Hydromagnetic Equilibria and Force-Free Fields

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INTRODUCTION — ELEMENTARY PROPERTIES

The equations governing the equilibrium of a perfectly conducting fluid in the presence of a magnetic field are

$$\begin{aligned} \nabla p &= \mathbf{J}\times\mathbf{B} \\ \nabla\times\mathbf{B} &= \mu\mathbf{J} \\ \nabla\cdot\mathbf{B} &= 0 \\ \text{or}\quad \nabla(p + B^2/2\mu) &= (\mathbf{B}\cdot\nabla)\mathbf{B}/\mu \\ \nabla\cdot\mathbf{B} &= 0, \end{aligned} \tag{1}$$

where p is the fluid pressure, $\mathbf{B}$ the magnetic field, and $\mathbf{J}$ the current density. These equations admit a large variety of solutions, i.e., of equilibrium configurations in which a conducting fluid is balanced by a magnetic field. It is our purpose to survey these possibilities with the expectation that, if a solution has been found to exist mathematically (and is, in addition, stable) it can actually be constructed by sufficient exercise of experimental ingenuity. In this context, it is extremely important to discover exactly what data should be specified in order to determine a solution uniquely.

In addition to certain general properties of these equations, we shall consider their solution in terms of arbitrary functions, the solution of well-posed boundary value problems, and several alternative formulations of the problem in terms of the calculus of variations. In principle, either the differential equations or the variational characterization can be taken as the definition of equilibrium; the two are only approximately equivalent. Moreover, a certain type of variational formulation (slightly different from that treated in this paper) can be instrumental in a stability analysis, which is essential to give physical meaning to an equilibrium configuration.[1]

By inspection of (1), since $\mathbf{B}$ and $\mathbf{J}$ are perpendicular to ∇, p, we see that p is constant on $\mathbf{B}$ lines and on $\mathbf{J}$ lines; equivalently, the $\mathbf{B}$ lines and $\mathbf{J}$ lines lie on constant p surfaces.

In the case of a unidirectional magnetic field (e.g., $B_z(x, y)$) the general solution is $p + B^2/2\mu$ = constant. The fluid pressure is balanced by the magnetic pressure; either p or B can be given arbitrarily and the other one "filled in".

The integral form of (1) is

$$\oint_S \{(p + B^2/2\mu)\mathbf{n} - \frac{1}{\mu}\mathbf{B}B_n\}dS = 0, \quad \oint_S B_n dS = 0. \tag{2}$$

These equations are more fundamental than the differential equations (1). They are equivalent to (1) when the functions are smooth. In addition, at a discontinuity surface, the integral relations (2) imply that B_n vanishes and that $p + B^2/2\mu$ is continuous. A discontinuity surface must be a flux surface; otherwise the Maxwell magnetic stress tensor would not be compatible with a scalar pressure.

A FLUID-DYNAMICAL ANALOGUE

Using an appropriate identification of variables, equations (1) become identical to the equations of steady incompressible inviscid flow. Setting the fluid density equal to unity, these equations are

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p^* = 0 \\ \nabla\cdot\mathbf{u} = 0 \end{cases} \tag{3}$$

where p^* denotes the fluid pressure. The identifications are $\mathbf{u} \sim \mathbf{B}/\sqrt{\mu}$ and $-p^* \sim p + B^2/2\mu$; the negative of the pressure p in the magnetic case then corresponds to the Bernoulli constant $p^* + u^2/2$. The interesting case is *rotational* flow, for which the vorticity $\nabla\times\mathbf{u} \neq 0$ corresponding to $\mathbf{J} \neq 0$.

It is interesting to consider the analogue of the fluid-dynamical *free boundary* or cavitation problem in which an irrotational flow is separated at a discontinuity surface (vortex sheet) from stagnant fluid or a cavity. The separation surface is determined by the extra boundary condition $|\mathbf{u}|$ = constant. This is mathematically equivalent to a vacuum magnetic field ($\mathbf{J} = 0$) separated at an interface (current sheet) from a field-free conducting fluid. Since p is constant in the conducting fluid, we obtain the free boundary condition $|\mathbf{B}|$ = constant.

Although the two problems are mathematically identical, it is perfectly possible for a significant fluid-dynamical problem to be uninteresting in the magnetic analogue (e.g., a jet from an orifice, Fig. 1*a*) and vice versa (e.g., the cusped equilibrium,[2] Fig. 1*b*).

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SOLUTION OF BOUNDARY VALUE PROBLEMS

The characteristics of a system of partial differential equations [3] give immediate qualitative and possibly quantitative information concerning relevant initial value or boundary value problems. An elementary calculation yields four characteristics for the system (1). Two of the characteristics are purely imaginary as for the potential equation. Corresponding to these two characteristics, one would expect to prescribe a single scalar boundary value on the entire boundary of the domain, e.g., the normal component B_n of the magnetic field.

