Forum Geometricorum
Volume 2 (2002) 175–182.

FORUM GEOM
ISSN 1534-1178

The Apollonius Circle as a Tucker Circle

Darij Grinberg and Paul Yiu

Abstract. We give a simple construction of the circular hull of the excircles of a triangle as a Tucker circle.

1. Introduction

The Apollonius circle of a triangle is the circular hull of the excircles, the circle internally tangent to each of the excircles. This circle can be constructed by making use of the famous Feuerbach theorem that the nine-point circle is tangent *externally* to each of the excircles, and that the radical center of the excircles is the Spieker point X_{10}, the incenter of the medial triangle. If we perform an inversion with respect to the radical circle of the excircles, which is the circle orthogonal to each of them, the excircles remain invariant, while the nine-point circle is inverted into the Apollonius circle. The points of tangency of the Apollonius circle, being the inversive images of the points of tangency of the nine-point circle, can be constructed by joining to these latter points to Spieker point to intersect the respective excircles again. [1] See Figure 1. In this paper, we give another simple construction of the Apollonius circle by identifying it as a Tucker circle.

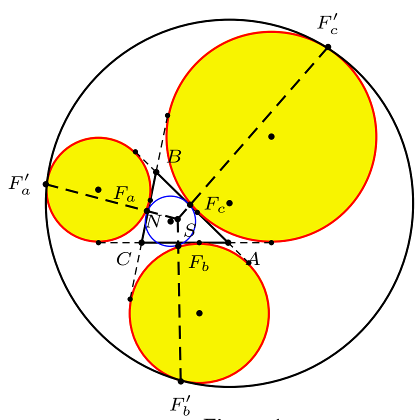

Figure 1

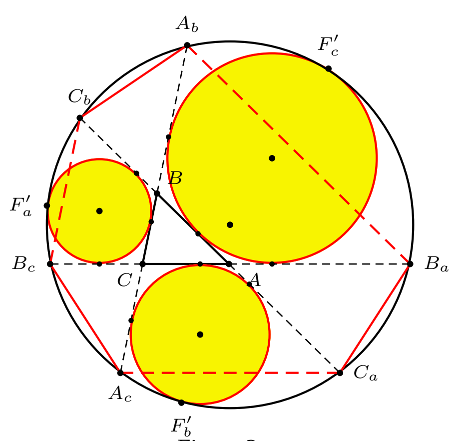

Figure 2

Theorem 1. *Let B_a and C_a be points respectively on the extensions of CA and BA beyond A such that B_aC_a is antiparallel to BC and has length s, the semiperimeter of triangle ABC. Likewise, let C_b, A_b be on the extensions of AB and CB beyond*

Publication Date: December 16, 2002. Communicating Editor: Jean-Pierre Ehrmann.

[1]The tangency of this circle with each of the excircles is internal because the Spieker point, the center of inversion, is contained in nine-point circle.

B, with C_bA_b antiparallel to CA and of length s, A_c, B_c on the extensions of BC and AC beyond C, with A_cB_c is antiparallel to AB and of length s. Then the six points A_b, B_a, C_a, A_c, B_c, C_b are concyclic, and the circle containing them is the Apollonius circle of triangle ABC.

The vertices of the Tucker hexagon can be constructed as follows. Let X_b and X_c be the points of tangency of BC with excircles (I_b) and (I_c) respectively. Since BX_b and CX_c each has length s, the parallel of AB through X_b intersects AC at C', and that of AC through X_c intersects AB at B' such that the segment $B'C'$ is parallel to BC and has length s. The reflections of B' and C' in the line I_bI_c are the points B_a and C_a such that triangle AB_aC_a is similar to ABC, with $B_aC_a = s$. See Figure 3. The other vertices can be similarly constructed. In fact, the Tucker circle can be constructed by locating A_c as the intersection of BC and the parallel through C_a to AC.

