Notes on Measure and Integration, and the underlying structures

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Part I

A vocabulary of structures

Chapter 1

1.1 Introduction

1.1.1 Structures on a set

1.1.1.1

A very general type of mathematical structures is obtained by equipping a set X with one or more subsets $\Gamma \subseteq F(X)$ where $F(X)$ is a set *naturally* associated with set X. 'Naturally' here means that any map $f\colon X \longrightarrow Y$ induces a map

$$f_*\colon F(X) \longrightarrow F(Y)$$

or a map

$$f^*\colon F(Y) \longrightarrow F(X)$$

1.1.1.2

In the first case we expect that

$$(f \circ g)_* = f_* \circ g_*,$$

and we speak of *covariant* dependence on X, in the second case we require that

$$(f \circ g)^* = g^* \circ f^*,$$

and we speak of *contravariant* dependence on X.

1.1.1.3

In modern Mathematics, such associations are called *covariant* and *contravariant functors* from the category of sets to the category of sets.

1.1.2 A few examples of such functors

1.1.2.1 Cartesian powers

Given a set I, consider the correspondence that associates with a set X its I-th Cartesian power

$$X \rightsquigarrow X^I := \{\{x_i\}_{i\in I} \mid x_i \in X\}\,.$$

The Cartesian power is a covariant functor, a map $f\colon X \longrightarrow Y$ induces the map

$$f_*\colon X^I \longrightarrow Y^I, \qquad f_*\left(\{x_i\}_{i\in I}\right) := \{f(x_i)\}_{i\in I}.$$

1.1.2.2 Exponents

Given a set A, consider the correspondence that associates with a set X the set of maps from X to A

$$X \rightsquigarrow A^X := \{\phi\colon X \longrightarrow A\}.$$

This functor is contravariant:

$$f^*\colon A^Y \longrightarrow A^X, \qquad f^*(\phi) := \phi \circ f. \tag{1.1}$$

1.1.2.3 The power set as a covariant functor

This is the functor that associates with a set X, the set $\mathscr{P}(X)$ of all of its subsets, and to $f\colon X \longrightarrow Y$, the *image-of-the-subset* map:

$$\mathscr{P}(X) \ni A \longmapsto f(A) := \{y \in Y \mid y = f(x) \text{ for some } x \in X\}.$$

1.1.2.4 The power set as a contravariant functor

This functor associates with a set X, the same set $\mathscr{P}(X)$, and to $f\colon X\longrightarrow Y$, the *preimage-of-the-subset* map:

$$\mathscr{P}(Y) \ni B \longmapsto f^{-1}(B) := \{x \in X \mid f(x) \in B\}.$$

1.1.2.5

For any set X, there exists a *natural* bijection[1]

$$\chi^X\colon \mathscr{P}(X)\longrightarrow 2^X, \qquad A \longmapsto \chi^X_A, \tag{1.2}$$

where

$$\chi^X_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

is the *characteristic* function of a subset $A \subseteq X$. In the interest of simplifying notation when possible, the superscript X is dropped when X is clear from the context.

1.1.2.6

'Naturality' of (1.2) means that, given a map $f\colon X\longrightarrow Y$, the following diagram commutes,

$$\begin{tikzcd}
\mathscr{P}(X) \arrow[d, "\chi^X"'] & \mathscr{P}(Y) \arrow[l, "f^{-1}"'] \arrow[d, "\chi^Y"] \\
2^X & 2^Y \arrow[l, "f^*"]
\end{tikzcd},$$

i.e., the composition of of arrows either way produces the same result

$$\chi^X \circ f^{-1} = f^* \circ \chi^Y.$$

In categorical language, we could say that χ is a *natural transformation* of the contravariant power-set functor $\mathscr{P}(\ )$ into the exponent functor $2^{(\ )}$ (in this case an *isomorphism* of functors, since all the maps χ^X are isomorphisms in the category of sets, i.e., they are invertible maps).

[1] In the language of sets, $0 = \varnothing$ and $n = \{0, \ldots, n-1\}$.

1.1.2.7

Besides the category of sets there are other categories of interest in Mathematics, and there exist several interesting functors between them. Categorical language allows one to see various 'natural' constructions in a clear light, and it facilitates noticing connections between seemingly distant concepts and subjects. For this reason, it became very popular in modern Mathematics to the point of being indispensible, and a 'must-learn' for a beginner. We shall use it too.

1.1.2.8

You are encouraged to familiarize yourself with the language of categories and functors as soon as possible and, after mastering the basics of categorical grammar, to learn also at least the concepts of an equivalence of categories and of a pair of adjoint functors, and study numerous fundamentally important examples these two concepts. To facilitate this, I include the most besic definitions below.

Like with any language, acquiring proficiency requires constant use, so you, after learning the basic concepts, should be constantly observing these concept at work in various branches of Mathematics.

1.2 First terms in the vocabulary

1.2.1 Families

1.2.1.1 Families of sets

The term a *family of sets* is used in two meanings: as a subset $\mathscr{F} \subseteq \mathscr{P}(U)$ of some set U or, as a map

$$I \longrightarrow \mathscr{P}(U), \qquad i \longmapsto X_i,$$

which assigns a set X_i to $i \in I$. In the latter case we speak of a family of subsets of U *indexed* by set I. The indexing set can be arbitrary and it may come equipped with additional structure like ordering.

1.2.1.2 Notation

It is customary to denote indexed families by $\{X_i\}_{i \in I}$.

1.2.1.3

A family of subsets of U viewed as a subset of $\mathscr{P}(U)$ is conceptually simpler, as its definition does not rely on the notion of a map yet it can be viewed as a special case of an indexed family, namely as a family indexed by itself:

$$\mathscr{F} \longrightarrow \mathscr{P}(U), \qquad X \longmapsto X.$$

1.2.1.4 Families of elements of a set

A family of elements of a set X will be always used in the sense of a family indexed by some set I. By definition it is a map

$$I \longrightarrow X, \qquad i \longmapsto x_i.$$

Conceptually, there is no difference between a *family of elements* of X and a *map* $I \longrightarrow X$. The difference is exclusively in notation and in the points of emphasis.

In the language of *families of elements* the focus is on X and its elements. The nature of the indexing set is secondary and generally not very important.

In the language of *maps*, the source and the target of a map are on equal footing, and the map itself is usually sufficiently important to merit its own symbol in notation.

1.2.1.5 Sequences

Families indexed by subsets of the set of natural numbers or, more generally, by ordered countable sets, are called *sequences*.

1.2.1.6 n-tuples

Families indexed by $I = \{1, \ldots, n\}$ are called *ordered n-tuples* of elements of X, and notation

$$(x_1, \ldots, x_n) \qquad \text{instead of} \qquad \{x_i\}_{i \in \{1,\ldots,n\}}$$

is generally used. Ordered 2-, 3-, 4-tuples are respectively called ordered *pairs*, *triples*, *quadruples*.

1.2.2 Operations involving families of sets

1.2.2.1 Union

The union of a family $\mathscr{F} \subseteq \mathscr{P}(U)$ is the set

$$\{u \in U \mid u \in X \text{ for some } X \in \mathscr{F}\}.$$

This set is denoted

$$\bigcup \mathscr{F} \qquad \text{or} \qquad \bigcup_{X \in \mathscr{F}} X.$$

1.2.2.2

The union of an indexed family $\{X_i\}_{i \in I}$ is defined similarly

$$\bigcup_{i \in I} X_i := \{u \in U \mid u \in X_i \text{ for some } i \in I\}.$$

1.2.2.3 Intersection

The intersection a family $\mathscr{F} \subseteq \mathscr{P}(U)$ is the set

$$\{u \in U \mid u \in X \text{ for every } X \in \mathscr{F}\}.$$

This set is denoted

$$\bigcap \mathscr{F} \qquad \text{or} \qquad \bigcap_{X \in \mathscr{F}} X.$$

1.2.2.4

The intersection of an indexed family $\{X_i\}_{i \in I}$ is defined similarly

$$\bigcap_{i \in I} X_i := \{u \in U \mid u \in X_i \text{ for every } i \in I\}.$$

1.2.2.5 Cartesian product

The Cartesian product of an indexed family $\{X_i\}_{i\in I}$ is the set of all families $\xi = \{x_i\}_{i\in I}$ of elements of $\bigcup_{i\in I} X_i$ such that $x_i \in X_i$:

$$\prod_{i\in I} X_i := \big\{\{x_i\}_{i\in I} \mid x_i \in X_i\big\}.$$

Equivalently,

$$\prod_{i\in I} X_i := \Big\{\xi\colon I \longrightarrow \bigcup_{i\in I} X_i \mid \xi(i) \in X_i\Big\}.$$

1.2.2.6 Notation

The Cartesian product of a finite family $(X_1, \ldots, X_n)$ is usually denoted

$$X_1 \times \cdots \times X_n.$$

1.2.2.7 Comment

It is important to observe that one can replace $\bigcup_{i\in I} X_i$ in the definition of the Cartesian product by *any* set that contains all X_i. The corresponding 'products' will be essentially identical sets. This is due to the observation that there exists a canonical identification between maps $A \longrightarrow B$ whose image is contained in a subset $B' \subseteq B$, and maps $A \longrightarrow B'$.

1.2.2.8 Canonical projections

The Cartesian product comes equipped with the family of surjective maps,

$$\pi_i\colon \prod_{j\in I} X_j \longrightarrow X_i \qquad \xi \longmapsto x_i \qquad (i \in I),$$

which send a map $\xi\colon I \longrightarrow \bigcup_{i\in I} X_i$ to its value at each i. When $I = \{1, \ldots, n\}$, then π_i is the i-th coordinate map

$$\pi_i\colon (x_1, \ldots, x_n) \longmapsto x_i \qquad (i = 1, \ldots, n).$$

1.2.2.9 A universal property of the Cartesian product

Given any set Y and a family $\{f_i\}_{i\in I}$ of maps $f_i\colon Y\longrightarrow X_i$, there exists a unique map $\tilde{f}\colon Y\longrightarrow \prod_{i\in I} X_i$ such that

$$f_i = \pi_i \circ \tilde{f} \qquad (i \in I). \tag{1.4}$$

Exercise 1 *Verify that the map*

$$\tilde{f}\colon y \longmapsto \{f_i(y)\}_{i\in I} \qquad (y \in Y)$$

satisfies (1.4)*, and that any map $g\colon Y\longrightarrow \prod_{i\in I} X_i$ which satisfies* (1.4) *coincides with $\tilde{f}$.*

1.2.2.10 Disjoint unions of sets

The disjoint union of an indexed family $\{X_i\}_{i\in I}$ should be thought of as the union of all sets X_i except that we keep as many distinct 'copies' of an element $x \in \bigcup_{i\in I} X_i$ as there are sets X_i which contain x. We achieve this by 'tagging' every element in $\bigcup_{i\in I} X_i$ by the index of the set it belongs to:

$$\coprod_{i\in I} X_i := \big\{(i,x) \in I \times \bigcup_{i\in I} X_i \mid x \in X_i\big\}.$$

1.2.2.11 Notation

The disjoint union of a finite family $(X_1, \ldots, X_n)$ is usually denoted

$$X_1 \sqcup \cdots \sqcup X_n.$$

Exercise 2 *Denote by p the composition of the inclusion map and the canonical projection*

$$\coprod_{i\in I} X_i \hookrightarrow I \times \bigcup_{i\in I} X_i \longrightarrow \bigcup_{i\in I} X_i. \tag{1.5}$$

Show that p is surjective. Show that the fiber $p^{-1}(x)$ at $x \in \bigcup_{i\in I} X_i$ is

$$p^{-1}(x) = \{(i,x) \mid x \in X_i\}.$$

In particular, $p^{-1}(x)$ is in on-to-one correspondence with the set

$$\{i \in I \mid x \in X_i\}.$$

1.2.2.12

It follows that the disjoint union of a family of sets $\{X_i\}_{i\in I}$ is canonically identified with their union if and only if sets X_i are disjoint for distinct $i \in I$:

$$X_i \cap X_j = \emptyset \qquad (i \neq j).$$

1.2.2.13 Canonical inclusions

The disjoint union comes equipped with the family of injective maps,

$$\iota_i \colon X_i \longrightarrow \coprod_{j\in I} X_j \qquad x \longmapsto (i, x) \qquad (i \in I).$$

1.2.2.14 A universal property of the disjoint union

Given any set Y and a family $\{f_i\}_{i\in I}$ of maps $f_i \colon X_i \longrightarrow Y$, there exists a unique map $\tilde{f} \colon \coprod_{i\in I} X_i \longrightarrow Y$ such that

$$f_i = \tilde{f} \circ \iota_i \qquad (i \in I). \tag{1.6}$$

Exercise 3 *Verify that the map*

$$\tilde{f} \colon (i, x) \longmapsto f_i(x) \qquad (i \in I; x \in X_i)$$

satisfies (1.6)*, and that any map $g \colon \coprod_{i\in I} X_i \longrightarrow Y$ which satisfies* (1.6) *coincides with $\tilde{f}$.*

1.2.2.15

Map p defined in (1.5) is precisely such universal map $\tilde{f}$ for the family of inclusion maps

$$f_i \colon X_i \hookrightarrow \bigcup_{j\in I} X_j \qquad (i \in I).$$

1.2.2.16

Note that the properties of the Cartesian product and of the disjoint union of a family of sets are *dual* to each other. We shall explain this concept of *duality* later.