The remaining two characteristics are real; *viz.*, the $\mathbf{B}$ lines counted twice. Corresponding to each real characteristic, one would expect to be able to specify a single scalar quantity at one end of each $\mathbf{B}$ line. In a geometry as in Fig. 2 in which every $\mathbf{B}$ line

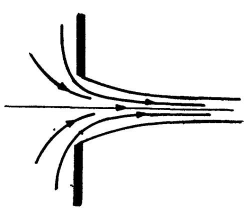

Figure 1*a*

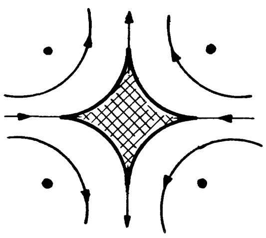

Figure 1*b*

intersects each end of the tube, one would expect to give both additional quantities at one end or one at each end. For example, one may conjecture that the following specification of boundary values (in addition to B_n on the whole surface) would be appropriate where t, n designate tangential and normal components:

Problem A_1: p given on S_1, p given on S_2 (compatibly);

Problem A_2: p given on S_1, J_n given on S_1;

Problem A_3: p given on S_1, J_n given on S_2;

Problem A_4: B_t given on S_1 (as well as B_n: the vector $\mathbf{B}$ is thus given).

In these problems, we prescribe p on boundary surfaces in such a way that the p lines are " simple " (not closed) as shown in Fig. 3. Moreover, in Problem A_1, since p is constant on $\mathbf{B}$ lines, matching magnetic flux on both ends imposes a simple compatibility requirement on p. For example, if p is given on S_1 and the lines of constant p are specified on S_2, this condition fixes the values of p on these given lines.

An iteration scheme for Problems A_2 and A_3 offering likelihood of convergence is as follows:

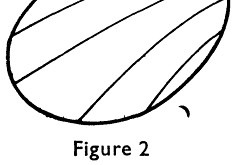

Figure 2

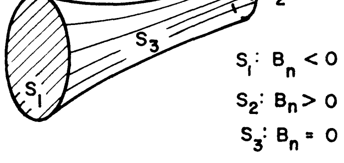

Figure 3

suppose that, at a certain stage of the iterations, we have a vector field $\mathbf{B}$ satisfying $\nabla\cdot\mathbf{B} = 0$ and the boundary condition for $\mathbf{B}_n$. We find p everywhere in the domain by carrying the boundary values of p along these $\mathbf{B}$ lines. The component $\mathbf{J}_\perp$ perpendicular to $\mathbf{B}$ is then obtained in the domain from the equation $\nabla\cdot p = \mathbf{J} \times \mathbf{B}$. We write $\mathbf{J}_\parallel = \sigma\mathbf{B}$ for the parallel component and employ the requirement $\nabla\cdot\mathbf{J} = 0$ to obtain along each $\mathbf{B}$ line the ordinary differential equation

$$\mathbf{B}\cdot\nabla\sigma + \nabla\cdot\mathbf{J}_\perp = 0 \tag{4}$$

for σ. The given " initial " condition for J_n at one end allows this equation to be solved uniquely on each line so that $\mathbf{J}$ is determined everywhere. We now solve for a new $\mathbf{B}$ from $\nabla\cdot\mathbf{B} = 0$, $\nabla \times \mathbf{B} = \mathbf{J}$ and the boundary condition for B_n, and then we continue the iterations.

We next consider two-dimensional problems; i.e., problems in which no quantity depends on z. There are several possibilities: (1) $\mathbf{B}$ can have the single component B_z and $\mathbf{J}$ the two components J_x and J_y; (2) $\mathbf{B}$ can have the two components B_x and B_y and $\mathbf{J}$ the single component J_z; (3) both $\mathbf{B}$ and $\mathbf{J}$ can be general (three-component) vectors depending on x and y alone.

The general solution in the first case has already been given explicitly, $p + B^2/2\mu =$ constant. In the second case, the number of characteristics is three rather than four; the $\mathbf{B}$ lines are counted only once. The two-dimensional analogue to Problem A_1 will be considered later. Corresponding to Problems A_2, A_3 and A_4 we have (see Fig. 4)

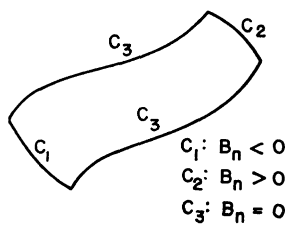

Figure 4

Problem B_1: p given on C_1 or C_2;

Problem B_2: $\mathbf{B}$ given on C_1 or C_2.

In the third (general two-dimensional) case, the characteristics are counted as in the full three-dimensional case. Correspondingly we list:

Problem C_2: p given on Γ_1, $\mathbf{J}_n$ given on Γ_1;

Problem C_3: p given on Γ_1, $\mathbf{J}_n$ given on Γ_2;

Problem C_4: $\mathbf{B}$ given on Γ_1 (as above, the analogue to Problem A_1 is left to later).

The two-dimensional Problems C_2 and C_3 are equivalent.

With axial symmetry, the θ coordinate corresponds to the z coordinate of the two-dimensional case. We have the same three subdivisions: (1) B_θ, J_r, J_z; (2) B_r, B_z, J_θ; (3) $\mathbf{B}$ and $\mathbf{J}$ general (three-component) vectors depending on r and z alone. In the first case, B_θ and p must be functions of r alone such that

$$\partial/\partial r(p + \tfrac{1}{2}\mu B_\theta^2) = -B_\theta^2/\mu r. \quad (5)$$

In the second and third cases, we identify Problems D_1, D_2 and E_2, E_3, E_4 in correspondence with B_1, B_2, and C_2, C_3, C_4, respectively.