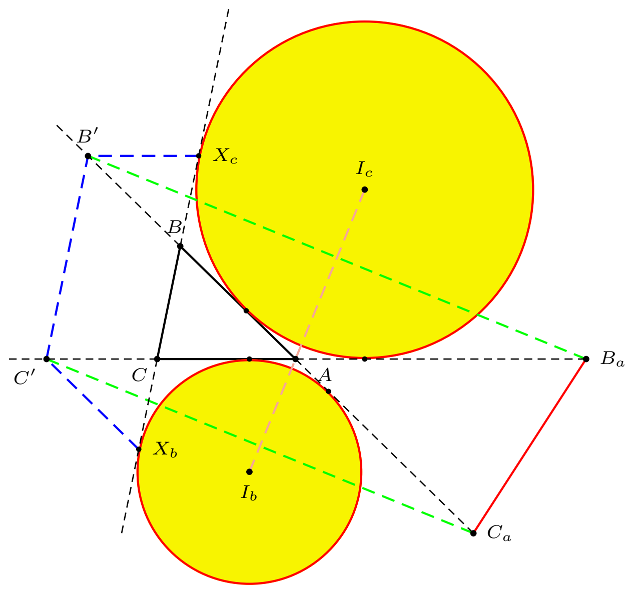

Figure 3

2. Some basic results

We shall denote the side lengths of triangle ABC by a, b, c.

R	circumradius
r	inradius
s	semiperimeter
$\triangle$	area
ω	Brocard angle

The Brocard angle is given by

$$\cot\omega = \frac{a^2 + b^2 + c^2}{4\triangle}.$$

Lemma 2. (1) $abc = 4Rrs$;
(2) $ab + bc + ca = r^2 + s^2 + 4Rr$;
(3) $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$;
(4) $(a + b)(b + c)(c + a) = 2s(r^2 + s^2 + 2Rr)$.

Proof. (1) follows from the formulae $\triangle = rs$ and $R = \frac{abc}{4\triangle}$.

(2) follows from the Heron formula $\triangle^2 = s(s - a)(s - b)(s - c)$ and

$$s^3 - (s - a)(s - b)(s - c) = (ab + bc + ca)s + abc.$$

(3) follows from (2) and $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$.

(4) follows from $(a + b)(b + c)(c + a) = (a + b + c)(ab + bc + ca) - abc$. $\square$

Unless explicitly stated, all coordinates we use in this paper are *homogeneous barycentric coordinates*. Here are the coordinates of some basic triangle centers.

circumcenter	O	$(a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2))$
incenter	I	$(a : b : c)$
Spieker point	S	$(b + c : c + a : a + b)$
symmedian point	K	$(a^2 : b^2 : c^2)$

Note that the sum of the coordinates of O is $16\triangle^2 = 16r^2s^2$. [2] We shall also make use of the following basic result on circles, whose proof we omit.

Proposition 3. *Let p_1, p_2, p_3 be the powers of A, B, C with respect to a circle $\mathcal{C}$. The power of a point with homogeneous barycentric coordinates $(x : y : z)$ with respect to the same circle is*

$$\frac{(x + y + z)(p_1x + p_2y + p_3z) - (a^2yz + b^2zx + c^2xy)}{(x + y + z)^2}.$$

Hence, the equation of the circle is

$$a^2yz + b^2zx + c^2xy = (x + y + z)(p_1x + p_2y + p_3z).$$

3. The Spieker radical circle

The fact that the radical center of the excircles is the Spieker point S is well known. See, for example, [3]. We verify this fact by computing the power of S with respect to the excircles. This computation also gives the radius of the radical circle.

Theorem 4. *The radical circle of the excircles has center at the Spieker point $S = (b + c : c + a : a + b)$, and radius $\frac{1}{2}\sqrt{r^2 + s^2}$.*