1.2.3 Associativity properties of operations on families of sets

1.2.3.1 Associativity of union

Suppose we have two families of sets

$$\{X_i\}_{i\in I} \qquad \text{and} \qquad \{X_k\}_{k\in K}.$$

The iterated union

$$\bigcup_{i\in I} X_i \ \cup \ \bigcup_{k\in K} X_k$$

and the union

$$\bigcup_{l\in I\sqcup K} X_l$$

are *equal* as sets. In the case of a pair of finite families $(A_1,\ldots,A_m)$ and $(B_1,\ldots,B_n)$, this equality acquires the form

$$(A_1\cup\cdots\cup A_m)\cup(B_1\cup\cdots\cup B_n) = A_1\cup\cdots\cup A_m\cup B_1\cup\cdots\cup B_n.$$

1.2.3.2 The total family

In general, given any family of families of sets

$$\Big\{\{X_{i_j}\}_{i_j\in I_j}\Big\}_{j\in J}, \tag{1.7}$$

the universal property of the disjoint union allows us to form the *total* family

$$\{X_l\}_{l\in L} \qquad \text{where} \qquad L = \coprod_{j\in J} I_j.$$

Indeed, regarding all sets to be subsets of a common set U, family of families of (1.7) is the same as a family of maps $I_j \longrightarrow \mathscr{P}(U)\}_{j\in J}$ and, by the universal property of disjoint union, there exists a unique map $L\longrightarrow\mathscr{P}(U)$ whose 'restrictions' to I_j are the component-families $I_j\longrightarrow\mathscr{P}(U)\}_{j\in J}$.

We shall refer to $L\longrightarrow\mathscr{P}(U)$ as the *total* family.

1.2.3.3

Now we are ready to make an observation about iterated unions of families. The following sets are equal

$$\bigcup_{j\in J}\bigcup_{i_j\in I_j} X_{i_j} = \bigcup_{l\in L} X_l.$$

Exercise 4 *Formulate the corresponding associativity laws for intersection of families.*

1.2.3.4 Associativity of Cartesian product

In the case of Cartesian product instead of equality we have a canonical identification between the iterated product of a family of families of sets and the product of the total family.

Let us consider first the case of a pair of families of sets

$$\{X_i\}_{i\in I} \qquad \text{and} \qquad \{X_k\}_{k\in K}.$$

The natural correspondence

$$(\{x_i\}_{i\in I}, \{x_k\}_{k\in K}) \;\leftrightarrow\; \{x_l\}_{l\in I\sqcup K}$$

identifies the iterated Cartesian product

$$\prod_{i\in I} X_i \;\times\; \prod_{k\in K} X_k$$

with

$$\prod_{l\in I\sqcup K} X_l.$$

In the case of a pair of finite families $(A_1, \ldots, A_m)$ and $(B_1, \ldots, B_n)$, this identification acquires the form

$$\big((a_1, \ldots, a_m), (b_1, \ldots, b_n)\big) \leftrightarrow (a_1, \ldots, a_m, b_1, \ldots, b_n).$$

1.2.3.5

In general, given any family of families of sets (1.7), the iterated Cartesian product and the product of the total family are naturally identified

$$\prod_{j\in J}\prod_{i_j\in I_j} X_{i_j} \longleftrightarrow \prod_{l\in L} X_l \qquad \text{where} \qquad L = \coprod_{j\in J} I_j. \tag{1.8}$$

Indeed, elements of $\prod_{i_j\in I_j} X_{i_j}$ are maps

$$\xi_j : I_j \longrightarrow \bigcup_{i_j\in I_j} X_{i_j}$$

such that $\xi_j(i_j) \in X_{i_j}$. By composing maps ξ_j with the inclusions

$$\bigcup_{i_j\in I_j} X_{i_j} \hookrightarrow U := \bigcup_{l\in L} X_l,$$

we can consider all ξ_j as being maps with the common target U. Thus, elements of

$$\prod_{j\in J}\prod_{i_j\in I_j} X_{i_j}$$

become families $\{\xi_j\}_{j\in J}$ of maps $\xi_j\colon I_j \longrightarrow U$. By the universal property of the disjoint union, there exists a unique map $\tilde{\xi}\colon L \longrightarrow U$ whose 'restrictions' to I_j are families maps $\xi_j\colon I_j \longrightarrow U$.

This map $\tilde{\xi}$ is an element of $\prod_{l\in L} X_l$. Since the correspondence between families $\{\xi_j\}_{j\in J}$ and maps $\tilde{\xi}$ is bijective, the correspondence in (1.8) is bijective.

Exercise 5 *Formulate and prove the corresponding associativity laws for disjoint union.*

1.2.3.6 Calculus of Cartesian powers of a set

For any sets A, B, and C, one has natural identifications

$$A^B \times A^C \longleftrightarrow A^{B\sqcup C} \tag{1.9}$$

and, more generally,

$$\prod_{j\in J} A^{B_j} \longleftrightarrow A^{\coprod_{j\in J} B_j}$$

which are special cases of identifications (1.8).

1.2.3.7

One has also the following natural identification

$$\left(A^B\right)^C \longleftrightarrow A^{B\times C} \tag{1.10}$$

given by the following pair of mutually inverse correspondences

$$\left(A^B\right)^C \ni\ f \longmapsto\ \phi \in A^{B\times C}, \qquad \text{where} \qquad \phi(b,c) := \big(f(c)\big)(b)$$

and

$$A^{B\times C} \ni \phi\ \longmapsto\ f \in \left(A^B\right)^C, \qquad \text{where} \qquad f(c) := \phi(\,\cdot\,,c).$$

1.2.3.8

Using the *families-of-elements* notation instead of *maps* notation, we can describe identification (1.10) also in this form

$$\left(X^I\right)^J \longleftrightarrow X^{I\times J}, \qquad \big\{\{x_{ij}\}_{i\in I}\big\}_{j\in J} \leftrightarrow \{x_{ij}\}_{(i,j)\in I\times J}.$$

1.3 The language of categories and functors

1.3.1 Categories

1.3.1.1 Objects and morphisms

A category $\mathcal{C}$ consists of two classes: $\mathcal{C}_0$ (the class of *objects*) and $\mathcal{C}_1$ (the class of *morphisms*, informally referred to as 'arrows' since they are visualized by drawing arrows in various diagrams).

Note that we are saying *classes*—not *sets*. Basic concepts of Category Theory require from foundations on which rests the edifice of Mathematics to allow talking about classes that are not sets, like the class of all sets, the class of all singleton sets, the class of all vector spaces over a given field of coefficients, etc.

We henceforth will be cautiously extending to classes certain terminology and notation usually associated with sets. For example, we may indicate that a is an object of category $\mathcal{C}$ by writing either $a \in \mathcal{C}_0$ or $a \in \operatorname{Ob}\mathcal{C}$. Similarly, we may say that α is a morphism of category $\mathcal{C}$ by writing either $\alpha \in \mathcal{C}_1$ or $\alpha \in \operatorname{Arr}\mathcal{C}$.

1.3.1.2

The class of morphisms is supposed to be related to the class of objects by the following two correspondences:

$$s\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_0 \qquad \text{and} \qquad t\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_0.$$

If α a morphism, one refers to $s(\alpha)$ as the *source* of α, and to $t(\alpha)$ as the *target* of α.

1.3.1.3 $\mathbf{Hom}_{\mathcal{C}}(\alpha, \beta)$

It was observed early that if one requires in the definition of a category that, for any pair of objects $a, b \in \mathcal{C}_0$, morphisms with a as their source and with t as their target form a *set* and not just a class, then one can avoid essentially all the potential dangers arising from presence of classes in foundations of Category Theory.

This set is usually denoted $\mathrm{Hom}_{\mathcal{C}}(\alpha, \beta)$ and its elements are referred as morphisms from a to b.

1.3.1.4 The class of composable pairs of morphisms

We say that a pair (α, β) of morphisms is *composable* if $s(\alpha) = t(\beta)$. Denote by $\mathcal{C}_2$ the class of composable pairs of morphisms. We assume that a correspondence

$$m\colon \mathcal{C}_2 \longrightarrow \mathcal{C}_1, \qquad (\alpha, \beta) \longmapsto \alpha \circ \beta, \tag{1.11}$$

is given. It is referred to as *composition* of morphisms, and is possibly the single most important element of the structure of a category.

1.3.1.5 The class of composable triples of morphisms

We say that a triple (α, β, γ) of morphisms is *composable* if $s(\alpha) = t(\beta)$ and $s(\beta) = t(\gamma)$. As can be expected, we denote the class of composable triples of morphisms by $\mathcal{C}_3$. (Binary) composition (1.11) induces two correspondences $\mathcal{C}_3 \longrightarrow \mathcal{C}_2$

$$m_1\colon (\alpha, \beta, \gamma) \longmapsto (\alpha \circ \beta, \gamma) \qquad \text{and} \qquad m_2\colon (\alpha, \beta, \gamma) \longmapsto (\alpha, \beta \circ \gamma).$$

By applying correspondence (1.11), we obtain two correspondences $\mathcal{C}_3 \longrightarrow \mathcal{C}_1$. We require them to be equal which means that

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma) \tag{1.12}$$

for any composable triple of morphisms. This condition is called *associativity* of the composition of morphisms.

1.3.1.6

Associativity identity (1.12) can be expressed as commutativity of the following diagram

$$\begin{tikzcd}
\mathcal{C}_3 \arrow[r, "m_1"] \arrow[d, "m_2"'] & \mathcal{C}_2 \arrow[d, "m"] \\
\mathcal{C}_2 \arrow[r, "m"] & \mathcal{C}_1
\end{tikzcd}$$

1.3.1.7 The identity morphisms

We could stop here and call the defined structures *categories*. The classical and still a 'default' definition of a category additionally requires presence of a correspondence

$$i \colon \mathcal{C}_0 \longrightarrow \mathcal{C}_1, \qquad a \longmapsto \mathrm{id}_a \in \mathrm{Hom}_{\mathcal{C}}(a, a),$$

such that

$$\alpha \circ \mathrm{id}_a = \alpha \qquad \text{and} \qquad \mathrm{id}_b \circ \alpha = \alpha \tag{1.13}$$

for any $\alpha \in \mathrm{Hom}_{\mathcal{C}}(a, b)$. Morphism id_a is referred to as the *identity* morphism of object a.

1.3.1.8

Each of the identities in (1.13) can be expressed as commutativity of a diagram of correspondences:

$$
\begin{tikzcd}
\mathcal{C}_1 \times \mathcal{C}_1 \arrow[r, "(\mathrm{id}_{\mathcal{C}_1}, i)"] & \mathcal{C}_2 \arrow[d, "m"] \\
\mathcal{C}_1 \arrow[u, "(\mathrm{id}_{\mathcal{C}_1}, s)"] \arrow[r, equal, "\mathrm{id}_{\mathcal{C}_1}"] & \mathcal{C}_1
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{C}_1 \times \mathcal{C}_1 \arrow[r, "(i, \mathrm{id}_{\mathcal{C}_1})"] & \mathcal{C}_2 \arrow[d, "m"] \\
\mathcal{C}_1 \arrow[u, "(t, \mathrm{id}_{\mathcal{C}_1})"] \arrow[r, equal, "\mathrm{id}_{\mathcal{C}_1}"] & \mathcal{C}_1
\end{tikzcd}
$$

1.3.1.9

There are very good reasons not to require presence of the identity morphisms in general, and to call the categories that possess such morphisms—*unital* categories.

1.3.1.10 Isomorphisms

We say that a morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ is an *isomorphism* if there exists $\beta \in \mathrm{Hom}_{\mathcal{C}}(b, a)$ such that

$$\alpha \circ \beta = \mathrm{id}_b \qquad \text{and} \qquad \beta \circ \alpha = \mathrm{id}_a\,. \tag{1.14}$$

Exercise 6 *Show that if there exist morphisms $\beta, \gamma \in \mathrm{Hom}_{\mathcal{C}}(b, a)$ such that*

$$\alpha \circ \beta = \mathrm{id}_b \qquad \textit{and} \qquad \gamma \circ \alpha = \mathrm{id}_a\,.$$

then $\beta = \gamma$.

1.3.1.11

In view of the above exercise, if there exists at least one *right* inverse and at least one *left* inverse for a morphism α, then they are equal, which implies that the two-sided inverse, (1.14), is unique when it exists. It is denoted α^{-1}.

1.3.1.12 Endomorphisms of an object

Morphisms $\alpha\colon a \longrightarrow a$ are called *endomorphisms* of object a. The set $\mathrm{Hom}_{\mathcal{C}}(a, a)$ is often denoted $\mathrm{End}_{\mathcal{C}}(a)$.

1.3.1.13 Automorphisms of an object

Isomorphisms $\alpha\colon a \longrightarrow a$ are called *automorphisms* of object a. The set of automorphisms is denoted $\mathrm{Aut}_{\mathcal{C}}(a)$.

1.3.1.14 Symmetries

Before categorical language was proposed and developed as means to describe and study underlying structure of numerous areas of Mathematics, automorphisms of various objects: geometric, physical systems, etc—were often called *symmetries*.

1.3.1.15 The category of sets

The category of sets usually takes pride of being presented as the first example of a category. We shall denote it Set.

Sets are its objects and morphisms $X \longrightarrow Y$ are maps $X \longrightarrow Y$:

$$\mathrm{Hom}_{\mathsf{Set}}(X, Y) = Y^X.$$

Isomorphisms in the category of sets coincide with the class of bijections.

1.3.1.16 Discrete categories

There are much simpler categories than the categories of sets. The simplest, are perhaps the categories with the *empty* class of morphisms. Such categories are referred to as *discrete*.

1.3.1.17 Discrete unital categories

Every unital category is supposed to have at least the identity morphisms for each object. For this reason, in the context of unital categories *discrete* means: *no morphisms besides the identity morphisms*.

1.3.1.18 Small categories

If the class of objects forms a set, such a category is called a *small* category. In this case, the class of morphisms is a set too. Indeed, it is the union

$$\mathcal{C}_1 = \bigcup_{(a,b)\in\mathcal{C}_0\times\mathcal{C}_0} \mathrm{Hom}_{\mathcal{C}}(a,b)$$

of the family of $\mathrm{Hom}_{\mathcal{C}}(a,b)$ which is indexed by the Cartesian square of the set of objects.

1.3.1.19

Several fundamentally important structures in Mathematics can be interpreted as small categories. We give here just one yet very important example of such structures: a *preordered* set. Other examples will appear later.