Additional justification of the above conjectures will be given later; here they are suggested merely by the counting of characteristics.

The problem of force-free fields, *viz.*, solution of the system

$$(\mathbf{\nabla}\times\mathbf{B})\times\mathbf{B} = 0 \qquad \mathbf{\nabla}\cdot\mathbf{B} = 0 \quad (6)$$

is a special case of the equilibrium problem obtained by specializing boundary values of pressure to p = constant. We identify Problem F_1 as the special case of A_2 or A_3, F_2 as the special case of C_2 or C_3, and F_3 as the special case of E_2 or E_3. Problems F_2 and F_3 are the two-dimensional and axially symmetric versions of F_1.

CHARACTERIZATION OF THE MAGNETIC FIELD IN A PRESSURE SURFACE

We first introduce the concept of a *surface harmonic* vector field on a surface S (e.g., a constant p surface). A tangential vector field $\mathbf{X}^{(2)} = \mathbf{\nabla}^{(2)}\phi$ is said to be a *surface gradient* (or irrotational vector field) if it is the projection of a three-dimensional gradient or if

$$\oint_C \mathbf{X}^{(2)}\cdot d\mathbf{x} = 0$$

for every closed curve C which bounds a portion of S. The *conjugate* vector field $\mathbf{n}\times\mathbf{X}^{(2)} = \mathbf{Y}^{(2)}$ (where $\mathbf{n}$ is the normal to S) is said to be a *surface curl* (or solenoidal vector field); we have

$$\oint_C Y_\nu^{(2)} ds = 0$$

where ν denotes the normal in S to the curve C.

If $\mathbf{X}^{(2)} = \mathbf{\nabla}^{(2)}\phi = \mathbf{n}\times\mathbf{\nabla}^{(2)}\psi$ (that is, if $\mathbf{X}^{(2)}$ is both irrotational and solenoidal), we call it *harmonic* and say that ϕ and ψ are conjugate surface harmonics. In the special case where S is a plane, ϕ and ψ satisfy the Cauchy-Riemann equations.

In a simply-connected plane domain, a harmonic vector $\mathbf{X}^{(2)}$ is uniquely determined by the boundary values of the normal or tangential components of $\mathbf{X}^{(2)}$. Specifying either at the boundary is equivalent (except for a trivial additive constant) to specifying ψ or ϕ at the boundary. Exactly the same facts hold on an arbitrary surface S.

In a multiply-connected domain, the solution to such a boundary-value problem is uniquely determined only when certain additional data called *periods* are prescribed. These may be specified values of the circulations $\oint\mathbf{X}^{(2)}\cdot d\mathbf{x}$ on each independent closed circuit or of the fluxes $\int X_\nu^{(2)}ds$ on arcs which cut these circuits (see Fig. 5). If these periods are nonzero, the functions ϕ and ψ are multiple-valued.

The above theory can be generalized to include *weighted* surface harmonics which satisfy the equation

$$\mathbf{X}^{(2)} = \mathbf{\nabla}^{(2)}\phi = \sigma\mathbf{n}\times\mathbf{\nabla}^{(2)}\psi; \quad (7)$$

here ϕ and ψ are conjugate harmonic with respect to a positive weight function σ.

Since $\mathbf{J}$ has no component normal to a pressure surface S_p, we conclude that $\mathbf{B}$ is a surface gradient on S_p; i.e.,

$$\mathbf{B} = \mathbf{\nabla}^{(2)}\phi = \mathbf{\nabla}\phi - \mathbf{n}(\partial\phi/\partial n). \quad (8)$$

In Appendix I, it is shown that, since $\mathbf{\nabla}\cdot\mathbf{B} = 0$ and p is constant on $\mathbf{B}$ lines, there exists a function ω such that

$$\mathbf{B} = \mathbf{\nabla}p\times\mathbf{\nabla}\omega = |\mathbf{\nabla}p|\mathbf{n}\times\mathbf{\nabla}^{(2)}\omega. \quad (9)$$

We see that $\mathbf{B}$ is a weighted surface harmonic with weight $|\mathbf{\nabla}p|$ on any S_p. The weight $|\mathbf{\nabla}p|$ arises in converting from an actual three-dimensional solenoidal vector with $\mathbf{\nabla}\cdot\mathbf{B} = 0$ to a two-dimensional area-weighted solenoidal vector. We note that $\mathbf{B}$ will be an actual three-dimensional gradient instead of a surface gradient only if $\mathbf{J} = 0$ and, hence, p is constant.

These remarks allow us to extend the previously conjectured existence theorems to cases in which the p lines on the boundary are closed curves (see Fig. 6). On a surface S_p, the magnetic field is determined only when, in addition to the normal component of B at each end, we give a period, e.g., the mmf

$$\tau(p) = \oint_C \mathbf{B}\cdot d\mathbf{x} \quad (10)$$

on a curve C circling S_p. This argument suggests the following modification of Problem A_1:

Problem G_1: p given on S_1, p given on S_1, τ given for each p.

