4^2$, $x_1^2+x_2^2+2x_3^2+2x_4^2$, $x_1^2+2x_2^2+2x_3^2+4x_4^2$, $x_1^2+2x_2^2+4x_3^2+8x_4^2$ is determined, and in case of the first three forms the remainder is reduced to $O(x^{5/2}\log x)$.

In paper [49] a method of Jarník, concerning double contour integrals, has been perfected and the estimation of the integral of [43] follows directly without recourse to the theory of Hecke's modular forms. What is more, in place of the former constant $\varkappa$, the singular series

$$\mathfrak{S}(Q) = \frac{\pi^2}{6D}\sum_{\substack{h,k=1\\(h,k)=1}}^{\infty}\frac{|S(h,k)|^2}{k^6h^2}$$

is introduced. Here D is the determinant of Q,

$$S(h,k) = \sum_{a_1,\ldots,a_4=0}^{k-1}\exp 2\pi i\cdot\frac{h}{k}Q(a_1,\ldots,a_4).$$

This series can be summed for all forms $Q = x_1^2+x_2^2+d(x_3^2+x_4^2)$ (d is a positive integer). Let $r_s(n)$ denote the number of representations of integer n by a form Q of $s = 3$ or $s = 4$ variables. It is proved in the paper that

$$\sum_{0\leqslant n\leqslant x} r_4^2(n) = \mathfrak{T}_4(Q)x^3 + O(x^2\log^3 x),$$

$$\sum_{0\leqslant n\leqslant x} r_3^2(n) = \mathfrak{T}_3(Q)x^2 + O(x^{3/2}\log x),$$

where $\mathfrak{T}_4(Q)$ and $\mathfrak{T}_3(Q)$ are certain singular series of type $\mathfrak{S}(Q)$. The first of these can be summed for all forms $Q = x_1^2+x_2^2+d(x_3^2+x_4^2)$, the second for $Q = x_1^2+d(x_2^2+x_3^2)$.

In [52], using the method of I. M. Vinogradov for trigonometric sums, Walfisz improves the remainder-term of the above-mentioned formula of [33] to $O(x\log^{4/5}x\log\log x)$.

In [59] the series $\mathfrak{S}(Q)$ and $\mathfrak{T}(Q)$ of [49] are summed for forms $Q = ax_1^2+bx_2^2+cx_3^2+dx_4^2$, $ax_1^2+bx_2^2+cx_3$. Further, one finds o-results for the error-term $P_k(x)$ corresponding to the number of lattice points of the k-dimensional sphere with even $k \geqslant 8$.

It is proved in [63] that for $k \equiv 0 \pmod 4$, $k \geqslant 8$,

$$P_k(x) = M_k\left\{(1-2^{-k/2})\zeta\left(\frac{k}{2}\right)\right\}^{-1}\left(-\sum_{u=1}^{\infty} u^{1-k/2}\psi\left(\frac{x}{u}\right)+\right.$$
$$\left.+(-1)^{k/4}\sum_{\substack{n=1\\ n\equiv 0 (\mathrm{mod}\, 2)}}^{\infty} n^{1-k/2}\left\{\psi\left(\frac{x}{n}\right)-2\psi\left(\frac{x}{2n}\right)\right\}\right)x^{k/2-1}+O(x^{(k-3)/2}),$$

and for $k \equiv 2 \pmod 4$, $k \geqslant 10$,

$$P_k(x) = M_k\left\{L\left(\frac{k}{2}\right)\right\}^{-1}\left(-\sum_{u=1}^{\infty}(-1)^{(u-1)/2}u^{1-k/2}\psi\left(\frac{x}{u}\right)+(-1)^{(k-2)/4}\sum_{n=1}^{\infty}(2n)^{1-k/2}\times\right.$$
$$\left.\times\left\{\psi\left(\frac{x}{n}\right)-\psi\left(\frac{x}{2n}\right)-2\psi\left(\frac{x-n}{4n}\right)\right\}\right)x^{k/2-1}+O(x^{(k-3)/2}),$$

where u is odd, $M_k = \pi^{k/2}\Gamma^{-1}\left(\frac{k}{2}\right)$, $\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$, $L(x) = \sum_{u=1}^{\infty}(-1)^{(u-1)/2}u^{-x}$, $\psi(y) = y-[y]-\frac{1}{2}$.