1.3.1.20 Preordered sets

We say that a binary relation $\mathrel{-\!\!3}$ on a set X is a *preorder* (the term *quasiorder* is used too), if it is *reflexive*,

$$x \mathrel{-\!\!3} x \qquad (x \in X),$$

and *transitive*

$$\textit{if} \ \ x \mathrel{-\!\!3} y \ \ \textit{and} \ \ y \mathrel{-\!\!3} z\,, \ \textit{then} \ \ x \mathrel{-\!\!3} z \qquad (x,y,z \in X). \tag{1.15}$$

Of these two properties transitivity is far more important.

A *preordered* set. i.e., a set equipped with a preorder gives rise to the category whose objects are elements of X, and $\mathrm{Hom}(x,y)$ consists of a single element, if $x \mathrel{-\!\!3} y$, and is empty otherwise. Since $\mathrm{Hom}(x,y)$ has at most one element, it does not matter how does one denote it. One may use, for example, symbol $\mathrel{-\!\!3}$ or, to indicate its source and target, $x \mathrel{-\!\!3} y$.

Note that in the associated category, objects x and y are isomorphic if and only if $x \mathrel{-\!\!3} y$ and $y \mathrel{-\!\!3} x$.

1.3.1.21

Vice-versa, any small category $\mathcal{C}$ with the property that, for any $a, b \in \mathcal{C}_0$,

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \textit{ has at most one element,} \tag{1.16}$$

is obtained this way.

Exercise 7 *For a small category that satisfies* (1.16)*, show that*

$$x \preceq y \qquad \textit{if} \qquad \mathrm{Hom}_{\mathcal{C}}(x, y) \neq \emptyset$$

defines a preorder relation on $X \coloneqq \mathcal{C}_o$.

1.3.1.22 Partially ordered sets

A *partial order* on a set X is a preorder which is *weakly antisymmetric*

$$\textit{if } \ x \preceq y \ \textit{ and } \ y \preceq x\,, \ \textit{ then } \ x = y. \tag{1.17}$$

1.3.1.23

Small discrete categories correspond to discrete partially ordered sets, i.e., the sets equipped with the smallest order relation—the *identity* relation:

$$x \preceq_{\mathrm{discr}} y \qquad \text{if} \qquad x = y.$$

1.3.2 Functors

1.3.2.1

A *functor* $F \colon \mathcal{C} \rightsquigarrow \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of two correspondences: between the classes of objects and between the classes of morphisms

$$F_0 \colon \mathcal{C}_0 \longrightarrow \mathcal{D}_0 \qquad \text{and} \qquad F_1 \colon \mathcal{C}_1 \longrightarrow \mathcal{D}_1$$

which are compatible with all the elements of the category structure. The latter means that the following diagrams of correspondences

$$
\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "F_0"] & \mathcal{D}_0 \\
\mathcal{C}_1 \arrow[u, "s"] \arrow[r, "F_1"] \arrow[d, "t"'] & \mathcal{D}_1 \arrow[u, "s"'] \arrow[d, "t"] \\
\mathcal{C}_0 \arrow[r, "F_0"] & \mathcal{D}_0
\end{tikzcd}
\tag{1.18}
$$

and

$$
\begin{tikzcd}
\mathcal{C}_2 \arrow[r, "F_2"] \arrow[d, "m"'] & \mathcal{D}_2 \arrow[d, "m"] \\
\mathcal{C}_1 \arrow[r, "F_1"] & \mathcal{D}_1
\end{tikzcd}
\tag{1.19}
$$

are commutative. Here, F_2 denotes the correspondence induced by F_1 on the classes of composable pairs:

$$F_2\colon \mathcal{C}_2 \longrightarrow \mathcal{D}_2, \qquad (\alpha, \beta) \longmapsto (F_1(\alpha), F_1(\beta)).$$

1.3.2.2 Unital functors

When the corresponding categories are *unital*, i.e., possess identity morphisms, then it is customary to require that a functor $F\colon \mathcal{C} \rightsquigarrow \mathcal{D}$ is compatible also with the identities. This means that the diagram

$$
\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "F_0"] \arrow[d, "\mathrm{id}"'] & \mathcal{D}_0 \arrow[d, "\mathrm{id}"] \\
\mathcal{C}_1 \arrow[r, "F_1"] & \mathcal{D}_1
\end{tikzcd}
\tag{1.20}
$$

is supposed to commute. We shall call such functors *unital*.

1.3.2.3

In the interest of keeping notation as transparent as possible it is customary to omit subscript indices and denote the correspondences between the objects, morphisms, composable pairs of morphisms, etc., using the same symbol F.

1.3.2.4

Commutativity of the two squares in diagram (1.18) then can be expressed as

$$s(F(\alpha)) = F(s(\alpha)) \qquad \text{and} \qquad t(F(\alpha)) = F(t(\alpha)) \qquad (\alpha \in \mathcal{C}_0),$$

while commutativity of diagram (1.19) expresses the fact that

$$F(\alpha) \circ F(\beta) = F(\alpha \circ \beta)$$

for any pair of composable morphisms α and β in $\mathcal{C}$.

Finally, commutativity of diagram (1.20) means that

$$\mathrm{id}_{F(a)} = F\left(\mathrm{id}_a\right) \qquad (a \in \mathcal{C}_0).$$

1.3.2.5 Contravariant functors

The functors we defined above are also called *covariant* functors. The *contravariant* variety is obtained if one requires instead

$$s(F(\alpha)) = F(t(\alpha)) \qquad \text{and} \qquad t(F(\alpha)) = F(s(\alpha)) \qquad (\alpha \in \mathcal{C}_0), \tag{1.21}$$

and

$$F(\alpha) \circ F(\beta) = F(\alpha \circ \beta) \tag{1.22}$$

for any pair of composable morphisms α and β in $\mathcal{C}$.

Exercise 8 *Express requirements* (1.21) *and* (1.22) *with help of diagrams analogous to* (1.18) *and* (1.19).

1.3.2.6

Functors very often encode natural constructions in Mathematics. We have already encountered a few functors in Section 1.1.2 of the Introduction, all being functors Set $\rightsquigarrow$ Set, the first and the third being covariant, the second and the fourth being contravariant.

1.3.2.7 Subcategories

For a category $\mathcal{C}$, suppose that, a pair of subclasses $\mathcal{C}'_0 \subseteq \mathcal{C}_0$ and $\mathcal{C}'_1 \subseteq \mathcal{C}_1$ is given such that the source and the target of any morphism in $\mathcal{C}'_1$ is a member of $\mathcal{C}'_0$ and the composition of any two such morphisms is a member of $\mathcal{C}'_1$.

If we equip the pair of classes $(\mathcal{C}_0, \mathcal{C}_1)$ with the source, target, and multiplication correspondences induced from category $\mathcal{C}$, we obtain a category on its own. Denote it $\mathcal{C}'$.

This situation arises frequently. We say that $\mathcal{C}'$ is a *subcategory* of $\mathcal{C}$.

1.3.2.8 Full subcategories

If

$$\mathrm{Hom}_{\mathcal{C}'}(a, b) = \mathrm{Hom}_{\mathcal{C}}(a, b) \qquad (a, b \in \mathcal{C}_0),$$

then we say that $\mathcal{C}'$ is a *full* subcategory of category $\mathcal{C}$.

1.3.2.9 The canonical inclusion functors

Given a subcategory $\mathcal{C}'$ of a category $\mathcal{C}$, the natural inclusion correspondences $\iota_0 \colon \mathcal{C}'_0 \longrightarrow \mathcal{C}_0$ and $\iota_1 \colon \mathcal{C}'_1 \longrightarrow \mathcal{C}_1$ define the *inclusion* functor $\iota \colon \mathcal{C}' \rightsquigarrow \mathcal{C}$.

1.3.2.10 The category of small categories

The category whose objects are small categories and morphisms are *covariant* functors between small categories is itself a category. It is denoted Cat and is called the category of (small nonunital) categories.

1.3.2.11 The category of small unital categories

If we consider only unital small categories and unital functors, then we obtain the category of small unital categories. We shall denote it here Cat_1. The reader should be warned that since categories are usually assumed to possess identity morphisms, the category of small unital categories is often denoted Cat.

1.3.2.12 The category of sets viewed as a subcategory of the category of small categories

Let us identify sets X with small discrete categories $\mathcal{X}$,

$$\mathcal{X}_0 = X, \qquad \mathcal{X}_1 = \emptyset.$$

Any map $f\colon X \longrightarrow Y$ defines a functor $F\colon \mathcal{X} \longrightarrow \mathcal{Y}$,

$$F_0 = f, \qquad F_1 = \mathrm{id}_\emptyset,$$

and every functor $F\colon \mathcal{X} \longrightarrow \mathcal{Y}$ is necessarily of this form since $\mathrm{id}_\emptyset$ is the only map from $\emptyset$ to $\emptyset$.

In particular, the category of sets can be viewed as a full subcategory of the category of small categories.

1.3.2.13 Set viewed as a subactory of Cat

In the unital case, we associate with any set X the category $\mathcal{X}'$,

$$\mathcal{X}'_0 = X, \qquad \mathcal{X}'_1 = X$$

with all the structural correspondences being id_X (note that $\mathcal{X}'_2 = \{(x, x) \mid x \in X\}$ is here naturally identified with set X).

Any map $f\colon X \longrightarrow Y$ defines a functor $F\colon \mathcal{X} \longrightarrow \mathcal{Y}$,

$$F_0 = f, \qquad F_1 = f, \tag{1.23}$$

Exercise 9 *Show that any* unital *functor* $F\colon \mathcal{X}' \longrightarrow \mathcal{Y}'$ *is of the form* (1.23).

It follows that Set, the unital category of sets, is a full subcategory of Cat, the category of small unital categories.

1.3.2.14

Since functors between unital categories do not necessarily respect the identity morphisms (an example will be given below), Cat_1 is a subcategory of Cat yet not a full subcategory.

1.3.2.15 Natural transformations of functors

Given two (covariant) functors F and G from a category $\mathcal{C}$ to a category $\mathcal{D}$, a natural transformation between them, denoted $\phi\colon F \Rightarrow G$, consists of a single correspondence $\phi\colon \mathcal{C}_0 \longrightarrow \mathcal{D}_1$ which is compatible with all the present structures. The latter means that

$$\phi(a) \in \mathrm{Hom}_{\mathcal{D}}(F(a), G(a)) \qquad (a \in \mathcal{C}_0), \tag{1.24}$$

and, for any morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(a, b)$, the following square commutes

$$\begin{tikzcd} F(a) \arrow[r, "\phi(a)"] \arrow[d, "F(\alpha)"'] & G(a) \arrow[d, "G(\alpha)"] \\ F(b) \arrow[r, "\phi(b)"] & G(b) \end{tikzcd} \tag{1.25}$$

1.3.2.16

In the language of correspondences, conditions (1.24) translates into commutativity of the following diagram

$$\begin{tikzcd} & \mathcal{D}_0 \\ \mathcal{C}_0 \arrow[ur, "F_0"] \arrow[r, "\phi"] \arrow[dr, "G_0"'] & \mathcal{D}_1 \arrow[u, "s"'] \arrow[d, "t"] \\ & \mathcal{D}_0 \end{tikzcd} \tag{1.26}$$

while conditions (1.25) expresses commutativity of the diagram

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{(\phi\circ t, F)} & \mathcal{D}_2 \\ {\scriptstyle (G,\phi\circ s)}\Big\downarrow & & \Big\downarrow{\scriptstyle m} \\ \mathcal{D}_2 & \xrightarrow{m} & \mathcal{D}_1 \end{array} \tag{1.27}$$

Exercise 10 *Formulate the definition of a natural transformation of* contravariant *functors analogous to* (1.24)-(1.25).

Exercise 11 *Formulate the definition of a natural transformation of* contravariant *functors analogous to diagrams* (1.26)-(1.27).

1.3.2.17

We have already encountered a natural transformation of contravariant functors $\chi\colon \mathscr{P}(\ ) \Rightarrow 2^{(\ )}$ in Section 1.1.2.5.

1.3.2.18

Many properties normally expressed as identities involving objects, morphisms, sets, maps, elements of various sets, etc, can be often expressed as commutativity of certain diagrams. This leads to proliferation of what some call 'diagrammatic thinking' in modern Mathematics. Employing diagrams often can significantly clarify the picture.

On some occasions information conveyed by diagrams may be more difficult to understand than the same information expressed differently. I would say that it is probably easier to understand the meaning of conditions (1.25) than the meaning of the commutativity of diagram (1.27). That is probably due to the fact that the conditions (1.25) are themselves expressed in terms of commutativity of some easy-to-understand diagrams.

1.3.3 The opposite category

1.3.3.1

Note that if one retains the clases of objects and arrows, $\mathcal{C}_0$ and $\mathcal{C}_1$, but exchanges the source and the target correspondences, $s\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_0$ and

$t\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_0$, then one obtains a category again. This is the *opposite* category $\mathcal{C}^{\mathrm{op}}$.

1.3.3.2

More precisely,

$$\mathcal{C}_0^{\mathrm{op}} = \mathcal{C}_0, \qquad \mathcal{C}_1^{\mathrm{op}} = \mathcal{C}_1, \qquad s^{\mathrm{op}} = t, \qquad \text{and} \qquad t^{\mathrm{op}} = s. \tag{1.28}$$

If an object a of $\mathcal{C}$ is considered as an object of $\mathcal{C}^{\mathrm{op}}$, then it should be denoted a^{op}. Similarly for morphisms: if $\alpha\colon a \longrightarrow b$ is a morphism in $\mathcal{C}$, then α considered as a morphism of the opposite category is a morphism $b^{\mathrm{op}} \longrightarrow a^{\mathrm{op}}$ and it should be denoted α^{op}.

1.3.3.3

The correspondences

$$a \longmapsto a^{\mathrm{op}} \qquad \text{and} \qquad \alpha \longmapsto \alpha^{\mathrm{op}} \qquad (a \in \mathcal{C}_0; \alpha \in \mathcal{C}_1),$$

define a *contravariant* functor

$$(\ )^{\mathrm{op}}_{\mathcal{C}}\colon \mathcal{C} \rightsquigarrow \mathcal{C}^{\mathrm{op}}.$$

1.3.3.4

Note that

$$(\ )^{\mathrm{op}}_{\mathcal{C}} \circ (\ )^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}} = \mathrm{id}_{\mathcal{C}^{\mathrm{op}}} \qquad \text{and} \qquad (\ )^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}} \circ (\ )^{\mathrm{op}}_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}}\,.$$

1.3.3.5 An example: a partially ordered set

If $\mathcal{C}$ is the category that corresponds to a partially ordered set $(X, \leq)$, cf. Section 1.3.1.23, then $\mathcal{C}^{\mathrm{op}}$ corresponds to set X equipped with the *reverse* order, $\leq^{\mathrm{rev}}$.