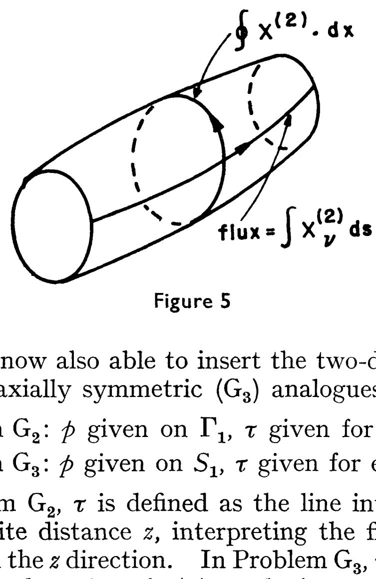

Figure 5

We are now also able to insert the two-dimensional (G_2) and axially symmetric (G_3) analogues of A_1:

Problem G_2: p given on Γ_1, τ given for each p;

Problem G_3: p given on S_1, τ given for each p.

In Problem G_2, τ is defined as the line integral of $\mathbf{B}$ over a finite distance z, interpreting the figure to be periodic in the z direction. In Problem G_3, $\tau = 2\pi B_\theta r$. In general, the value of $\tau(p)$ can be interpreted as the degree of "twist" of the magnetic lines on each tubular p surface. It should be noted that J_n is an alternative way of specifying this twist.

SOLUTION IN TERMS OF ARBITRARY FUNCTIONS

First we consider the general two-dimensional case in which B_x, B_y, B_z depend only on x, y. Since

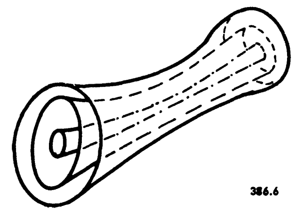

Figure 6

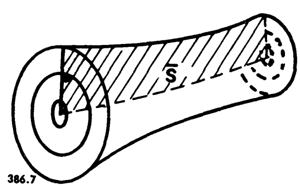

Figure 7

$\nabla \cdot \mathbf{B} = 0$, we can introduce a stream function $\psi(x, y)$ for the B_x, B_y components of the field; we then have

$$B_x = -\frac{\partial \psi}{\partial y}, \quad B_y = \frac{\partial \psi}{\partial x} \tag{11}$$

so that

$$\mathbf{B} = \mathbf{n} \times \nabla \psi + \mathbf{n} B_z \tag{12}$$

where $\mathbf{n} = (0, 0, 1)$. It can be seen that p and B_z are constant on the curves of constant ψ in the x, y plane. However, p and B_z may be multiple-valued functions of ψ. The stream function $\psi(x, y)$ satisfies the non-linear potential equation

$$\Delta \psi + \mu p'(\psi) + \mu f'(\psi) = 0 \tag{13}$$

where

$$f(\psi) = \frac{1}{2\mu} B_z^2; \tag{14}$$

p' and f' refer to derivatives with respect to ψ. In (13), $p(\psi)$ and $f(\psi)$ are arbitrary functions of ψ. Hence, our system of four first-order equations has been reduced to a single second-order equation containing two arbitrary functions. One can expect a solution to be determined by specifying the arbitrary functions and boundary values for ψ as for the standard two-dimensional potential equation. For solutions depending on x only, the equation

$$\frac{d^2\psi}{dx^2} + \mu p'(\psi) + \mu f'(\psi) = 0 \tag{15}$$

can be integrated explicitly giving many interesting equilibrium configurations.

In the case of axial symmetry in which B_r, B_θ, B_z depend only on r, z, we introduce a stream function $\psi(r, z)$ such that

$$B_r = \frac{1}{r}\frac{\partial \psi}{\partial z} \qquad B_z = -\frac{1}{r}\frac{\partial \psi}{\partial r} \tag{16}$$

and, hence

$$\mathbf{B} = \frac{1}{r}\mathbf{n} \times \nabla \psi + \mathbf{n} B_\theta \tag{17}$$

where $\mathbf{n} = (n_r, n_\theta, n_z) = (0, 1, 0)$. Here $\psi\ (r, z)$ satisfies the equation

$$\Delta^* \psi + \mu r^2 p'(\psi) + \mu f'(\psi) = 0 \tag{18}$$

where

$$\Delta^* = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{19}$$

and

$$f(\psi) = \frac{1}{2\mu} r^2 B_\theta^2. \tag{20}$$

In the case of cylindrical symmetry, in which there is dependence on r alone, we may give any two of p, ψ and B_θ as functions of r and immediately compute the other.

Interesting special solutions of Eq. (15) and Eq. (18) can be found by separation of variables after taking f' and p' to be linear in ψ. In some cases eigenvalue problems arise. It is thus clear that uniqueness cannot be expected in general. However, it is possible to prove uniqueness as well as existence even with quite arbitrary $f(\psi)$ and $p(\psi)$ for domains which are not too large.

The force-free special cases are obtained by the simple expedient of setting $p'\ (\psi) = 0$.[4]

The reduction in terms of arbitrary functions offers support for the conjectured existence of solutions to Problems B_1, C_2 or C_3, D_1, E_2 or E_3, F_2, and F_3. In each case, the boundary data over and above B_n serve to determine the functions $p(\psi)$ and $f(\psi)$.