---

[2] This is equivalent to the following version of Heron's formula:

$$16\triangle^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4.$$

Proof. We compute the power of $(b+c:c+a:a+b)$ with respect to the A-excircle. The powers of A, B, C with respect to the A-excircle are clearly

$$p_1 = s^2, \qquad p_2 = (s-c)^2, \qquad p_3 = (s-b)^2.$$

With $x = b+c$, $y = c+a$, $z = a+b$, we have $x+y+z = 4s$ and

$$\begin{aligned}
&(x+y+z)(p_1x+p_2y+p_3z) - (a^2yz+b^2zx+c^2xy)\\
=&4s(s^2(b+c)+(s-c)^2(c+a)+(s-b)^2(a+b))\\
&-(a^2(c+a)(a+b)+b^2(a+b)(b+c)+c^2(b+c)(c+a))\\
=&2s(2abc+(a+b+c)(a^2+b^2+c^2)) - 2s(a^3+b^3+c^3+abc)\\
=&2s(abc+a^2(b+c)+b^2(c+a)+c^2(a+b))\\
=&4s^2(r^2+s^2),
\end{aligned}$$

and the power of the Spieker point with respect to the A-excircle is $\frac{1}{4}(r^2+s^2)$. This being symmetric in a, b, c, it is also the power of the same point with respect to the other two excircles. The Spieker point is therefore the radical center of the excircles, and the radius of the radical circle is $\frac{1}{2}\sqrt{r^2+s^2}$. □

We call this circle the Spieker radical circle, and remark that the Spieker point is the inferior of the incenter, namely, the image of the incenter under the homothety $\mathsf{h}(G, -\frac{1}{2})$ at the centroid G.

4. The Apollonius circle

To find the Apollonius circle it is more convenient to consider its superior, *i.e.*, its homothetic image $\mathsf{h}(G,-2)$ in the centroid G with ratio -2. This homothety transforms the nine-point circle and the Spieker radical circle into the circumcircle $O(R)$ and the circle $I(\sqrt{r^2+s^2})$ respectively.

Let d be the distance between O and I. By Euler's theorem, $d^2 = R^2 - 2Rr$. On the line OI we treat I as the origin, and O with coordinate R. The circumcircle intersects the line IO at the points $d \pm R$. The inversive images of these points have coordinates $\frac{r^2+s^2}{d\pm R}$. The inversive image is therefore a circle with radius

$$\frac{1}{2}\left|\frac{r^2+s^2}{d-R} - \frac{r^2+s^2}{d+R}\right| = \left|\frac{R(r^2+s^2)}{d^2-R^2}\right| = \frac{r^2+s^2}{2r}.$$

The center is the point Q' with coordinate

$$\frac{1}{2}\left(\frac{r^2+s^2}{d-R} + \frac{r^2+s^2}{d+R}\right) = \frac{d(r^2+s^2)}{d^2-R^2} = -\frac{r^2+s^2}{2Rr}\cdot d.$$

In other words,

$$IQ' : IO = -(r^2+s^2) : 2Rr.$$

Explicitly,

$$Q' = I - \frac{r^2+s^2}{2Rr}(O-I) = \frac{(r^2+s^2+2Rr)I - (r^2+s^2)O}{2Rr}.$$

From this calculation we make the following conclusions.

(1) The radius of the Apollonius circle is $\rho = \frac{r^2+s^2}{4r}$.
(2) The Apollonius center, being the homothetic image of Q' under $\mathsf{h}(G, -\frac{1}{2})$, is the point [3]

$$Q = \frac{1}{2}(3G - Q') = \frac{6Rr \cdot G + (r^2+s^2)O - (r^2+s^2+2Rr)I}{4Rr}.$$

Various authors have noted that Q lies on the Brocard axis OK, where the centers of Tucker circles lie. See, for example, [1, 9, 2, 7]. In [1], Aeppli states that if d_A, d_B, d_C are the distances of the vertices A, B, C to the line joining the center of the Apollonius circle with the circumcenter of ABC, then

$$d_A : d_B : d_C = \frac{b^2-c^2}{a^2} : \frac{c^2-a^2}{b^2} : \frac{a^2-b^2}{c^2}.$$

It follows that the barycentric equation of the line is

$$\frac{b^2-c^2}{a^2}x + \frac{c^2-a^2}{b^2}y + \frac{a^2-b^2}{c^2}z = 0.$$

This is the well known barycentric equation of the Brocard axis. Thus, the Apollonius center lies on the Brocard axis. Here, we write Q explicitly in terms of O and K.

Proposition 5. $Q = \frac{1}{4Rr}\big((s^2-r^2)O - \frac{1}{2}(a^2+b^2+c^2)K\big)$.