Further, in case of $k \equiv 0 \pmod 8$ there are obtained exact values of the numbers

$$P_k = \limsup_{n=\infty} \frac{2P_k(n)}{M_k n^{k/2-1}}, \qquad \varrho_k = \liminf_{n=\infty} \frac{2P_k(n)}{M_k n^{k/2-1}}.$$

The corresponding values for the remaining k's are not known. Furthermore, there is no method known enabling one, given a k, to find these values approximately with an arbitrarily prescribed accuracy. Paper [65] contains rather good approximations to P_k, ϱ_k for $k \equiv 4 \pmod 8$, $k \geqslant 12$.

Paper [66] gives approximations to P_k, ϱ_k for $k \equiv 2 \pmod 4$, $k \geqslant 10$, with greater error than those obtained for $k \equiv 4 \pmod 8$.

Papers [67] and [68] contain simpler proofs of estimates of $P_k(x)$ dealt with in [63].

A formula of [63] displays the main oscillatory term of order x^{k-1} of the function $P_{2k}(x)$; the next one, of order x^{k-2}, is found in [69].

In [79] we find in a certain sense approximations to the numbers P_k and ϱ_k for odd k's.

In [85] it is proved, using a theorem of L. K. Hua, that for four-dimensional rational ellipsoids the following estimate holds

$$P(x) = O\left(x\log^{3/4}x(\log\log x)^{1/2}\right).$$

This estimate is refined to $O(x\log^{2/3}x)$ in [87] on using a certain new method of I. M. Vinogradov (1958) for trigonometric sums.

In the monographs [4] and [5], using methods of Vinogradov and Korobov (1958) concerning exponential sums, Walfisz improves his estimate of [12] to

$$A_4(x) = \frac{\pi^2}{2}x^2+O(x\log^{2/3}x).$$

Paper [19] provides simple proofs of certain theorems by Petersson concerning lattice points of multi-dimensional spheres. In [88] and [89], using these theorems by Petersson and estimates of Lursmanaschwili generalizing those of [63], Walfisz found new O-results for $P_{2k}(x)$ and $P_{2k+1}(x)$, where $k \geqslant 3$, with remainders $O(x^{k/2}\log x)$ and $O(x^{k/2+1/4}\log x)$ respectively.

All the above-quoted results of Walfisz and along with them a number of closely related results of other scholars, have been incorporated with detailed proofs in a lengthy book published in German [3] by the Polish Academy of Sciences in Warsaw and in Russian [4] by the Georgian Academy of Sciences in Tbilissi.

Walfisz devoted a considerable series of papers to questions on the additive theory of numbers including the additive theory of primes. Thus in [40] he displays the formula

$$N_{r,s}(n) = \pi^{r/2}\Gamma^{-1}(r/2+s)n^{r/2+s-1}\log^{-s}n\,\mathfrak{S}_{r,s}(n)+o(n^{r/2+s-1}\log^{-s}n),$$

where $N_{r,s}(n)$ stands for the number of representations of n as a sum of r squares of integers and s primes ($r \geqslant 5$, $s \geqslant 1$) and $\mathfrak{S}_{r,s}(n)$ is the corresponding singular series. The formula was obtained previously by G. K. Stanley, however, only on the extended Riemann-hypothesis.

In [42], it is proved that for $r \geqslant 4$ and an arbitrary $\varepsilon > 0$

$$N_{r,1}(n) = \pi^{r/2}\Gamma^{-1}(r/2-1)\mathfrak{S}_{r,1}(n)\int_2^n (n-u)^{r/2-2}\,\mathrm{Li}(u)\,du+O(n^{r/2}\log^{-1/\varepsilon}n)$$

and

$$Q(n) = \prod_p\left(1-\frac{1}{p(p-1)}\right)\prod_{p|n}\left(\frac{p^2-p}{p^2-p-1}\right)\mathrm{Li}(n)+O(n\log^{-1/\varepsilon}n),$$

where $Q(n)$ denotes the number of representations of n as a sum of a square-free number and a prime number. The first estimate improves its particular case $s = 1$ given in [40]. The second estimate is incomparably better than the one obtained previously by A. Page. This improvement has been effected by the so-called lemma of Siegel-Walfisz about primes in an arithmetical progression, which plays such a prominent rôle in the additive theory of primes.