VARIATIONAL ANALYSIS. ADDITIONAL BOUNDARY VALUE PROBLEMS

It is well known that solutions of the fluid free-surface problem can be described as those vector functions $\mathbf{u}(x)$ which make stationary the functional $\int \frac{1}{2} u^2 dV$ when the fluid volume is held fixed. Analogously, in the magnetic case we vary $\int B^2/2\mu dV$ holding the volume of the magnetic domain fixed. Or we can drop the restriction to constant volume and, as suggested by the Lagrange multiplier rule, vary

$$\begin{aligned} \int_{V_m} \frac{1}{2\mu} B^2 dV + p_0 \int_{V_m} dV &= \int_{V_m} \frac{1}{2\mu} B^2 dV \\ &\quad - \int_{V_f} p_0 dV + \text{constant} \\ &= \int_{V_f + V_m} \left(\frac{1}{2\mu} B^2 - p\right) dV + \text{constant.} \end{aligned} \tag{21}$$

Here V_m represents the vacuum (magnetic) domain and V_f the conducting fluid domain; the sum $V_m + V_f$ is fixed. The pressure p takes the value zero in the vacuum and the constant value p_0 in the fluid. We shall see that, if interpreted properly, the same functional,

$$Q = \int \left(\frac{1}{2\mu} B^2 - p\right) dV \tag{22}$$

serves for the general equilibrium problem with fluid and magnetic field mixed; here p as well as $\mathbf{B}$ will be a function of x, y, z. The class of admissible pairs $(\mathbf{B}, p)$ allowed to compete in the variation of the given functional is restricted by the conditions that $\mathbf{B}$ be solenoidal and p be constant on the $\mathbf{B}$ lines of the associated $\mathbf{B}$,

$$\begin{aligned}\nabla\cdot\mathbf{B}&=0\\ \mathbf{B}\cdot\nabla p&=0,\end{aligned}\tag{23}$$

as well as by boundary conditions that will vary from problem to problem.

A simple way of incorporating both constraints (23) is to represent $\mathbf{B}$ in the form

$$\mathbf{B}=\nabla p\times\nabla\omega \tag{24}$$

(see Appendix I). The function ω may be multiple-valued if the p surfaces are not simple.

We perform the variation, obtaining

$$\begin{aligned}\delta\int_V(\tfrac{1}{2}B^2-p)dV&=\int_V(\mathbf{B}\cdot\delta\mathbf{B}-\delta p)dV\\ &=\int_V\{\mathbf{B}\cdot(\nabla\delta p\times\nabla\omega+\nabla p\times\nabla\delta\omega)-\delta p\}dV\\ &=\int_V\{\delta p(\mathbf{J}\cdot\nabla\omega-1)-\delta\omega(\mathbf{J}\cdot\nabla p)\}dV\\ &-\oint_S\{\delta p(\mathbf{B}\times\nabla\omega)-\delta\omega(\mathbf{B}\times\nabla p)\}\cdot\mathbf{n}dS=0.\end{aligned}\tag{25}$$

For arbitrary variations of δp and $\delta\omega$ in the volume integral, we conclude

$$\begin{aligned}\mathbf{J}\cdot\nabla\omega&=1\\ \mathbf{J}\cdot\nabla p&=0\end{aligned}\tag{26}$$

from which we easily obtain $\nabla p=\mathbf{J}\times\mathbf{B}$ as the Euler equation.

Next we turn to the boundary variation and first consider the case of simple p surfaces (Fig. 3), with p given at both ends S_1 and S_2. It can easily be verified that there is no contribution to the variation (25) from S_3 on which $B_n=0$. On S_1 and S_2, from $\delta B_n=0$ we conclude that $\delta\omega$ is a function of p; however, we are certainly free to fix the value of ω at one end of each p line on S_1, and this makes $\omega\delta=0$ everywhere. We then have :

Theorem 1: In the tubular geometry of Fig. 2, if p is given (compatibly) at the ends S_1 and S_2 in such a way that the p lines are simple, then Q is made stationary by any solution of the system (1) which satisfies these boundary conditions. This theorem corresponds to the conditions of Problem A_1.

Next consider tubular p surfaces (Fig. 6). Since ω and ϕ (see Eq. (8)) may be multiple-valued, we cut the domain on a surface S which extends from the axis of the nested surfaces S_p to the outer boundary S_3 (Fig. 7). The boundary integral in (25) must now be taken over both sides of $\bar{S}$ as well as S_1, S_2 and S_3. As in the previous case, the contribution from S_3 vanishes. A little manipulation shows that the contribution from the cut $\bar{S}$ also vanishes. On S_1 or S_2, since $\delta B_n=0$, we must have $\sigma\omega=f(p)$. We obtain for this part of the variation,

$$\begin{aligned}\int_{S_1+S_2}f(p)(\mathbf{B}\times\nabla p)\cdot d\mathbf{S}&=\int_{S_2-S_1}f(p)\tau(p)dp\\ &=\int_{S_2}(f_2(p)-f_1(p))\tau(p)dp\end{aligned}\tag{27}$$

using Eq. (II.11) of Appendix II. The difference $\delta\omega_2$—$\delta\omega_1$ represents the variation of the twist from S_1 to S_2 on a given pressure surface. In order to have Q stationary we must either fix this twist beforehand so that $\delta\omega_2-\delta\omega_1=0$ or else specify that $\tau(p)=0$; the latter is a natural boundary condition in this problem. To specify the twist for a class of admissible vector fields $\mathbf{B}$, we require the magnetic lines which originate on a ray C' on S_1 to end on another given ray C'' on S_2 (Fig. 8). We have:

Theorem 2: With tubular p surfaces, p given compatibly at the ends S_1 and S_2 and two given rays C' and C'' identified, Q is stationary if $\mathbf{B}$ and p satisfy (1). If the twist is not specified, Q is stationary for a solution of (1) which satisfies $\tau(p)=0$. There are really two ways of specifying twist, since the specification of $\tau(p)$ or the identification of two rays C' and C'' are, in an intuitive sense, equivalent.