1.3.3.6

One of the uses of the concept of the opposite category is that it allows to consider any *contravariant* functor $F\colon \mathcal{C} \rightsquigarrow \mathcal{D}$ as a *covariant* functor either $\mathcal{C}\longrightarrow\mathcal{D}^{\mathrm{op}}$ or $\mathcal{C}^{\mathrm{op}}\longrightarrow\mathcal{D}$. Formally speaking, this is done by composing F with $(\ )^{\mathrm{op}}_{\mathcal{D}}$ or $(\ )^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}}$,

$$(\ )^{\mathrm{op}}_{\mathcal{D}} \circ F\colon \mathcal{C}\longrightarrow\mathcal{D}^{\mathrm{op}} \qquad \text{or} \qquad F \circ (\ )^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}}\colon \mathcal{C}^{\mathrm{op}} \rightsquigarrow \mathcal{D}.$$

1.3.3.7

Any functor $F\colon \mathcal{C} \rightsquigarrow \mathcal{D}$, induces also a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}^{\mathrm{op}}$

$$F^{\mathrm{op}} := (\ )^{\mathrm{op}}_{\mathcal{D}} \circ F \circ (\ )^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}}. \tag{1.29}$$

Note that F^{op} is covariant (respectively, contravariant) when F is covariant (respectively, contravariant).

1.3.3.8

Assigning to any category $\mathcal{C}$ its opposite category $\mathcal{C}^{\mathrm{op}}$ is *natural* in $\mathcal{C}$, so one can expect that it gives rise to a functor on the category of (small) categories. This is so indeed, the correspondences

$$\mathcal{C} \longmapsto \mathcal{C}^{\mathrm{op}} \qquad \text{and} \qquad F \longmapsto F^{\mathrm{op}} \qquad (\mathcal{C} \in \mathrm{Cat}_0; F \in \mathcal{C}_1), \tag{1.30}$$

defined by (1.28) and (1.29), yield a functor $(\ )^{\mathrm{op}}\colon \mathrm{Cat} \rightsquigarrow \mathrm{Cat}$.

Exercise 12 *Is functor* (1.30) *covariant or contravariant?*

1.3.3.9 Importance of the opposite category concept

Any diagram in a category $\mathcal{C}$ can be interpreted as the same diagram—but with the *direction of all arrows reversed*—in the opposite category.

An immediate corollary of this simple observation yields the following *Duality Principle*:

> For any categorical concept or construction involving one or more diagrams, there is a *dual* concept or construction.

1.3.4 Categories of arrows

1.3.4.1

For any category there are several naturally associated categories whose objects are morphisms. We shall mention here three.

1.3.4.2 The category of arrows

For a category $\mathcal{C}$, let $\mathcal{C}^{\to}$ be the category whose objects are morphisms of $\mathcal{C}$,

$$(\mathcal{C}^{\to})_0 := \mathcal{C}_1,$$

and morphisms $\phi\colon \alpha \longrightarrow \beta$ are pairs of morphisms $\phi = (\phi_s, \phi_t)$ in $\mathcal{C}$,

$$\phi_s\colon s(\alpha) \longrightarrow s(\beta), \qquad \phi_t\colon t(\alpha) \longrightarrow t(\beta),$$

such that the following diagram commutes

$$\begin{tikzcd} \bullet \arrow[r, dashed, "\phi_s"] \arrow[d, "\alpha"'] & \bullet \arrow[d, "\beta"] \\ \bullet \arrow[r, dashed, "\phi_t"'] & \bullet \end{tikzcd}$$

1.3.4.3

Category of arrows $\mathcal{C}^{\to}$ is sometimes also denoted $\operatorname{Arr}\mathcal{C}$. One should be advised however, that $\operatorname{Arr}\mathcal{C}$ may also be used to denote the *class* of morphisms in $\mathcal{C}$.

1.3.4.4 Two comma categories

For any object a in a category $\mathcal{C}$, one can consider two categories: one, $\mathcal{C}^{a\to}$, whose objects are morphisms in $\mathcal{C}$ with source a,

$$(\mathcal{C}^{a\to})_0 := \{\alpha \in \mathcal{C}_1 \mid s(\alpha) = a\},$$

and another one, $\mathcal{C}^{\to a}$, whose objects are morphisms with target a,

$$(\mathcal{C}^{a\to})_0 := \{\alpha \in \mathcal{C}_1 \mid t(\alpha) = a\}.$$

1.3.4.5

Morphisms $\phi\colon \alpha \longrightarrow \beta$ in $\mathcal{C}^{a\rightarrow}$ are morphisms $\phi\colon t(\alpha) \longrightarrow t(\beta)$ such that the following diagram commutes

$$\begin{tikzcd} & \bullet \arrow[dd, dashed, "\phi"] \\ a \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\ & \bullet \end{tikzcd}$$

1.3.4.6

Morphisms $\phi\colon \alpha \longrightarrow \beta$ in $\mathcal{C}^{\rightarrow a}$ are morphisms $\phi\colon s(\alpha) \longrightarrow s(\beta)$ such that the following diagram commutes

$$\begin{tikzcd} \bullet \arrow[dr, "\alpha"] \arrow[dd, dashed, "\phi"'] & \\ & a \\ \bullet \arrow[ur, "\beta"'] & \end{tikzcd}$$

1.3.5 Categories of diagrams

1.3.5.1

(Covariant) functors from a *small* category Γ to an arbitrary category $\mathcal{C}$ form a category, denoted $\mathcal{C}^{\Gamma}$, with morphisms $\phi\colon F \longrightarrow G$ being natural transformations of functors.

1.3.5.2 Diagrams as functors

Such functors are often called *diagrams in* $\mathcal{C}$ and the reason will become clear when we look at a series of simple examples.

1.3.5.3 $\mathcal{C}$

Consider the category with a single object, 0, with empty class of morphisms. Denote this category by $\mathbf{1}$. Functors from $\mathbf{1}$ to $\mathcal{C}$ correspond to single objects in $\mathcal{C}$, and $\mathcal{C}^{\mathbf{1}}$ becomes naturally identified with category $\mathcal{C}$ itself.

1.3.5.4 $\mathcal{C}^{\rightarrow}$

Consider the category with two objects, 0 and 1, and a single morphism

$$0 \longrightarrow 1.$$

Denote this category by **2**. Functors from **2** to $\mathcal{C}$ correspond to single morphisms in $\mathcal{C}$, and $\mathcal{C}^{\mathbf{2}}$ becomes naturally identified with the category of arrows, $\mathcal{C}^{\rightarrow}$.

1.3.5.5 The category of composable pairs of arrows

Consider the category with three objects, 0, 1 and 2, and just three morphisms, the following two

$$0 \longrightarrow 1 \longrightarrow 2,$$

and their composition. Denote this category by **3**. Functors from **3** to $\mathcal{C}$ correspond to single morphisms in $\mathcal{C}$, and $\mathcal{C}^{\mathbf{2}}$ becomes naturally identified with the category of composable pairs of arrows in $\mathcal{C}$.

Exercise 13 *The category of composable pairs of arrows in $\mathcal{C}$ has class $\mathcal{C}_2$ as its class of objects. Knowing that morphisms $\phi\colon (\alpha_0, \alpha_1) \longrightarrow (\beta_0, \beta_1)$ are defined in a natural manner, give the definition of morphisms.*

1.3.5.6

Categories **1**, **2** and **3** correspond to the linearly ordered sets $\{0,\}$, $\{0,1\}$, $\{0,1\,2\}$. Let **n** be the category with n objects

$$0, 1, \ldots, \text{ n-1}$$

which corresponds to the linearly ordered set $\{0, \ldots, n-1\}$.

Exercise 14 *Find the number of morphisms in* **n**.

Exercise 15 *Provide a description of $\mathcal{C}^{\mathbf{n}}$ which generalizes to arbitrary n the descriptions given above for $n = 1, 2, 3$.*

1.3.5.7 The category of commuting squares

Consider the category with four objects

$$00, \quad 01, \quad 10, \quad \text{and} \quad 11,$$

and just five morphisms

$$\begin{tikzcd} 00 \arrow[r, "\top"] \arrow[d, "\vdash"'] \arrow[dr, "\triangle"] & 01 \arrow[d, "\dashv"] \\ 10 \arrow[r, "\bot"'] & 11 \end{tikzcd}$$

Denote this category by $\square$. Objects of $\mathcal{C}^{\square}$ are commuting squares in $\mathcal{C}$.

Exercise 16 *Describe morphisms in* $\mathcal{C}^{\square}$.

1.3.5.8 The category of families of objects

Let $\mathbf{I}$ be the category with a set I as its class of objects and empty class of morphisms. Objects of $\mathcal{C}^{\mathbf{I}}$ are families $\{a_i\}_{i\in I}$ of objects of category $\mathcal{C}$ indexed by set I.

Exercise 17 *Describe morphisms* $\phi\colon \{a_i\}_{i\in I} \longrightarrow \{b_i\}_{i\in I}$.

Chapter 2

Sets equipped with a family of subsets

2.1 Two categories of pairs

2.1.0.9

Pairs $(X, \mathscr{F})$, where X is a set and $\mathscr{F} \subseteq \mathscr{P}(X)$, form a category in two natural ways. In both cases, pairs $(X, \mathscr{F})$ provide the objects.

The difference between these two categories is their morphisms: in the first case we consider the power-set functor $\mathscr{P}(\ )$ as a covariant functor, in the second—as a contravariant functor.

2.1.0.10 The first category of pairs $(X, \mathscr{F})$

Morphisms $(X, \mathscr{F}) \longrightarrow (Y, \mathscr{G})$ are maps $f\colon X \longrightarrow Y$ such that

$$f(F) \in \mathscr{G} \qquad \text{for any} \qquad F \in \mathscr{F}.$$

2.1.0.11 The second category of pairs $(X, \mathscr{F})$

Morphisms $(X, \mathscr{F}) \longrightarrow (Y, \mathscr{G})$ are maps $f\colon X \longrightarrow Y$ such that

$$f^{-1}(G) \in \mathscr{F} \qquad \text{for any} \qquad G \in \mathscr{G}. \tag{2.1}$$

2.1.0.12

One could profitably refer to the first as the *covariant* category of pairs $(X, \mathscr{F})$, and to the second—as the *contravariant* category of pairs $(X, \mathscr{F})$. Be warned however that the words 'covariant' and 'contravariant' are here used strictly as names that allow us to clearly indicate which of the two categories of pairs we mean. As *concepts*, 'covariant' and 'contravariant' apply to functors, not categories.

2.2 Topological spaces

2.2.1 Topologies

2.2.1.1

A family $\mathscr{T} \subseteq \mathscr{P}(X)$ is called a *topology* on a set X if it is closed under arbitrary unions and finite intersections, which means that, for any family $\{U_i\}_{i \in I}$ of elements of $\mathscr{T}$, one has

$$\bigcup_{i \in I} U_i \in \mathscr{T} \tag{2.2}$$

and

$$\bigcap_{i \in I} U_i \in \mathscr{T}, \tag{2.3}$$

where the indexing set, I, is arbitrary in (2.2), while in (2.3) it is supposed to be *finite*. It is also assumed that the smallest and the largest elements of $\mathscr{P}(X)$ belong to $\mathscr{T}$:

$$\emptyset \in \mathscr{T} \qquad \text{and} \qquad X \in \mathscr{T}.$$

2.2.1.2 The set of topologies on a set

The set $\mathrm{Top}(X)$ of topologies on a set X is a subset of the set of all families of subsets of X, i.e., of $\mathscr{P}(\mathscr{P}(X))$. In particular, it is ordered by inclusion. It possesses the smallest element

$$\mathscr{T}^{\mathrm{triv}} := \{\emptyset, X\}$$

which is called the *trivial* topology, and the largest element

$$\mathscr{T}^{\text{discr}} := \mathscr{P}(X)$$

which is called the *discrete* topology.

Exercise 18 *Show that the intersection of any family of topologies $\mathfrak{T}$,*

$$\bigcap \mathfrak{T} = \bigcap_{\mathscr{T} \in \mathfrak{T}} \mathscr{T}, \tag{2.4}$$

is a topology on X.

2.2.1.3 The topology generated by a family of subsets

For any family of subsets $\mathscr{F} \subseteq \mathscr{P}(X)$ of a set X, the intersection of the family of all topologies $\mathscr{T}$ containing $\mathscr{F}$,

$$\mathscr{T}_{\mathscr{F}} := \bigcap_{\substack{\mathscr{T} \in \mathrm{Top}(X) \\ \mathscr{T} \supseteq \mathscr{F}}} \mathscr{T},$$

is the smallest topology that contains $\mathscr{F}$. We shall call it the topology *generated* by $\mathscr{F}$.

2.2.1.4

Since (2.4) is the largest family of subsets of X, which is contained in every member $\mathscr{T}$ of the family, it follows from Exercise 18 that any subset $\mathfrak{T}$ of partially ordered set $\mathrm{Top}(X)$ has infimum, and that this infimum coincides with the infimum of $\mathfrak{T}$ when viewed as a subset of $\mathscr{P}(\mathscr{P}(X))$:

$$\inf_{\mathrm{Top}(X)} \mathfrak{T} = \bigcap \mathfrak{T} = \inf_{\mathscr{P}(\mathscr{P}(X))} \mathfrak{T}.$$

2.2.1.5

Recall that in any partially ordered set $(S, \leq)$, if s is the infimum of the set $U(E)$ of upper bounds of a set $E \subseteq S$, then $s \in U(E)$ which means that

$$\inf U(E) = \min U(E),$$

and $\min U(E)$ is, by definition, $\sup E$. In particular, if every subset E has infimum in S, it has also supremumem in S.