It is proved in [47] that 1) almost every number $\equiv 4 \pmod{24}$ can be expressed as a sum of four prime squares; 2) almost every number $\equiv 3, 27, 51, 99 \pmod{120}$ can be expressed as a sum of three prime squares; 3) for every class of residues (mod 120) there can be found a σ with $3 \leqslant \sigma \leqslant 8$ such that almost every number of this class can be expressed as a sum of σ prime squares, but almost no number of the class can be expressed as a sum of less than σ prime squares. The proof rests on an estimate of I. M. Vinogradov and on the above-quoted lemma of Siegel-Walfisz. The disadvantage of this lemma is that it depends on a theorem of Siegel whose proof is rather intricate and uses tools from the theory of algebraic fields, while all of its applications refer to rational primes. In order to overcome this disadvantage Walfisz proves [48] a new lemma concerning primes in an arithmetical progression; this new lemma is more complicated than that of Siegel-Walfisz, but it dispenses with Siegel's theorem. Using this new lemma, Walfisz obtains all basic results of the additive theory of prime numbers, admittedly with slightly weaker remainders, previously proved by I. M. Vinogradov, T. Estermann, H. Davenport, H. Heilbronn and himself.

Paper [50] is the first to give an elementary proof of formulae due to B. Boulyguine for the number of representations of integers as a sum of 18, 20, 22 and 24 squares.

In papers [53], [54] and [58] there are obtained, using an elementary method based on certain trigonometrical identities, exact formulae for the number of representations by 73 quaternary quadratic forms.

In [57], it is proved starting from the familiar theorem of Goldbach-Vinogradov, that for $r \geqslant 3$,

$$N_r(n) = \frac{n^{r-1}}{(r-1)!\log^r n} S_r(n) + o\left(\frac{n^{r-1}}{\log^r n}\right),$$

where $N_r(n)$ stands for the number of representations of n as a sum of r primes.

Paper [74] contains this asymptotic formula: for $k \geqslant 8$

$$r_k(m, n) = \pi^{(k-1)/2} k^{1-k/2} \Gamma^{-1}\left(\frac{k-1}{2}\right) \Delta_k^{(k-3)/2} S_k(m, n) + O(n^{k/2-2}\log n),$$

where $r_k(m, n)$ denotes the number of representations of two given integers m and n as a sum of k integers and their squares respectively, $S_k(m, n)$ — the corresponding singular series, $\Delta_k = kn - m^2$. An accessible proof has been published in the expository paper [72].

Let $\nu(P)$ be the number of representations of an odd $P \geqslant 6$ as a sum of three primes and let $S(P)$ be the corresponding singular series. It is proved in [82] and [84] that for arbitrary integer $m \geqslant 3$

$$\nu(P) = S(P)P^2 \sum_{q=3}^{m} c_q (\log P)^{-q} + O\left(P^2 (\log P)^{-m-1}\right),$$

where c_q's are real numbers depending only upon q. Furthermore, for $q = 3, 4, 5, 6, 7$ the constants c_q have been calculated. The formula itself is more precise than the well-known one due to I. M. Vinogradov.

A number of Walfisz' papers have been concerned with the zeta-function and other functions defined by Dirichlet series. A long time ago it was proved in paper [1] that functions determined in the half-plane $\sigma > 1$ by the Dirichlet series $\sum_p p^{-s}$ and $\sum_p \log p \cdot p^{-s}$ (p — a prime) cannot be continued over the line $\sigma = 0$. This result was previously found by Landau but only on the Riemann hypothesis concerning the zeta-zeros.

Walfisz's Doctor Thesis [2] provides the following theorem: let $\zeta_{\mathfrak{K}}(s)$ be the Dedekind zeta-function of the algebraic number-field $\mathfrak{K}$ of degree $\varkappa \geqslant 2$, $H(x)$ — the summatorial function of the corresponding Dirichlet series, h — the class-number of the field, $h\lambda$ — the residue of $\zeta_{\mathfrak{K}}(s)$ at $s = 1$; then

$$H(x) - h\lambda x = \Omega_{\pm}(x^{\frac{1}{2} - \frac{1}{2\varkappa}}).$$

This theorem improves a theorem of Landau which asserts that for $\vartheta < \frac{1}{2} - 1/2\varkappa$ no formula

$$H(x) - h\lambda x = O(x^{\vartheta})$$

holds. Further, paper [2] gives an expansion of $H(x)$ in a series of hyper-Bessel functions and it is proved that the series converges for $\varkappa = 2$ and can be summed by the Riesz method $\left(n, \left[\frac{\varkappa - 1}{2}\right]\right)$ for $\varkappa \geqslant 3$ (n are norms of ideals of the field $\mathfrak{K}$). For an imaginary quadratic field $\mathfrak{K}$ the expansion includes the corresponding results of G. Voronoï, Hardy and Landau.