It is easy to alter the problem by the use of the Lagrange multiplier rule so as to be able to specify τ instead of the twist. This variational formulation corresponds to Problems G_1, G_2 and G_3.

Some modifications of these variational problems are necessary to obtain force-free fields. We drop the term p in the variational functional, and obtain

$$M=\int\frac{1}{2\mu}B^2dV. \tag{28}$$

Instead of the representation $\mathbf{B}=\nabla p\times\nabla\omega$ we now take

$$\mathbf{B}=\nabla\pi\times\nabla\omega \tag{29}$$

where π is no longer identified with the pressure. It is now easy to verify the following theorems:

Theorem 3: For admissible sets $(\mathbf{B},\pi)$ with simple π surfaces and π given compatibly on S_1 and S_2, M is stationary when $\mathbf{B}$ is force-free.

Theorem 4: For admissible sets $(\mathbf{B},\pi)$ with tubular π surfaces and π given compatibly on S_1 and S_2 as well as the twist from S_1 to S_2, M is stationary when $\mathbf{B}$ is force-free.

If one is willing to place a large burden on the verification of the compatibility of given boundary data, then it is possible to formulate problems in which p (or π) is given at both ends S_1 and S_2 without regard to the simplicity of p lines. We have:

Theorem 5: Consider admissible classes $(\mathbf{B},p)$ [or $(\mathbf{B},\pi)$] which satisfy $\nabla\cdot\mathbf{B}=0$ and p [or π] constant on $\mathbf{B}$ lines, p [or π] given compatibly at both ends, and a fixed (compatible) identification of the ends of each $\mathbf{B}$ line. Then Q [or M] is rendered stationary when $(\mathbf{B},p)$ satisfies $\nabla p=\mathbf{J}\times\mathbf{B}$ [$\mathbf{J}\times\mathbf{B}=0$].

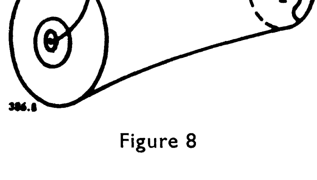
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Figure 8

This theorem follows when we note that assignment of B_n and p [or π] at both ends uniquely fixes the correspondence between $\mathbf{B}$ lines if the p surface [π surfaces] are simple, and if they are not simple, the additional specification of twist serves the same purpose. Physically, the boundary condition that the ends of each $\mathbf{B}$ line be fixed corresponds to a perfectly conducting fluid in contact with perfectly conducting end walls S_1 and S_2. In the force-free case, the reference is to a so-called "pressureless plasma" in which the gas pressure is negligible compared to the magnetic pressure, but which is, nevertheless a good conductor; that is physically realizable since the conductivity of a plasma is independent of its density.

It is interesting to compare the variational problem for the force-free field with the classical Dirichlet's principle which states that M is minimized subject to given B_n (and $\nabla\cdot\mathbf{B}=0$) when $\nabla\times\mathbf{B}=0$. The additional requirement that the ends of each $\mathbf{B}$ line be fixed prevents this minimum from being attained and yields $(\nabla\times\mathbf{B})\times\mathbf{B}=0$ as the Euler equation instead of $\nabla\times\mathbf{B}=0$.

As a final example, we take as our domain the interior of a topological torus and look for solutions which have the outer boundary of the torus as a constant pressure surface. No boundary values can be given since no $\mathbf{B}$ lines are accessible. However, the variational formulation itself suggests what data are required to obtain a well-posed problem. As before, we take admissible sets $(\mathbf{B}, p)$ which satisfy $\mathbf{B}=\nabla p\times\nabla\omega$, where the p surfaces now are nested toruses about some closed curve C_0 as axis. For simplicity we take p to be monotone, $p_0>p>p_1$, where p_0 is the value taken on the axis C_0 and p_1 is the value taken on the outer boundary. To make ω single valued, it is necessary to introduce two surfaces $\bar{S}_1$ and $\bar{S}_2$ as cuts; $\bar{S}_1$ is a transverse cut across the torus (leaving ends similar to S_1 and S_2 of Fig. 2), and $\bar{S}_2$ extends from the axis C_0 to the outer boundary. The only contribution to the variation (25) is on the cuts; we have

$$\delta Q=\int_{\bar{S}_1+\bar{S}_2}[\delta p](\nabla\omega\times\mathbf{B})\cdot\mathbf{n}dS \qquad (30)$$
$$-\int_{\bar{S}_1+\bar{S}_2}[\delta\omega]\nabla p\times\mathbf{B})\cdot\mathbf{n}dS.$$