Applying this to $S = \mathrm{Top}(X)$, we see that any family of topologies $\mathcal{T}$ on X has the supremum. Unlike the corresponding infima, the supremum of $\mathcal{T}$ in $\mathrm{Top}(X)$ generally does not coincide with the supremum of $\mathcal{T}$ in $\mathscr{P}(\mathscr{P}(X))$ because the union of a family of topologies is only rarely a topology.

Exercise 19 *Show that* $\sup_{\mathrm{Top}(X)} \mathcal{T}$ *is the topology generated by* $\sup_{\mathscr{P}(\mathscr{P}(X))} \mathcal{T}$.

2.2.2 Topological spaces

2.2.2.1

Pairs $(X, \mathscr{T}_X)$, where $\mathscr{T}_X$ is a topology on a set X, are called *topological spaces*. Topological spaces naturaly form a subcategory of the category of pairs, and we have two possibilities: to consider topological spaces as a full subcategory of the *covariant* category of pairs, cf. 2.1.0.10, or of the *contravariant* category of pairs, cf. 2.1.0.11

2.2.2.2 Open maps

In the first case, morphisms $(X, \mathscr{T}_X) \longrightarrow (Y, \mathscr{T}_Y)$ are called *open maps*.

2.2.2.3 Continuous maps

In the second case, morphisms $(X, \mathscr{T}_X) \longrightarrow (Y, \mathscr{T}_Y)$ are called *continuous* maps.

2.2.2.4 The category of topological spaces

Since continuous maps are considered to be far more important than open maps, the established practice is to apply the name *the category of topological spaces* to the category whose morphisms are continuous maps. This category is usually denoted Top.

2.2.2.5 The category of sets viewed as a subcategory of the category of topological spaces

Any map between discrete topological spaces is continuous:

$$\mathrm{Hom}_{\mathrm{Top}}\Big(\big(X,\mathscr{T}^{\mathrm{discr}}\big),\big(Y,\mathscr{T}^{\mathrm{discr}}\big)\Big)=\mathrm{Hom}_{\mathrm{Set}}(X,Y).$$

This observation allows us to consider Set as a subcategory of Top.

2.2.3 Measurable spaces

2.2.4 σ-algebras of subsets

2.2.4.1

A family $\mathfrak{M}\subseteq\mathscr{P}(X)$ of subsets of a set X is called a *σ-algebra* if it is closed under *countable* unions and the operation of taking the complement

$$A\longmapsto A^c:=X\setminus A. \tag{2.5}$$

Additionally, it is assumed that $X\in\mathfrak{M}$.

2.2.4.2 The set of σ-algebras on a set

The set $\sigma\text{-alg}(X)$ of σ-algebras on a set X is a subset of the set of all families of subsets of X, i.e., of $\mathscr{P}(\mathscr{P}(X))$. In particular, it is ordered by inclusion. It possesses the smallest element

$$\mathfrak{M}^{\mathrm{triv}}:=\{\emptyset,X\}$$

which is called the *trivial* σ-algebra, and the largest element

$$\mathfrak{M}^{\mathrm{discr}}:=\mathscr{P}(X)$$

which will be called the *discrete* σ-algebra.

Exercise 20 *Show that the intersection of any family of σ-algebras $\mathcal{M}$,*

$$\bigcap\mathcal{M}=\bigcap_{\mathfrak{M}\in\mathcal{M}}\mathfrak{M},$$

is a σ-algebra on X.

2.2.4.3 The σ-algebra generated by a family of subsets

For any family of subsets $\mathscr{F} \subseteq \mathscr{P}(X)$ of a set X, the intersection of the family of all σ-algebras $\mathfrak{M}$ containing $\mathscr{F}$,

$$\mathscr{F}^* := \bigcap_{\substack{\mathfrak{M} \in \sigma\text{-alg}(X) \\ \mathfrak{M} \supseteq \mathscr{F}}} \mathfrak{M},$$

is the smallest σ-algebra on X which contains $\mathscr{F}$. We shall call it the σ-algebra *generated* by $\mathscr{F}$.

2.2.4.4 Measurable spaces

2.2.4.5

Pairs $(X, \mathfrak{M})$ are referred to as *measurable spaces*.

2.2.4.6 Measurable maps

As in the case of topological spaces, we have two choices what to consider to be a morphism $(X, \mathfrak{M}) \longrightarrow (Y, \mathfrak{N})$. And again, we condition (2.1) is the more important one. Maps $f \colon X \longrightarrow Y$ such that

$$\text{for any } B \in \mathfrak{N}, \text{ one has } f^{-1}(B) \in \mathfrak{M},$$

will be called *measurable*.

2.2.4.7 The category of measurable spaces

Below, *the category of measurable spaces* will always mean the full subcategory of the *contravariant* category of pairs, cf. 2.1.0.11. We shall denote it Meas.

2.2.5 Borel σ-algebra

2.2.5.1 Borel subsets of a topological space

For a topological space $(X, \mathscr{T})$, the σ-algebra $\mathscr{T}^*$ generated by the topology is called the *Borel* σ-algebra, and its members—Borel subsets of X.

2.2.5.2 Borel maps between topological spaces

A map $f\colon X \longrightarrow Y$ between topological spaces is called a *Borel* map if it is a morphism of the corresponding Borel measurable spaces

$$\big(X, (\mathscr{T}_X)^*\big) \longrightarrow \big(Y, (\mathscr{T}_Y)^*\big)\,.$$

Any continuous map $f\colon\ (X, \mathscr{T}_X) \longrightarrow (Y, \mathscr{T}_Y)$ is a Borel map.

2.2.5.3

A map $f\colon X \longrightarrow Y$ from a measurable space to a topological space is said to be *measurable* if it is a morphism $(X, \mathfrak{M}) \longrightarrow \big(Y, (\mathscr{T}_Y)^*\big)$. Thus, we will be also talking of *measurable functions* $f\colon X \longrightarrow \mathbf{R}$, $f\colon X \longrightarrow [0, \infty]$, etc.

Chapter 3

Sets equipped with one or more relations

3.1 Introduction

3.1.1 Relations on a set

3.1.1.1 I-ary relations

Let I be a set. An I-ary relation on a set X is the same as as a subset $R \subseteq X^I$.

3.1.1.2 Morphisms

A natural notion of a morphism $(X, R) \longrightarrow (Y, S)$ is that it is a map $X \longrightarrow Y$ such that the induced map $f_* : X^I \longrightarrow Y^I$ sends R to S:

$$f_*(R) \subseteq S.$$

Explicitly, this means that if $\{x_i\}_{i \in I} \in R$, then $\{f(x_i)\}_{i \in I} \in S$.

Exercise 21 *Formulate the notion of a set with* two *relations, and define the appropriate notion of a morphism.*

3.1.1.3 Notation: binary relations

If $R \subseteq X^2$ is a binary relation on a set X, an alternative notation may be used to denote the fact that $(x, x') \in R$:

$$x \sim_R x'$$

or simply

$$x \sim x',$$

when the relation is clear from the context. Here $\sim$ is a generic symbol for a pair of elements 'being in relation'. In specific situations special symbols may be used. For example, when R is a *partial order* relation, then the symbols $\leq$ or $\preceq$ are generally used.

3.1.1.4

In this notation, a morphism $(X, \sim_X) \longrightarrow (Y, \sim_Y)$ is a map $f\colon X \longrightarrow Y$ such that

$$x \sim_X x' \qquad \text{implies} \qquad f(x) \sim_Y f(x') \qquad (x, x' \in X).$$

3.1.1.5 Terminology: isotone maps

Morphisms $(X, \leq_X) \longrightarrow (Y, \leq_Y)$ are referred to as *order-preserving*, or *isotone* maps. The latter is common in literature on partially ordered sets.

3.1.1.6 Restriction to a subset

If $Y \subseteq X$ is a subset, then Y^I can be naturally identified with the subset of X^I of those functions from I to X whose values belong to Y. In particular, $R \cap Y^I$ becomes an I-ary relation on Y. We shall call it the *restriction* of relation R to Y, and denote it $R_{|Y}$.

3.2 Sets equipped with an operation

3.2.1 I-ary operations

3.2.1.1

An I-ary operation on a set X is a map

$$\mu\colon X^I \longrightarrow X. \tag{3.1}$$

3.2.1.2 Commutativity

We say that operation (3.1) is *commutative* if the following diagram commutes

$$\begin{tikzcd} X^I \arrow[d, "\rho^*"'] \arrow[dr, "\mu_I"] & \\ X^I \arrow[r, "\mu_I"'] & X \end{tikzcd} \tag{3.2}$$

for any bijection $\rho\colon I \longrightarrow I$.[1] Note that $\rho^*\colon X^I \longrightarrow X^I$ is the induced map, introduced in (1.1).

3.2.2 n-ary operations

3.2.2.1

An n-ary operation on a set X is a map

$$\mu\colon X^n \longrightarrow X. \tag{3.3}$$

It can be viewed as an $(n+1)$-ary relation

$$R_\mu = \{(x_0, x_1, \ldots, x_n) \in X^{n+1} \mid x_0 = \mu(x_1, \ldots, x_n)\}.$$

Exercise 22 *Let X be a set and $R \subseteq X^{n+1}$. Show that there exists an n-ary operation, (3.3), such that $R = R_\mu$ if and only if R satisfies the following property*

$$\text{for any } x_1, \ldots, x_n \in X, \text{ there exists a unique element } x_0 \in X, \text{ such that } (x_0, x_1, \ldots, x_n) \in R. \tag{3.4}$$

[1] Self-bijections of I are called *permutations* of elements of set I.

3.2.2.2

Sets with an n-ary operation are sometimes called *n-ary structures*. They form a full subcategory of the category of sets with an $(n+1)$-ary relation.

Exercise 23 *Show that* $f\colon (X,\mu)\longrightarrow(Y,\nu)$ *is a morphism if and only if*

$$f\left(\mu\left(x_1,\ldots,x_n\right)\right)=\nu\left(f\left(x_1\right),\cdots,f\left(x_n\right)\right)\qquad\left(x_1,\cdots,x_n\in X\right). \tag{3.5}$$

3.2.2.3

For a subset $Y\subseteq X$ of a set with an n-ary operation (X,μ), the restriction of R_μ to Y is an n-ary relation on Y which does not need to satisfy property (3.4).

Exercise 24 *Show that* $R_{|Y}=R_\nu$ *for some* n*-ary operation* ν *on* Y *if and only if*

for any $y_1,\ldots,y_n\in Y$*, one has* $\mu(y_1,\ldots,y_n)\in Y$.

Show that, for all $y_1,\ldots,y_n\in Y$,

$$\nu(y_1,\ldots,y_n)=\mu(y_1,\ldots,y_n).$$

3.2.2.4

In this case, we shall denote ν by μ_Y, call it the operation on Y *induced* by μ, and (Y,μ_Y), the *subset-with-operation* of (X,μ).

3.2.3 The category of sets with an n-ary operation

3.2.3.1 Homomorphisms

Traditionally, maps $f\colon X\longrightarrow Y$ between sets equipped with an n-ary operation which satisfy identity (3.5) are referred as *homomorphisms*. This is where the term *morphism* originated.

3.2.3.2

Identity (3.5) is equivalent to the commutativity of the following diagram

$$\begin{array}{ccc} X^n & \xrightarrow{f_*} & Y^n \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} \\ X & \xrightarrow{f} & Y \end{array}$$

where $f_*(x_1, \ldots, x_n) := (f(x_1), \ldots, f(x_n))$.

3.2.3.3 Induced operations

3.2.3.4

An n-ary operation on a set induces several other n-ary operations on related sets. We shall consider here just two examples.

3.2.3.5 The induced operation on Y^X

The set of maps from a set X to a set Y which is equipped with an n-ary operation ν, is itself naturally equipped with a n-ary operation that is induced by ν.

For maps $f_1, \ldots, f_n$, we define $\nu(f_1, \ldots, f_n)$ as the map $X \longrightarrow Y$ whose value at $x \in X$ is calculated by applying ν to the values of $f_1, \ldots, f_n$ at x:

$$\nu(f_1, \ldots, f_n)(x) := \nu(f_1(x), \ldots, f_n(x)) \qquad (x \in X).$$

3.2.3.6 The induced operation on $\mathscr{P}(X)$

The set of subsets of a set X which is equipped with an n-ary operation μ, is itself naturally equipped with an n-ary operation that is induced by μ.

For subsets $A_1, \ldots, A_n$ of X, we define $\mu(A_1, \ldots, A_n)$ as the set obtained by applying μ to every n-tuple $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$:

$$\mu(A_1, \ldots, A_n) := \{\mu(a_1, \ldots, a_n) \mid (a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n\}$$

Exercise 25 *Suppose that subsets $A_1, \dots, A_n$ are finite. Show that $\mu(A_1, \dots, A_n)$ is finite by demonstrating the inequality*

$$|\mu(A_1, \dots, A_n)| \leq |A_1| \cdots |A_n| .$$

3.2.4 0-ary operations

3.2.4.1

For any set X, there is just a single map $\emptyset \longrightarrow X$, namely the canonical inclusion map ι that embeds the empty set into X. Thus, the zeroth Cartesian power of any set X has a single element, namely ι, and therefore any 0-ary operation on a set X,

$$X^0 \longrightarrow X, \tag{3.6}$$

is the same as selecting a single element $\xi \in X$, the latter being the only value of map (3.6).

3.2.4.2 The category of sets with a distinguished element

In particular, sets equipped with a 0-ary operation are just sets with a distinguished element. Morphisms $(X, \xi) \longrightarrow (Y, v)$ are the maps $f\colon X \longrightarrow Y$ which are compatible with the distinguished elements, i.e.,

$$f(\xi) = v.$$

3.2.5 Unary operations

3.2.5.1

A unary operation on a set X is the same as a map $\phi\colon X \longrightarrow X$. Such maps are often referred to as *selfmaps* on X.