A more general question, namely the so-called problem of Piltz in number-fields, is considered in [8] and [16] where the exact order of summation is obtained. Thus the infinite series for $H(x)$ can be summed by the Riesz method $\left(n, \frac{\varkappa - 3}{2} + \varepsilon\right)$ but not by $\left(n, \frac{\varkappa - 3}{2}\right)$.

Also paper [3] deals with Riesz summability of Dirichlet series.

Papers [14] and [15], in case of Piltz problem, provide an estimate generalizing and improving the above-mentioned estimate of paper [2].

For a quadratic number field, paper [11] gives

$$H(x) = \varrho x + O(x^{163/494}),$$

where ϱ is the residue of the zeta function at the pole. Van der Corput proved previously that

$$H(x) = \varrho x + O(x^{\theta}), \quad \theta < \tfrac{1}{3}.$$

It is proved in [4] that

$$\zeta(\tfrac{1}{2}+it) = O(t^{163/988}).$$

This estimate improves Landau's previous one.

In paper [35] it is shown by using the theorem of Schottky-Landau, that for arbitrarily given $\delta > 0$ and $\varepsilon > 0$, there exists in

$$t > 16, \quad 1 - \frac{(\log\log\log t)^{1+\varepsilon}}{\log\log t} < \sigma < 1 + \frac{(\log\log\log t)^{1+\varepsilon}}{\log\log t}$$

a set of points $M = M(\delta, \varepsilon, \mathfrak{K})$ of measure $\leqslant \delta$ on which the Dedekind zeta function of an arbitrary algebraic field $\mathfrak{K}$ takes every value, with a possible exception of one, non-enumerably many times.

Paper [83] gives abscissae of convergence and of absolute convergence of the Dirichlet series of the Epstein zeta functions.

In [85] it is proved by using a theorem of L. K. Hua, that

$$\zeta(1+ix) = O\left(\log^{3/4} x (\log\log x)^{1/2}\right),$$

and that

$$\pi(x) - \int_2^x \frac{du}{\log u} = O\left\{x \exp\left(-C \log^{4/7} x (\log\log x)^{-2/7}\right)\right\}.$$

In the book [5] Walfisz proves, using methods of Vinogradov and Korobov, that

$$\pi(x; k, l) = \frac{1}{\varphi(k)} \int_2^x \frac{du}{\log u} + O\left\{x \exp\left(-C \log^{3/5} x (\log\log x)^{-1/5}\right)\right\}.$$

Some of Walfisz's papers deal with the divisor problem. He proves in [12], using H. Weyl's method of estimation of exponential functions, that

$$\sum_{1 \leqslant n \leqslant x} \sigma(n) - \frac{\pi^2}{6} x + \frac{1}{2} \log x = O\left(\frac{\log x}{\log\log x}\right),$$

$$\sum_{1 \leqslant n \leqslant x} n\sigma(n) - \frac{\pi^2}{12} x^2 = O\left(\frac{x \log x}{\log\log x}\right),$$

where $\sigma(n)$ is the sum of reciprocals of the divisors of n. These estimates improve the previously known estimates of Wigert and Dirichlet respectively.

In papers [27] and [45], there are obtained estimates for integrals over the squared left-hand side members of the quoted inequalities.

In the books [4] and [5] Walfisz reduces, using methods of Vinogradov and Korobov, the remainders of the above-mentioned formulae of [12] to $O(\log^{2/3} x)$ and $O(x \log^{2/3} x)$ respectively.

Papers [73] and [76] are the first to improve the well-known estimate of Mertens for the Euler function $\varphi(n)$. It is proved that

$$\sum_{1 \leqslant n \leqslant x} \varphi(n) = \frac{3}{\pi^2} x^2 + O\left(x \log^{3/4} x (\log\log x)^2\right).$$

The error-term of this formula is reduced to $O\left(x \log^{3/4} x (\log\log x)^{3/2}\right)$ in [85], while in the book [5] it is improved, by methods of Vinogradov and Korobov, to $O\left(x \log^{2/3} x (\log\log x)^{4/3}\right)$.