Since p (therefore δp) is single valued, $[\delta p]=0$. The periods $[\omega]$ are given by

$$\sigma_i(p)=\int_{C_i}\nabla\omega\cdot d\mathbf{x}. \qquad (31)$$

where C_i are the independent closed curves on the torus S_p. In Appendix II it is shown that

$$\begin{cases}\sigma_1(p)=\dfrac{d}{dp}\Phi_2(p)\\ \sigma_2(p)=-\dfrac{d}{dp}\Phi_1(p)\end{cases} \qquad (32)$$

where $\Phi_i(a)$ represents the magnetic flux across that part of $\bar{S}_i$ for which $p_1<p<a$. From Eq. (II.11) of Appendix II, we obtain the form

$$\delta Q=\int_{p_0}^{p_1}\{\tau_1\delta\sigma_2-\tau_2\delta\sigma_1\}dp \qquad (33)$$

where p_0 and p_1 are the values taken by p on the axis C_0 and on the outer boundary of the torus, respectively. We now state:

Theorem 6: In the class of admissible $(\mathbf{B}, p)$ defined by $\nabla\cdot\mathbf{B}=0$, $\mathbf{B}\cdot\nabla p=0$ and given $\Phi_1(p)$, $\Phi_2(p)$ (or $\sigma_1(p)$, $\sigma_2(p)$) for $p_0>p>p_1$, Q is stationary when $\nabla p=\mathbf{J}\times\mathbf{B}$.[5]

We can obtain a formulation with given $\tau_i(p)$ rather than $\Phi_i(p)$ by modifying Q according to the conventional Lagrange multiplier rule.

For force-free fields, exactly the same analysis yields the result that Q is stationary if Φ_1 is a given function of Φ_2 and Φ_2 attains the fixed values Φ_2 and 0 on C_0 and on the outer boundary; in parametric form we give $\Phi_1(\pi)$ and $\Phi_2(\pi)$.

As a special case of a toroidal geometry, we can take a problem with cylindrical symmetry (dependence on r alone) and introduce periodicity to provide the toroidal topology. In this case, the above conjectures are trivially proved.

We summarize various sets of data which are believed to define definite equilibrium configurations. In the tubular domain of Fig. 2, we specify B_n as shown and we also specify p and J_n at one end or p at one end and J_n at the other. The justification lies in the counting of characteristics, in a heuristically appealing iteration scheme and in a direct verification by integrating in terms of arbitrary functions in certain special cases. Or we can specify p at one end and the corresponding terminal points of each B line in a manner compatible with the given values of B_n. This is confirmed by counting of characteristics supplemented by the known properties of surface harmonics, by variational analysis and by integration in terms of arbitrary functions in certain special cases. In the toroidal geometry, we conclude that the two fluxes $\Phi_i(p)$ or mmfs $\tau_i(p)$ can be specified arbitrarily; the justification is proved by variational analysis and by explicit solution in special cases.

Appendix I

REPRESENTATION OF AN INCOMPRESSIBLE VECTOR FIELD IN TERMS OF TWO STREAM FUNCTIONS

We note, for arbitrary $\phi(x,y,z)$, $\psi(x,y,z)$ and arbitrary $\alpha(\phi,\psi)$ (which means that α is constant on the intersection of $\phi=$ constant and $\psi=$ constant), that

$$\nabla\cdot(\alpha\nabla\phi\times\nabla\psi)=0. \qquad (I.1)$$

We remark further that, if (I.1) holds for $\alpha(x,y,z)$, $\phi(x,y,z)$, $\psi(x,y,z)$ in a domain where ϕ and ψ are independent functions (i.e., $\nabla\phi\times\nabla\psi\neq 0$), then it follows that $\alpha=\alpha(\phi,\psi)$.

Next consider a solenoidal vector field **B** and a small region of space which is simply covered by the **B** lines which intersect a transverse surface S. We have:

Theorem I.1: There exist functions ϕ, ψ, $\alpha(\phi, \psi)$ such that $\mathbf{B} = \alpha \nabla\phi \times \nabla\psi$.

To prove Theorem I.1 we choose any independent ϕ and ψ on S (i.e., $\phi =$ constant and $\psi =$ constant form a coordinate system). We then carry the values of ϕ and ψ off S on the **B** lines. Since $\nabla\phi \times \nabla\psi$ is parallel to **B**, we conclude that $\mathbf{B} = \alpha\nabla\phi \times \nabla\psi$ and, hence, $\alpha = \alpha(\phi, \psi)$ by the above remark.

We note that the flux through S,

$$d\Phi \equiv B_n dS = d\phi d\psi, \tag{I.2}$$

which justifies the terminology " stream functions ".

We again take an arbitrary vector field **B** with $\nabla \cdot \mathbf{B} = 0$ and a small region and assert:

Theorem 2: There exist ϕ, ψ such that $\mathbf{B} = \nabla\phi \times \nabla\psi$. In fact, if ϕ is any given function which is constant on **B** lines, then there exists ψ such that $\mathbf{B} = \nabla\phi \times \nabla\psi$.

Choose ϕ arbitrarily on S so that the ϕ curves are simple. Introduce s as the arc length on a ϕ curve, ν as the normal to the ϕ curve in S. Note that if $\mathbf{B} = \nabla\phi \times \nabla\psi$ then $B_n = (\partial\phi/\partial\nu/(\partial\phi/\partial S)$. This suggests that we construct ψ as

$$\psi(s) = \int \frac{1}{B_n} \frac{\partial\phi}{\partial\nu} dS, \tag{I.3}$$

integrated along each ϕ line (the value of ψ at one end of each ϕ line can be arbitrarily assigned). As before, we carry the values of ϕ and ψ off S as constants on **B** lines. From the previous theorem, $\mathbf{B} = a\nabla\phi \times \nabla\psi$. By the construction of ψ, $a = 1$ on S; since a is constant on each **B** line, $a = 1$ throughout.