3.2.5.2 The category of sets with a self-map

Morphisms $(X,\phi)\longrightarrow(Y,\psi)$ are the maps $f\colon X\longrightarrow Y$ which are compatible with the selfmaps, i.e., such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & Y \arrow[d, "\psi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes which translates into the identity $f\circ\phi=\psi\circ f$.

3.2.5.3

Certain sets possess *natural* unary operations, e.g., $\mathscr{P}(X)$ comes equipped with the 'complement-of-a-subset' self-map, cf. (2.5).

3.3 Binary structures

3.3.1 General binary structures

3.3.1.1 Binary structures

Sets equipped with a single binary operation are sometimes called *binary structures*. They form a full subcategory of the category of sets equipped with a binary relation, cf. Section 3.2.3. We shall denote it Bin.

3.3.1.2 Notation

Traditionally, for a binary operation on a set X an alternative notation is used:

$$x*y \qquad \text{instead of} \qquad \mu(x,y)$$

where $*$ here stands for any symbol denoting the operation. You may see here $+$, $\times$, $\cdot$, $\otimes$, and many other symbols.

3.3.1.3 Simplified notation

A frequent practice is to omit the symbol for the operation altogether and to write xy for $\mu(x,y)$.

3.3.1.4 Identity elements

An element $e \in X$ is a *left identity* if

$$\mu(e, x) = x \qquad (x \in X).$$

Exercise 26 *Formulate the notion of a* right identity *in a set with a binary operation, and show that, if e is a left identity and e' is a right identity, then $e = e'$. In particular, any set with a binary operation has no more than one two-sided identity.*

3.3.1.5 Sink elements

An element $z \in X$ is a *left sink* if

$$\mu(e, x) = x \qquad (x \in X).$$

Exercise 27 *Formulate the notion of a* right sink *in a set with a binary operation, and show that, if z is a left sink and z' is a right sink, then $z = z'$.*

In particular, any set with a binary operation has no more than one two-sided sink.

3.3.1.6 Idempotents

An element x of a binary structure is called an *idempotent* if

$$\mu(x, x) = x.$$

3.3.1.7 Commutative binary operations

In the special case of a binary operation, the general notion of commutativity introduced in Section 3.2.1.2 takes on the following form. A binary operation μ is *commutative* if it satisfies the identity

$$\mu(x, y) = \mu(y, x) \qquad (x, y \in X). \tag{3.7}$$

The same in simplified notation:

$$xy = yx \qquad (x, y \in X).$$

3.3.1.8

Identity (3.7) can be expressed as commutativity of the following diagram

$$\begin{tikzcd} X \times X \arrow[d, "\tau"'] \arrow[dr, "\mu"] & \\ X \times X \arrow[r, "\mu"'] & X \end{tikzcd}$$

where $\tau\colon X \times X \longrightarrow X \times X$ is the *flip*

$$\tau\colon (x,y) \longmapsto (y,x) \qquad (x,y \in X).$$

3.3.1.9 Additive notation and terminology

A commutative binary operation is often referred to as *addition*. In that case *additive notation* $x+y$ is used rather than $\mu(x,y)$ or xy.

3.3.1.10 Additive maps

In additive notation homomorphisms betwen commutative binary structures $f\colon X \longrightarrow Y$ are just *additive* maps

$$f\left(x+x'\right) = f(x) + f\left(x'\right) \qquad \left(x, x' \in X\right).$$

3.3.1.11

Given a binary structure (X,μ), define a binary relation $\sim_\mu$ on X by

$$x \sim_\mu y \qquad \text{if} \qquad \mu(x,y) = y \qquad (x,y \in X). \tag{3.8}$$

Exercise 28 *Show that relation $\sim_\mu$ defined in* (3.8) *is reflexive if and only if every element in (X,μ) is idempotent.*

Exercise 29 *Show that relation $\sim_\mu$ defined in* (3.8) *is* weakly antisymmetric*, cf.* (1.17)*, if μ is commutative.*

3.3.1.12 Associative binary operations

A binary operation is said to be *associative* if it satisfies the identity

$$\mu\left(\mu(x,y),z\right) = \mu\left(x,\mu(y,z)\right) \qquad (x,y,z \in X). \tag{3.9}$$

The same in simplified notation:

$$(xy)z = x(yz) \qquad (x,y,z \in X).$$

Exercise 30 *Show that relation* $\sim_\mu$ *defined in* (3.8) *is* transitive*, cf.* (1.15)*, if* μ *is associative.*

3.3.1.13

Identity (1.12) can be expressed as commutativity of the following diagram

$$\begin{tikzcd} X \times X \times X \arrow[r, "\mu \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_X \times \mu"'] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"] & X \end{tikzcd}$$

3.3.2 Binary versus I-ary operations

3.3.2.1

A binary operation $\mu\colon X \times X \longrightarrow X$ allows to convert 2 elements of a set into a single element. What about 3 or more elements? Given a list of n elements

$$x_1, \ldots, x_n,$$

we have to apply it first to a single pair of consecutive elements, say x_i and x_{i+1}, and replace that pair with the result. The new list has length $n-1$. By iterating this procedure $n-1$ times we eventually obtain a list consisting of a single element. This is the final result.

3.3.2.2 Iterated n-ary operations

If we recorded the sequence of steps performed, we can use the same recipe to any n-tuple of elements of X. The resulting map $X^n \longrightarrow X$ is what we call an *iterated* n-ary operation induced by a binary operation.

A binary operation induces two ternary operations,

$$(x_1, x_2, x_3) \longmapsto \mu(\mu(x_1, x_2), x_3) \qquad (x_1, x_2, x_3 \in X) \tag{3.10}$$

and

$$(x_1, x_2, x_3) \longmapsto \mu(x_1, \mu(x_2, x_3)) \qquad (x_1, x_2, x_3 \in X), \tag{3.11}$$

five quaternary operations $X^4 \longrightarrow X$, etc. There are exactly

$$\frac{1}{n}\binom{2n-2}{n-1}$$

induced iterated n-ary operations in total, and all of them are different in general. The number of induced iterated n-ary operations coincides with the number of nested sequences of $n-1$ pairs of parentheses, and is known as the $(n-1)$-st *Catalan* number.

3.3.2.3

Associativity of μ states that the two ternary operations above, (3.10)–(3.11), coincide. Using this fact, one can show by induction on n that all the iterated n-ary operations coincide. Thus, an associative binary operation induces exactly one n-ary operation for each $n \geq 2$.

3.3.2.4

Commutativity of an associative binary operation has one more advantage: it allows one to extend the operation to families of elements $\{x_i\}_{i \in I}$ of X indexed by *arbitrary* finite nonempty sets I, producing I-ary operations

$$\mu_I \colon X^I \longrightarrow X. \tag{3.12}$$

For a general associative operation, performing the iterated operation requires that the indexing set be ordered, which amounts to providing a bijection between the set $\{1, \ldots, n\}$ and I

$$I = \{i_1, \ldots, i_n\}.$$

Then family $\{x_i\}_{i\in I}$ becomes a list

$$x_{i_1}, \ldots, x_{i_n}$$

and we apply μ to that list as explained above.

If I has n elements, there are exactly $n!$ different orderings of I, and therefore an associative binary operation induces exactly $n!$ operations (3.12). They all coincide if μ is commutative and we denote this unique I-ary operation μ_I.

3.3.2.5 Associativity seen through the induced I-ary operations

Given a *finite* family $\{\xi_j\}_{j\in J}$ of *finite* families $\xi_j = \{x_{i_j}\}_{i\in I_j}$ of elements of X, we can evaluate μ_{I_j} on each ξ_j to get

$$\{\mu_{I_j}(\xi_j)\}_{j\in J} \in X^J$$

and subsequently evaluate μ_J on it. We can also apply μ_L to the *total* family

$$\{x_l\}_{l\in L}$$

indexed by the disjoint sum

$$L = \coprod_{j\in J} I_j,$$

cf. Section 1.2.3.2. The results are equal

$$\mu_J\left(\{\mu_{I_j}(\{x_{i_j}\}_{i\in I_j})\}_{j\in J}\right) = \mu_L(\{x_l\}_{l\in L}). \tag{3.13}$$

If we simplify notation by omitting indexing sets, then identity (3.13) becomes a little easier to read

$$\mu_J\left(\{\mu_{I_j}(\{x_{i_j}\})\}\right) = \mu_L(\{x_l\}).$$

3.3.2.6

Note that identity (3.13) holds even if some of the indexing sets I_j have a single element, provided that

$$\mu_I\colon X^I \longrightarrow X$$

is to be understood as the canonical bijection that identifies $X^{\{\bullet\}}$ with X:

$$X^{\{\bullet\}} \ni \{\bullet \longmapsto x\} \quad \longleftrightarrow \quad x \in X.$$

3.3.2.7 A comment on notation

Above I is a set with a single element and we denoted that single element $\bullet$. Mathematical notation employs symbolic 'names' like a, b, c, etc. in order to distinguish between different elements of a set. There is no need to do that when the set has a single element. We denoted the single element of set I by $\bullet$ to indicate the fact that we do not need to 'name' it first before we can make a reference to it.

3.3.2.8 A comment on the meaning of identity (3.13)

Identity (3.13) expresses compatibility of the system of induced operations μ_I and is a manifestation of associativity of the original binary operation.

3.3.2.9

'Associativity' identity (3.13) becomes more legible when we express it as commutativity of the diagram

$$\begin{tikzcd}
\prod_{j\in J} X^{I_j} \arrow[r, "\prod_{j\in J}\mu_{I_j}"] & \prod_{j\in J} X \arrow[d, "\mu_J"] \\
X^L \arrow[u, leftrightarrow, "(1.8)"] \arrow[r, "\mu_L"] & X
\end{tikzcd} \tag{3.14}$$

where the left vertical arrow is the canonical identification of Cartesian products discussed in Sections 1.2.3.5 and 1.2.3.6.

3.3.2.10 Iterated operations in additive notation

If we use additive notation and terminology, then

$$\mu_I\big(\{x_i\}_{i\in I}\big) \qquad \text{becomes} \qquad \sum_{i\in I} x_i$$

and identity (3.13) becomes

$$\sum_{j\in J}\sum_{i_j\in I_j} x_{i_j} = \sum_{l\in L} x_l.$$

3.3.2.11

In this form associativity identity (3.13) seems much easier to comprehend than in the original form which emplys functional notation for operations. This illustrates the fact that the notation we use indeed is either aiding or hindering our human comprehension.

> *It is one of the first duties of a professional mathematician to pay due respect to proper notation, and to always strive for notation that is simultaneously precise, clear, and suitable. One has to constantly negotiate between these goals which are at times not easy to reconcile.*

3.3.3 Semigroups

3.3.3.1

An associative binary structure (X, μ) is called a *semigroup*.

3.3.3.2 The category of semigroups

Semigroups form a full subcategory of the category of sets with a binary operation, and therefore also a full subcategory of the category of sets with a ternary relation.[2] The category of semigroups will be denoted $\mathrm{Semigrp}$.

[2] *Ternary* means $n = 3$.

3.3.3.3 Subsemigroups

Subsets-with-operation (Y, μ_Y) of a semigroup (X, μ) are called *subsemigroups*. The canonical inclusion of Y into X is then a homomorphism of semigroups.

Exercise 31 *Let $\{T_i\}_{i \in I}$ be a family of subsemigroups of a semigroup S. Show that*

$$\bigcap_{i \in I} T_i$$

is a subsemigroup of S.

3.3.3.4 The subsemigroup generated by a subset

The set of subsemigroups of a semigroup S is contained in $\mathscr{P}(S)$ and thus ordered by inclusion. It follows from Exercise 31 that, for any subset $X \subseteq S$,

$$\langle X \rangle := \bigcap_{\substack{T \text{ a subsemigroup of } S \\ T \supseteq X}} T,$$

is the *smallest* subsemigroup of A which contains X. We call it the subsemigroup *generated* by subset X.

Exercise 32 *Show that $\langle X \rangle = X$ if and only if X is a subsemigroup.*

Exercise 33 *Show that $\langle \langle X \rangle \rangle = \langle X \rangle$.*

3.3.3.5 A set of generators

We say that $X \subseteq A$ *generates* semigroup A, or *is a set of generators* for A, if $\langle X \rangle = A$.

3.3.3.6 Semigroups as categories with a single object

When a category $\mathcal{C}$ has a single object, the structure of the category is uniquely determined by the set

$$\mathcal{C}_1 = \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet)$$

where $\bullet$ denotes the only object of $\mathcal{C}$, and the associative composition map

$$\mathcal{C}_2 = \mathcal{C}_1^2 \longrightarrow \mathcal{C}_1.$$

In other words, the set of morphisms forms a semigroup under composition. Vice-versa, given any semigroup (X, μ), one can associate with it the following category

$$\mathcal{C}_0 := \{\bullet\}, \qquad \mathcal{C}_1 = \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) := X,$$

with μ playing the role of the composition map.

Exercise 34 *Show that functors between categories with a single object are in one-to-one correspondence with homomorphisms of semigroups.*

3.3.4 Examples of semigroups

3.3.4.1

Exercise 35 *Let X be a set. Show that the canonical projections*

$$p_1 \colon (x, y) \longmapsto x \qquad (x, y \in Y)$$

and

$$p_2 \colon (x, y) \longmapsto y \qquad (x, y \in Y),$$

are associative.

Exercise 36 *Show that every element in X is a right identity for (X, p_1) and a left identity for (X, p_2), and that (X, p_1) has a two-sided identity precisely when X has a single element.*

3.3.4.2 Semilattices

A partially ordered set $(S, \leq)$ is called a *semilattice* if, for any $s, t \in S$, the set $\{s, t\}$ has supremum.

Exercise 37 *Show that the operation*

$$(s, t) \longmapsto s \vee t := \sup\{s, t\} \qquad (s, t \in S),$$

is associative.

Exercise 38 *Show that the semigroup* $(S, \vee)$ *has an identity element if and only if semilattice* S *has the smallest element.*

Exercise 39 *Show that the semigroup* $(S, \vee)$ *has a sink element if and only if semilattice* S *has the largest element.*

3.3.4.3

The semigroup $(S, \vee)$ associated with a semilattice $(S, \leq)$ is commutative and every element is an idempotent.