Proof.

$$\begin{aligned}
Q =& \frac{1}{4Rr}\big((r^2+s^2)O + 6Rr \cdot G - (r^2+s^2+2Rr)I\big)\\
=& \frac{1}{4Rr}\big((s^2-r^2)O + 2r^2 \cdot O + 6Rr \cdot G - (r^2+s^2+2Rr)I\big)\\
=& \frac{1}{16Rrs^2}\big(4s^2(s^2-r^2)O + 8r^2s^2 \cdot O + 24Rrs^2 \cdot G - 4s^2(r^2+s^2+2Rr)I\big).
\end{aligned}$$

Consider the sum of the last three terms. By Lemma 2, we have

$$\begin{aligned}
& 8r^2s^2 \cdot O + 24Rrs^2 \cdot G - 4s^2(r^2+s^2+2Rr)I\\
=& 8r^2s^2 \cdot O + abc \cdot 2s \cdot 3G - 2s(a+b)(b+c)(c+a)I\\
=& \frac{1}{2}(a^2(b^2+c^2-a^2), b^2(c^2+a^2-b^2), c^2(a^2+b^2-c^2))\\
& + (a+b+c)abc(1,1,1) - (a+b)(b+c)(c+a)(a,b,c).
\end{aligned}$$

[3] This point is X_{970} of [7].

Consider the first component.

$$\begin{aligned}
&\frac{1}{2}\left(a^2(b^2+c^2-a^2)+2abc(a+b+c)-2(a+b)(b+c)(c+a)a\right)\\
=&\frac{1}{2}\left(a^2(b^2+2bc+c^2-a^2)+2abc(a+b+c)-2a((a+b)(b+c)(c+a)+abc)\right)\\
=&\frac{1}{2}\left(a^2(a+b+c)(b+c-a)+2abc(a+b+c)-2a(a+b+c)(ab+bc+ca)\right)\\
=&s(a^2(b+c-a)+2abc-2a(ab+bc+ca))\\
=&s(a^2(b+c-a)-2a(ab+ca))\\
=&a^2s(b+c-a-2(b+c))\\
=&-a^2\cdot 2s^2.
\end{aligned}$$

Similarly, the other two components are $-b^2\cdot 2s^2$ and $-c^2\cdot 2s^2$. It follows that

$$\begin{aligned}
Q=&\frac{1}{16Rrs^2}\left(4s^2(s^2-r^2)O-2s^2(a^2,b^2,c^2)\right)\\
=&\frac{1}{4Rr}\left((s^2-r^2)O-\frac{1}{2}(a^2+b^2+c^2)K\right).
\end{aligned} \tag{1}$$

□

5. The Apollonius circle as a Tucker circle

It is well known that the centers of Tucker circles also lie on the Brocard axis. According to [8], a Tucker hexagon/circle has three principal parameters:

- the chordal angle $\phi\in\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right]$,
- the radius of the Tucker circle

$$r_\phi=\left|\frac{R}{\cos\phi+\cot\omega\sin\phi}\right|,$$

- the length of the equal antiparallels

$$d_\phi=2r_\phi\cdot\sin\phi.$$

This length d_ϕ is negative for $\phi<0$. In this way, for a given d_ϕ, there is one and only one Tucker hexagon with d_ϕ as the length of the antiparallel segments. In other words, a Tucker circle can be uniquely identified by d_ϕ. The center of the Tucker circle is the isogonal conjugate of the Kiepert perspector $K\left(\frac{\pi}{2}-\phi\right)$. Explicitly, this is the point

$$\frac{4\triangle\cot\phi\cdot O+(a^2+b^2+c^2)K}{4\triangle\cot\phi+(a^2+b^2+c^2)}.$$

Comparison with (1) shows that $4\triangle\cot\phi=-2(s^2-r^2)$. Equivalently,

$$\tan\phi=-\frac{2rs}{s^2-r^2}.$$

This means that $\phi = -2\arctan\frac{r}{s}$. Clearly, since $s > r$,

$$\cos\phi = \frac{s^2 - r^2}{r^2 + s^2}, \qquad \sin\phi = -\frac{2rs}{r^2 + s^2}.$$