If the ϕ lines are taken as closed curves, then the construction yields a ψ which is not single valued, and a cut should be introduced.

APPENDIX II

INTEGRATION FORMULAS

We shall use two basic formulas. First, given two functions $f(x)$ and $g(x)$ defined in a space $x = (x_1, \ldots, x_n)$, we have

$$\int_{a<g<b} f(x)dx = \int_a^b \langle f \rangle_g \, dg \tag{II.1}$$

where

$$\langle f \rangle_g = \int_{S_g} f(dS/|\nabla g|). \tag{II.2}$$

The integral (II.1) over the shell $a < g < b$ is written as an iterated integral first on the surface S_g, then with respect to g.[6] The second identity is,

$$\int_S \nabla\phi \times \nabla\psi \cdot dS = \int_S d\phi d\psi = \int_C \phi d\psi = -\int_C \psi d\phi; \tag{II.3}$$

here S is an arbitrary surface in three-space with C as its boundary.[7] The surface S may have to be cut to make ϕ and ψ single valued. If S is a torus, on which there are the two independent closed curves C_1 and C_2, this formula reduces to

$$\int_S \nabla\phi \times \nabla\psi \cdot dS = [\phi]_1[\psi]_2 - [\phi]_2[\psi]_1 \tag{II.4}$$

where

$$[\phi]_i = \int_{C_i} d\phi, \quad [\psi]_i = \int_{C_i} d\psi \tag{II.5}$$

are the periods of ϕ and ψ, respectively.

We apply these formulas to obtain the identity

$$\int_{a<p<b} B^2 dV = \int_a^b dp \int_{Sp} d\phi d\omega \tag{II.6}$$

for an equilibrium **B**, i.e., for **B** satisfying

$$\mathbf{B} = \nabla^{(2)}\phi = \nabla p \times \nabla\omega. \tag{II.7}$$

Using (II.2) and then (II.3) we verify

$$\begin{aligned} \langle B^2 \rangle_p &= \int_{S_p} \nabla^{(2)}\phi \cdot (\nabla p \times \nabla\omega) \frac{dS}{|\nabla p|} \\ &= \int_{S_p} \nabla\omega \times \nabla^{(2)}\phi \cdot d\mathbf{S} = \int d\omega d\phi. \end{aligned} \tag{II.8}$$

In the special case of a torus, we have

$$\int_{a<b<p} B^2 dV = \int \{\tau_1(p)\sigma_2(p) - \tau_2(p)\sigma_1(p)\} dp \tag{II.9}$$

where

$$\left\{ \begin{aligned} \sigma_i(p) &= \oint_{C_i} \nabla\omega \cdot d\mathbf{x} \\ \tau_i(p) &= \oint_{C_i} \mathbf{B} \cdot d\mathbf{x}. \end{aligned} \right. \tag{II.10}$$

On a transverse surface S, we compute

$$\int_{a<p<b} (\mathbf{B} \times \nabla p \cdot \mathbf{n}) dS = -\int^b \tau(p) dp. \tag{II.11}$$

We have $\mathbf{n}$ as the normal to S, and introduce $\boldsymbol{\nu}$ as the normal in S to a p curve, and S as arc length along a p curve (Fig. 9). The proof is

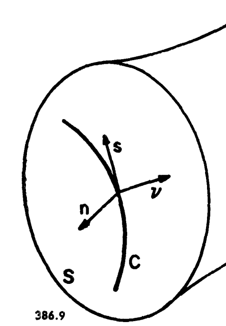

Figure 9

$$\langle \mathbf{B} \times \nabla p \cdot \mathbf{n} \rangle \mathbf{v} = \int_{C_p} (\mathbf{B} \times \nabla p \cdot \mathbf{n}) \frac{ds}{|\nabla^{(2)} p|}$$

$$= \int_{C_p} (\mathbf{B} \times \nabla \cdot \mathbf{n}) dS = - \oint_{C_p} \mathbf{B} \cdot d\mathbf{x} = - \tau(p). \quad \text{(II.12)}$$

Again, on a transverse surface S,

$$\langle B_n \rangle_p = \int_{C_p} (\nabla p \times \nabla \omega \cdot \mathbf{n}) \frac{ds}{|\nabla^{(2)} p|}$$

$$= \oint_{C_p} \nabla \cdot \omega d\mathbf{x} = \sigma(p). \quad \text{(II.13)}$$

Consequently,

$$\Phi(b) - \Phi(a) = \int_{a<p<b} B_n dS = \int_a^b \sigma(p) dp \quad \text{(II.14)}$$

or

$$\sigma(p) = \Phi'(p). \quad \text{(II.15)}$$

Taking signs into account, we can rewrite (II.9) as

$$\int_{a<p<b} B^2 dV = \int^b \{\tau_1 \Phi_1' + \tau_2 \Phi_2'\} dp \quad \text{(II.16)}$$

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