3.3.4.4

Vice-versa, if (X, μ) is a commutative semigroup with the property that every element is idempotent, then X equipped with relation $\sim_\mu$ introduced in Section (3.8), becomes a partially ordered set, cf. Exercises 28–30.

Exercise 40 *Show that* $\sim_\vee$ *coincides with the original partial order relation* $\leq$.

Exercise 41 *Show that* $(X, \sim_\mu)$ *is a semilattice. More precisely, show that*

$$\sup\{x, y\} = \mu(x, y) \qquad (x, y \in X). \tag{3.15}$$

3.3.4.5

Identity (3.15) means that the $\vee$-operation corresponding to $\sim_\mu$ is the original μ-operation.

By combining everything together, we arrive at the following observation.

Proposition 3.3.1 *For any set* X*, there exists a natural correspondence between partial order relations which make* X *into a semilattice, and binary operations which make* X *into a commutative semigroup where every element is idempotent.*

3.3.4.6 A word of caution

Proposition 3.3.1 seems to say that there is a natural isomorphism between the category semilattices and the category commutative semigroups in which every element is idempotent.

This is indeed so if one properly understands what to consider to be a morphism of semilattices. Note that a map $f\colon X \longrightarrow X'$ is a homomorphism of semigroups $(X, \vee) \longrightarrow (X', \vee')$ if and only if f is a morphism of partially ordered sets $(X, \leq) \longrightarrow (X', \leq')$ such that

$$f\left(\sup_{(X,\leq)} A\right) = \sup_{(X',\leq')} f(A) \tag{3.16}$$

for any nonempty finite subset $A \subseteq X$.

3.3.4.7 Right-exact maps

Let us call a map $f\colon X \longrightarrow X'$ between partially ordered sets *right-exact* if it preserves the suprema of nonempty finite sets, i.e., it satisfies (3.16) whenever $\sup A$ exists and $A \subseteq X$ is nonempty and finite.

Exercise 42 *Show that any right-exact map is automatically* monotonic, *i.e.,*

$$\textit{if} \;\; x \leq x', \;\; \textit{then} \;\; f(x) \leq f\left(x'\right) \qquad \left(x, x' \in X\right).$$

3.3.4.8 The category of semilattices

If by the category of semilattices we understand the subcategory of the category of partially ordered sets with morphisms being right-exact maps, then the category of semilattices is naturally isomorphic to the full subcategory of the category of commutative semigroups formed by semigroups where every element is idempotent.

3.3.4.9

Note that the identity

$$\mu(x, x) = x \qquad (x \in X)$$

which expresses the fact that every element is idempotent, like many other such identities can be also expressed, without resorting to elements of X, as commutativity of a certain diagram, in this case:

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[dd, equal] & \\
& X \times X \arrow[dl, "\mu"] \\
X &
\end{tikzcd}$$

where $\Delta\colon X \longrightarrow X \times X$ denotes the *diagonal* embedding of X into $X \times X$:

$$\Delta\colon x \longmapsto (x, x) \qquad (x \in X).$$

3.3.4.10 Subsemilattices

A partially ordered subset $\left(A, \leq_{|A}\right)$ of a semilattice $(X, \leq)$ does not need to be a semilattice. Indeed, the set $A = \{x, y\}$, for any two elements $x, y \in X$ which are not comparable, lacks both $\sup\{x, y\}$ and $\inf\{x, y\}$.

If, however, it is, we should consider it to be a *subsemilattice* of $(X, \leq)$ only if the inclusion map $A \hookrightarrow X$ is a morphism in the category of semilattices, i.e., is a right-exact map.

3.3.4.11

What we described above should be called sup-semilattices. By replacing sup with inf, one obtains the concept of an inf-semilattice. The theories are of course identical, since X with the reverse order,

$$x \leq^{\mathrm{rev}} x' \qquad \text{if} \qquad x' \leq x \qquad \left(x, x' \in X\right),$$

is a sup-semilattice precisely when $(X, \leq)$ is an inf-semilattice.[3]

3.3.4.12 The semigroup of maps with values in a semigroup

The set of maps S^X from a set X into a semigroup S is naturally a semigroup: the binary operation is applied pointwise to the values, cf. Section

[3] sup-semilattices are also called *join-semilattices*, or $\vee$*-semilattices*; inf-semilattices are also called *meet-semilattices*, or $\wedge$*-semilattices*.

3.2.3.5, and associativity is an immediate consequence of associativity of the operation in S.

3.3.4.13

When X is equipped with a binary operation of its own, we can consider the subset of S^X formed by homomorphisms from X to S. In general, the product of two homomorphisms is not a homomorphism, unless they *commute*:

$$gf = gf.$$

Exercise 43 *Show that the product fg of two homomorphisms from a binary structure X to a semigroup S is a homomorphism if f commutes with g.*

3.3.4.14

It follows that if S is a commutative semigroup, then $\mathrm{Hom}_{\mathrm{Bin}}(X, S)$ is a subsemigroup of S^X.

3.3.4.15 The set of endomorphisms of an object

The set of endomorphisms $\mathrm{End}_{\mathcal{C}}(a)$ of an object a in an arbitrary category $\mathcal{C}$ is a monoid.

3.3.5 Monoids

3.3.5.1

Semigroups with a two-sided identity are called monoids. A homomorphism of semigroups does not necessarily send the identity element to the identity element, as the following simple example demonstrates:

$$X = M_2(\mathbf{Z}), \qquad Y = \left\{ \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} \middle| \, m \in \mathbf{Z} \right\},$$

and the operation is the multiplication of 2×2-matrices. Since Y is a subsemigroup of X, the inclusion of Y in X is a homomorphism of semigroups. However, the identity element of Y,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is not the identity element of X,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

3.3.5.2

In view of this, one additionally requires from a morphism of *monoids* that it respects the identity elements. In particular, the category of monoids is a subcategory of the category of semigroups, yet not a full subcategory. We shall denote it Mon.

3.3.5.3 An example: $\mathbf{End}_{\mathcal{C}}(a)$

In a unital category, the set of endomorphisms $\mathrm{End}_{\mathcal{C}}(a)$ of any object a is a monoid.

3.3.5.4 Submonoids

For the same reason, submonoids of (X, μ) are not just subsemigroups (Y, μ_Y) which happen to be monoids, but the subsemigroups which contain the identity element

3.3.5.5

Note that $\{e\}$ is the smallest submonoid (frequently referred to as a *trivial* submonoid) while $\emptyset$ is the smallest subsemigroup.

3.3.5.6

Intersection of a family of submonoids is a submonoid. In particular, for any subset X of a monoid there exists the smallest submonoid that contains X. We call it the submonoid *generated* by X. e of (X, μ).

3.3.5.7 Monoids as categories with a single object

Monoids correspond to unital categories with a single object, and homomorphisms between monoids correspond to unital functors.

3.3.5.8 Invertible elements

An element u of a monoid X is said to be a *left* inverse of an element x if

$$ux = e$$

where e is the identity element.

Exercise 44 *Formulate the notian of a* right *inverse and show that in a monoid, if u is a left inverse of x, and v is a right inverse of x, then $u = v$.*

3.3.5.9

In particular, every element x in a monoid has no more than one two-sided inverse. This unique element is denoted x^{-1} (if one uses *multiplicative* notation for the operation), and x is said to be *invertible*.

3.3.5.10

Invertible elements in a monoid correspond to isomorphisms in the associated category.

Exercise 45 *Show that any homomorphism of monoids $f\colon (X,\mu)\longrightarrow(Y,\nu)$, sends invertible elements in X to invertible elements in Y. More precisely, show that for any such element, $(f(x))^{-1} = f\left(x^{-1}\right)$.*

3.3.6 Groups

3.3.6.1

A monoid (X,μ) is called a *group*, if *every* element $x \in X$ is invertible. In view of the above exercise, it is natural to consider groups as the full subcategory of the category of monoids. This category is often denoted Grp.

In contrast to monoids, groups form a full subcategory of the category of sets with a binary operation. In particular, Grp is a full subcategory of the category of semigroups.

Exercise 46 *Let (G, μ) and (H, ν) be two groups, and $f : (G, \mu) \longrightarrow (H, \nu)$ be a homomorphism of sets with a binary operation. Show that $f(e_G) = e_H$.*

3.3.6.2 Abelian groups

Commutative groups are called *abelian groups* in view of a long established tradition that predates nearly all the other terminology employed here.[4] Abelian groups form, of course, a full subcategory of Grp. It is denoted Ab.

3.3.6.3 Groupoids

Groups correspond to categories with a single object and the property that any morphism is an isomorphism. For this reason, categories with the same property are called *groupoids*.

3.3.6.4 A comment about notation

If (X, μ) is a semigroup, monoid, or a group, it is customary to refer to X alone as a semigroup, monoid or, respectively, a group. This rarely leads to terminological confusion if the operation is clear from the context and often greatly simplifies notation. We shall follow this convention in the future.

3.3.6.5 A comment about terminology

The binary operation in a general semigroup, monoid, or a group, X, is often referred to as the *multiplication* in X.

[4]Named after Niels Henrik Abel (1802-1829), a Norwegian mathematician who proved impossibility of solving by radicals a general polynomial equation of degree greater than 4. Abel also proved that the equation was solvable by radicals if the group of automorphisms of the equation was commutative.

Introduced by Camille Jordan (1838-1922) in 1870, who used the term *groupe ablien* to denote some specific groups of matrices, this terminology is applied to general commutative groups later, first perhaps in a 1882 article by Heinrich Martin Weber (1842-1913).

Exercise 47 *Show that in any monoid M, the set of invertible elements $G(M)$ is a group with respect to the operation induced by the multiplication in M.*

3.3.6.6

Combined Exercises 47 and 45 show that associating with a monoid X the group of its invertible elements $G(X)$ defines a functor $\mathrm{Mon} \longrightarrow \mathrm{Grp}$.

Exercise 48 *Let M be a monoid and $\iota\colon G(M) \hookrightarrow M$ denote the canonical inclusion of the group of invertible elements of M into M. Note that ι is a homomorphism of monoids.*

Show that, for any group G and any homomorphism of monoids $f : G \longrightarrow M$, there exists a unique homomorphism of groups $\tilde{f} : G \longrightarrow G(M)$ such that $f = \iota \circ \tilde{f}$.

3.4 Sets with a pair of binary operations

3.4.1 Introduction

3.4.1.1

Structures involving a pair of binary operations on a given set are both very common and very important. An essential feature of such structures are 'compatibility' conditions that relate one of the two operations to the other one. These conditions are usually expressed in the form of identities.

3.4.1.2 Distributivity

The most important of all of these conditions is *distributivity*. Given two binary operations $\bullet$ and $\circ$ on a set X, we say that operation $\circ$ is *left-distributive over* operation $\bullet$ if the following identity holds

$$x \circ (y \bullet z) = (x \circ y) \bullet (x \circ z) \qquad (x, y, z \in X).$$

Exercise 49 *Formulate the definition of* right-distributivity *of $\circ$ over $\bullet$.*

Exercise 50 *Consider union and intersection as binary operation on* $\mathscr{P}(X)$*. Show that* $\cap$ *distributes over* $\cup$ *and* $\cup$ *distributes over* $\cap$*.*[5]

3.4.1.3

Theory of *Lie algebras* is founded on *Jacobi identity*:

$$x \bullet (y \bullet z) + y \bullet (z \bullet x) + z \bullet (x \bullet y) = 0 \qquad (x, y, z \in X)$$

One of the two operations is referred to as *addition*, the other—as the *Lie bracket* operation: the standard notation for $x \bullet y$ is $[x, y]$.

3.4.1.4

In the context of lattices we shall encounter the pair of *absorption identities*, and the *modular identity*.

3.4.2 Semirings

3.4.2.1 Biadditive pairings

Suppose that commutative semigroups S, T, and U be given. We shall use additive notation and terminology throughout.

A map

$$\mu\colon S \times T \longrightarrow U$$

is said to be *biadditive*, or a *biadditive pairing*, if it is additive in each argument:

$$\mu\left(s + s', t\right) = \mu(s, t) + \mu\left(s', t\right) \qquad \left(s, s' \in S; t \in T\right), \tag{3.17}$$

and

$$\mu\left(s, t + t'\right) = \mu(s, t) + \mu\left(s, t'\right) \qquad \left(s \in S; t, t' \in T\right). \tag{3.18}$$

3.4.2.2

Left-additivity of (3.17) expresses the fact that μ right-distributes over addition. Similarly, Right-additivity condition (3.18) expresses the fact that μ right-distributes over addition.

[5]This is a very rare situation when two binary operations distribute over each other.

3.4.2.3 Semirings

A commutative semigroup S equipped with a biadditive binary operation

$$\mu \colon S \times S \longrightarrow S \tag{3.19}$$

is called a *semiring*.

3.4.2.4 The additive semigroup of a semiring

In a semiring the original semigroup operation is referred to as *addition* and the corresponding semigroup as the *additive* semigroup of the semiring We shall refer to semigroup $(S, +)$ as the *additive* semigroup of the semiring, and will denote it S^{+} in order to distinguish it from S viewed as a semiring.

3.4.2.5 Multiplication

We will refer to biadditive operation (3.19) as the *multiplication*, and will usually denote $\mu(s,t)$ by $s \cdot t$ or st. Equipped with multiplication S is just a binary structure. We will denote it $S^{\times}$.

3.4.2.6 The category of (nonassociative) semirings

Morphisms $(S, +, \cdot) \longrightarrow (T, +, \cdot)$ are maps $S \longrightarrow T$ which are simultaneously homomorphisms of the additive semigroups $S^{+} \longrightarrow T^{+}$ and of multiplicative binary structures $S^{\times} \longrightarrow T^{\times}$. Traditionally, such maps are called *homomorphisms* of semirings.

3.4.2.7

Terminology like an *associative* (resp. *commutative*, *unital*) semiring always refers to the corresponding properties of the multiplication. The identity element for multiplication is usually called *identity* or *unit*, and is most of the time denoted 1.