Walfisz's book [5], published recently in Berlin (German Democratic Republic), expounds the methods of H. Weyl, Vinogradov and Korobov for exponential sums and applies them to the divisor problem, to the Euler function, the zeta function of Riemann, questions of the distribution of primes and other problems. The book contains all the recent results of Walfisz mentioned above and also those of other mathematicians.

A great number of Walfisz's papers have been concerned with diophantine approximations in connection with the investigation of arithmetical character of irrational numbers. In [24], modifying the familiar metrical theorem of Khintchine, he shows that the inequality

$$\left|\theta - \frac{h}{k}\right| < \frac{f(k)}{k},$$

under certain conditions imposed on $f(k)$, has for almost all real θ's an infinity of integer solutions h, k, where $k > 0$, $k \not\equiv 2 \pmod 4$, $(h, k) = 1$.

In papers [26] and [34], Walfisz estimates the trigonometric sums

$$R_k(x) = \sum_{0 \leqslant n \leqslant x} r_k(n) e^{2n\pi\theta i}, \quad D(x) = \sum_{1 \leqslant n \leqslant x} d(n) e^{2n\pi\theta i},$$

the estimates depending on the character of the real θ, where $r_k(n)$ denotes the number of representations of the positive integer n as a sum of $k \geqslant 2$ squares of integers and $d(n)$ denotes the number of divisors of n.

Paper [29] contains a simple proof of Hardy-Littlewood's estimate

$$\sum_{1\leqslant n\leqslant x} \operatorname{cosec} n\theta\pi = O(x),$$

which is true providing that the denominators of the continued fraction of the irrational θ are bounded. In [31], it is proved on the same assumption about θ, that

$$\sum_{1\leqslant n\leqslant x} \{n(n\theta-[n\theta]-\tfrac{1}{2})\}^{-1} = O(\log x).$$

Riemann asserted in his brilliant paper on trigonometric series that for rational θ the following identity holds

$$\sum_{n=1}^{\infty}\frac{1}{n}\psi(n\theta+\tfrac{1}{2}) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}t(n)\sin 2n\pi\theta,$$

where $t(n) = \sum_{d|n}(-1)^d$ and $\psi(u) = u-[u]-\frac{1}{2}$ for non-integer u and $\psi(u) = 0$ for integer u. In [39] Walfisz proves this identity 1) for all rational θ's 2) for almost all real θ's 3) for all algebraic irrationalities.

Ramanujan proved that

$$\sum_{\substack{1\leqslant m\leqslant x\\ m\equiv m_0 (\mathrm{mod}\, k)}} \sum_{d|m} 1 = \alpha x(\log x+2\gamma-1)+\beta x+O(x^{1/3}\log x),$$

where α, β are constants depending only on m_0 and k, γ — the Euler constant. In [20] the remainder-term has been reduced to $O(x^{27/82}\log^{11/41}x)$.

It is proved in [51] that for almost all n, the Ramanujan function $\tau(n)$ is divisible by $2^5\cdot 3^2\cdot 5^2\cdot 7\cdot 691$. Previously, it was known only that for almost all n, the function $\tau(n)$ was divisible by 691.

In [61] using some relations between class-numbers of positive binary quadratic forms, it is shown that

$$\limsup_{k=\infty}\frac{L_k}{\log\log k} \geqslant e^{\gamma}, \qquad \frac{1}{L_k} = \Omega\{(\log\log k)^{1/2}\},$$

where k runs through those positive integers for which $-k$ is a fundamental discriminant, γ is the Euler constant and $L_k = \sum_{n=1}^{\infty}\left(\frac{-k}{n}\right)\frac{1}{n}$.

The first of these inequalities had been proved by Littlewood on the hypothesis concerning the zeros of the Dirichlet L-functions. These estimates imply also results for the number of classes of positive Gaussian binary quadratic forms.

In [70] Walfisz derives by an elementary method, involving no limiting process, the results of the classical theory of Pell's equation. He gives an expository treatment in his tract [2].

In [71] Pell's equation in an arbitrary imaginary field has been studied. In the Gaussian field, it had been investigated previously by Dirichlet.

The present short paper gives only a partial account on the versatile activity of Walfisz in the field of number theory. It should be also added that we owe to him a number of papers on the algebraic theory of ideals, on various questions of the theory of functions and the theory of modular forms.