Now, the radius of the Tucker circle with chordal angle $\phi = -2\arctan\frac{r}{s}$ is given by

$$r_\phi = \left|\frac{R}{\cos\phi + \cot\omega\sin\phi}\right| = \frac{r^2 + s^2}{4r}.$$

This is exactly the radius of the Apollonius circle. We therefore conclude that the Apollonius circle is the Tucker circle with chordal angle $-2\arctan\frac{r}{s}$. The common length of the antiparallels is

$$d_\phi = 2r_\phi \cdot \sin\phi = 2 \cdot \frac{r^2 + s^2}{4r} \cdot \frac{-2rs}{r^2 + s^2} = -s.$$

This proves Theorem 1 and justifies the construction in Figure 3.

6. Concluding remarks

We record the coordinates of the vertices of the Tucker hexagon.[4]

$$\begin{aligned} B_c &= (-as : 0 : as + bc), & C_b &= (-as : as + bc : 0), \\ A_b &= (0 : cs + ab : -cs), & B_a &= (cs + ab : 0 : -cs), \\ C_a &= (bs + ca : -bs : 0), & A_c &= (0 : -bs : bs + ca). \end{aligned}$$

From these, the power of A with respect to the Apollonian circle is

$$-\frac{cs}{a}\left(b + \frac{as}{c}\right) = \frac{-s(bc + as)}{a}.$$

Similarly, by computing the powers of B and C, we obtain the equation of the Apollonius circle as

$$a^2yz + b^2zx + c^2xy + s(x + y + z)\sum_{\text{cyclic}}\frac{bc + as}{a}x = 0.$$

Finally, with reference to Figure 1, Iwata and Fukagawa [5] have shown that triangles $F_a'F_b'F_c'$ and ABC are perspective at a point P on the line IQ with $IP : PQ = -r : \rho$.[5] They also remarked without proof that according to a Japanese wooden tablet dating from 1797,

$$\rho = \frac{1}{4}\left(\frac{s^4}{r_ar_br_c} + \frac{r_ar_br_c}{s^2}\right),$$

which is equivalent to $\rho = \frac{r^2+s^2}{4r}$ established above.

---

[4] These coordinates are also given by Jean-Pierre Ehrmann [2].

[5] This perspector is the Apollonius point $X_{181} = \left(\frac{a^2(b+c)^2}{s-a} : \frac{b^2(c+a)^2}{s-b} : \frac{c^2(a+b)^2}{s-c}\right)$ in [7]. In fact, the coordinates of F_a' are $(-a^2(a(b+c)+(b^2+c^2))^2 : 4b^2(c+a)^2s(s-c) : 4c^2(a+b)^2s(s-b))$; similarly for F_b' and F_c'.

References

[1] A. Aeppli: Das Taktionsproblem von Apollonius, angewandt auf die vier Berührungskreise eines Dreiecks, *Elemente der Mathematik*, 13 (1958) 25–30.
[2] J.-P. Ehrmann, Hyacinthos message 4620, January 1, 2002.
[3] E. Rouché et Ch. de Comberousse, *Traité de Géométrie*, Gauthier-Villars, Paris, 1935.
[4] R. A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint.
[5] C. Kimberling, S. Iwata, H. Fukagawa and T. Seimiya, Problem 1091 and solution, *Crux Math.*, 11 (1985) 324; 13 (1987) 128–129, 217–218.
[6] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.
[7] C. Kimberling, *Encyclopedia of Triangle Centers*, 2000 http://www2.evansville.edu/ck6/encyclopedia/.
[8] F. M. van Lamoen, Some concurrencies from Tucker hexagons, *Forum Geom.*, 2 (2002) 5–13.
[9] R. Stärk, Ein Satz über Eckentfernungen beim Dreieck, *Elemente der Mathematik*, 45 (1990) 155–165.

Darij Grinberg: Geroldsäckerweg 7, D-76139 Karlsruhe, Germany
E-mail address: darij_grinberg@web.de

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida, 33431-0991, USA
E-mail address: yiu@fau.edu