3.4.2.8 A comment about terminology

Semirings form a full subcategory of the category of sets with *two* binary operations. Associativity, however, is such an important property that a common practice is to tacitly assume it when speaking of semirings. From now on, the phrase *nonassociative* ring will refer to semirings that are *not assumed* to be associative. Note that such a reference does not preclude associativity.

3.4.2.9 The category of associative semirings

We shall denote the category of associative semirings by Semiring and will refer to its object simply as 'semirings'. It is a full subcategory of the category of *nonassociative* semirings.

3.4.2.10 The multiplicative semigroup of an associative semiring

When S is an associative semiring, $S^{\times}$ is a semigroup. We shall refer to it as the *multiplicative semigroup* of S.

3.4.2.11 Zero

If the additive semigroup of a semiring is a monoid, its identity element is denoted 0 and referred to as *zero*.

3.4.2.12 Semirings with zero

A *semiring with zero* is a semiring whose additive semigroup is a monoid and zero satisfies the following identity

$$0 \cdot s = 0 = s \cdot 0 \qquad (s \in S). \tag{3.20}$$

Identity (3.20) means that 0 is a sink of the multiplicative semigroup, cf. Section 3.3.1.5.

3.4.3 Examples of semirings

3.4.3.1 $[0, \infty)$ and $[0, \infty]$

The set $[0, \infty)$ of nonnegative real numbers equipped with usual addition and multiplication of real numbers, forms an associative and commutative semiring with zero.

The set $[0,\infty] := [0,\infty) \cup \infty$ of *extended* nonnegative real numbers can be equipped with a semiring structure by extending addition and multiplication of real numbers as follows

$$a + \infty = \infty = \infty + a \qquad \text{and} \qquad a \cdot 0 = 0 = 0 \cdot a \qquad (a \in [0,\infty])\,.$$

Note that ∞ is a sink of the additive monoid of $[0,\infty]$ while 0 is a sink of the multiplicative monoid of $[0,\infty]$.

3.4.3.2 The near-semiring S^S

The set of selfmaps $S \longrightarrow S$ possesses two semigroup structures when S is a semigroup. The first one is obtained when we consider S^S as the set of all maps from *set* S to *semigroup* S: the operation is pointwise multiplication multiplication $\cdot$, as defined in Section 3.2.3.5. The other operation is composition of maps which endows S^S with a structure of a monoid.

Exercise 51 *Show that $\circ$ is right-distributive over $\cdot$.*

3.4.3.3

Composition in S^S is practically never left-distributive over $\cdot$ as even the simplest examples demonstrate.

3.4.3.4 Example demonstrating that S^S is not left-distributive

The two-element set $S = \{\pm 1\}$ equipped with usual multiplication of integers is a group. Let $f \colon S \longrightarrow S$ be the constant map that sends both 1 and -1 to -1. Then

$$f \circ (f \cdot f) = f$$

while

$$(f \circ f) \cdot (f \circ f) = f \cdot f$$

is the constant map that sends both 1 and -1 to 1.

3.4.3.5

Equipped with pointwise multiplication and composition, the S^S is an example of a *near-semiring*, a structure more general than a semiring. As we shall discover in a moment, under additional hypothesis that S is commutative, there is a true semiring inside of S^S.

3.4.3.6

For any semigroup S, the set $\mathrm{End}_{\mathrm{Semigrp}}(S)$ is a submonoid of $\left(S^S, \circ\right)$. As we noted in Section 3.3.4.14, it is also a subsemigroup of $\left(S^S, \cdot\right)$ *when* S is commutative.

Exercise 52 *Assuming $(S, +)$ to be a commutative semigroup, show that composition left-distributes over addition in* $\mathrm{End}_{\mathrm{Semigrp}}(S)$.

3.4.3.7 The semiring of endomorphisms of a commutative semigroup

It follows from Exercises 51 and 52 that $\left(\mathrm{End}_{\mathrm{Semigrp}}(S), +, \circ\right)$ is a semiring. It is unital: the identity morphism id_S is its multiplicative identity. It is a semiring with zero precisely when $(S, +)$ is a monoid.

3.4.4 Rings

3.4.4.1

Semirings whose additive semigroup is a group are called *rings*.

3.4.4.2 One more comment about terminology

The remarks made in Section 3.4.2.8 apply here too: it is a common practice to tacitly assume associativity when speaking of rings, and to use the designation *nonassociative ring* when associativity is not assumed.

3.4.4.3 The category of associative rings

Nonassociative rings form a full subcategory of the category of nonassociative semirings. Similarly, associative rings form a full subcategory of the category of associative semirings.

The category of associative rings will be denoted Ring and we will refer to its objects as 'rings'.

3.4.5 Examples of rings

3.4.5.1 The ring of endomorphisms of an abelian group

For an abelian group $(A,+)$, the semiring $\mathrm{End}_{\mathrm{Ab}}(A)$ which was introduced in Section 3.4.3.7 is a unital ring.

3.4.6 Lattices

3.4.6.1

A partially ordered set $(L,\leq)$ is said to be a *lattice* if any nonempty finite subset $A\subseteq L$ has both supremum and infimum.

3.4.6.2

In particular, we obtain the binary operations on L,

$$l\vee m := \sup\{l,m\} \qquad (\text{'join'}) \tag{3.21}$$

and

$$l\wedge m := \inf\{l,m\} \qquad (\text{'meet'}), \tag{3.22}$$

are commutative, associative, and every element $l\in L$ is an idempotent.

3.4.6.3 Absorption identities

The two operations are related to each other through the following pair of identities

$$l\wedge(l\vee m) = l \qquad (l,m\in L) \tag{3.23}$$

and

$$l\vee(l\wedge m) = l \qquad (l,m\in L). \tag{3.24}$$

Exercise 53 *Prove identities* (3.23)–(3.24).

3.4.6.4 Consequences of absorption identities

Let L be a set equipped with two binary operations $\vee$ and $\wedge$ which satisfy the above pair of absorption identities. Then, for any $l \in L$,

$$l \wedge l \overset{(3.23)}{=} l \wedge (l \vee (l \wedge l)) \overset{(3.24)}{=} l.$$

In other words, $\wedge$-idempotence of all elements is a consequence of the absorption identities.

Exercise 54 *Using just absorption identies show that*

$$l \vee l = l \qquad (l \in L).$$

Exercise 55 *Using just absorption identities show that*

$$l = l \wedge m \qquad \text{if and only if} \qquad l \vee m = m \qquad (l, m \in L). \tag{3.25}$$

3.4.6.5

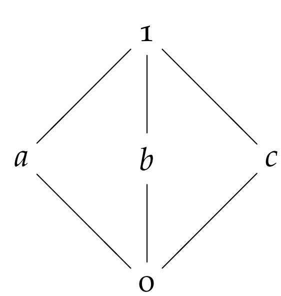
None of the two distributivity identities

$$l \wedge (m \vee n) = (l \wedge m) \vee (l \wedge m) \quad (l, m, n \in L) \tag{3.26}$$

or

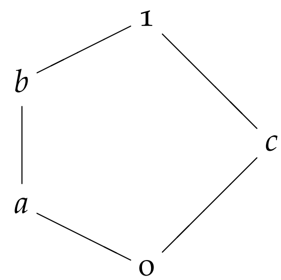
$$l \vee (m \wedge n) = (l \vee m) \wedge (l \vee n) \quad (l, m, n \in L) \tag{3.27}$$

holds in general. The minimal examples are two lattice structures on a set with 5-elements whose Hasse diagrams are



(M_5)

and



(N$_5$)

(we denoted $\inf L$ by 0 and $\sup L$ by 1).[6]

Exercise 56 *Show that in any lattice the following two distributivity inequalities always hold*

$$(l \wedge m) \vee (l \wedge m) \leq l \wedge (m \vee n) \quad (l, m, n \in L)$$

and

$$(l \vee m) \wedge (l \vee n) \leq l \vee (m \wedge n) \quad (l, m, n \in L).$$

3.4.6.6

Remarkably, if either one of identities (3.26) or (3.27) holds, the other one holds too. This is yet another consequence of the pair of absorption identities in combination with commutativity of $\wedge$ and associativity of $\vee$.

To derive identity (3.27) from (3.26), we note that

$$\begin{aligned}
(l \vee m) \wedge (l \vee n) &= \big((l \vee m) \wedge l\big) \vee \big((l \vee m) \wedge n\big) && \text{identity (3.26)} \\
&= l \vee \big((l \vee m) \wedge n\big) && (\vee)\wedge\text{-absorption} \\
&= l \vee \big((l \wedge n) \vee (m \wedge n)\big) && \text{identity (3.26)} \\
&= \big(l \vee (l \wedge n)\big) \vee (m \wedge n) && \text{associativity of } \vee \\
&= l \vee (m \wedge n) && \vee(\wedge)\text{-absorption}
\end{aligned}$$

Exercise 57 *Derive identity* (3.26) *from* (3.27).

[6] *Hasse* diagrams are visual presentations of partially ordered sets as graphs whose nodes correspond to the elements of the partially ordered set, and $a < b$ if there exists a 'path upwards' from node a to node b. Thus, in lattice N5, $a < b$ and neither one is comparable with c.

3.4.6.7

We saw that to any lattice structure on a set L corresponds a pair of commutative and associative binary operations on L which satisfy the pair of absorption identities. In fact, this is a bijective correspondence: every such pair of binary operations arises from a unique lattice structure on L.

Proposition 3.4.1 *For any set L, there exists a natural correspondence between partial order relations that make L into a lattice, and pairs of commutative and associative binary operations which satisfy absorption identities.*

Proof. Suppose $\wedge$ and $\vee$ is a pair of commutative and associative operations on a set L, satisfying absorption identities. If we consider the associated relations $\sim_\wedge$ and $\sim_\vee$, then $(L, \sim_\wedge)$ and $(L, \sim_\vee)$ are partially ordered sets. Moreover,

$$\sup_{(L,\sim_\wedge)}\{l, m\} = l \wedge m \quad \text{and} \quad \sup_{(L,\sim_\vee)}\{l, m\} = l \vee m \qquad (l, m \in L),$$

cf. Exercise 41. Equivalence of equalities in (3.25) means that partial order $\sim_\vee$ is the reverse of $\sim_w$, and in the reverse partial order infimum and supremum are switched. Thus,

$$\inf_{(L,\sim_\vee)}\{l, m\} = \sup_{(L,\sim_\wedge)}\{l, m\} = l \wedge m \qquad (l, m \in L)$$

completing the proof that $(L, \sim_v ee)$ is a lattice.

In the opposite direction, we already demonstrated that, for any lattice $(L, \leq)$, operations (3.21)-(3.22) are commutative, associative, and satisfy absorption identities (Exercise 53).

The fact that the correspondences

$$\leq \;\rightsquigarrow\; (\vee, \wedge) \qquad \text{and} \qquad (\vee, \wedge) \;\rightsquigarrow\; \sim_\vee$$

are mutually inverse follows from the corresponding fact established earlier for semilattices, combined with the fact that partials orders $\sim_\wedge$ and $\sim_\vee$ are reverse for each other. $\square$

3.4.6.8 The category of lattices

If we understand by the category of lattices the subcategory of the category of partially ordered sets whose objects are lattices and morphisms are supposed to be *exact maps*, i.e., to preserve suprema *and* infima of nonempty finite subsets, then it is a corollary of Propostion 3.4.1 that the category of lattices is isomorphic to the category of sets equipped with a pair of commutative and associative binary operations satisfying the absorption identities.

3.4.6.9 Sublattices

COmments made for semilattices apply here as well: A partially ordered subset $\left(X, \leq_{|X}\right)$ of a lattice $(L, \leq)$ will be called a *sublattice* of $(L, \leq)$ if the inclusion map $X \hookrightarrow L$ is a morphism in the category of lattices, i.e., is an exact map.

3.4.6.10 Bounded lattices

A *bounded lattice* is a lattice that contains the largest and the smallest elements.

3.4.6.11 Complete lattices

A partially ordered set is a *complete lattice* if *every* subset has supremum and infimum.

3.4.7 Distributive lattices

3.4.7.1

We saw above that if $\wedge$ distributes over $\vee$, then $\vee$ distributes over $\wedge$.

3.4.7.2

The class of distributive lattices constitutes an 'intersection' between the class of commutative semirings and the class of lattices.

3.4.8 Examples of distributive lattices

3.4.8.1 $\mathscr{P}(X)$

The set of subsets of a set X provides perhaps the most important example of a complete distributive lattice. Note that semirings $(\mathscr{P}(X), \cup, \cap)$ and $(\mathscr{P}(X), \cap, \cup)$ are isomorphic: the 'complement-of-a-subset' map, (2.5) provides an isomorphism between the two.

3.4.8.2 Linearly ordered sets

A partially ordered set $(L, \leq)$ is said to be *linearly ordered* if any two elements are comparable, i.e.,

$$l \leq m \qquad \text{or} \qquad m \leq l \qquad (l, m \in L).$$

In a linearly ordered set, $l \vee m = \max\{l, m\}$ and $l \wedge m = \min\{l, m\}$.

3.4.8.3 The set of natural numbers ordered by divisibility

Consider the relation of divisibility on the set of natural numbers:

$$l \leq m \qquad \text{if} \qquad l \mid m \qquad (l, m \in \mathbf{N}).$$

Here $l \vee m$ is the *greatest common multiple* of l and m, while $l \wedge m$ is their *greatest common divisor*. Note that natural number 1 is the smallest element while 0 is the largest element, thus $(\mathbf{N}, |)$ is an example of a bounded distributive lattice. In fact, lattice $(\mathbf{N}, |)$ is complete.

Exercise 58 *Show that* $(\mathbf{N}, |)$ *is a complete lattice.*

3.4.9 Algebraic structures

3.4.9.1

The general notion of an *algebraic structure* on a set X is usually formulated as a sequence of operations $(\mu_1, \ldots, \mu_l)$ on X satisfying an explicit list of properties that can be expressed as identities involving any number of those operations and arbitrary elements of set X.

Associativity and commutativity of a single binary operation are examples of such properties, as is left- and right-distributivity of one binary operation over another one.

3.4.9.2

For every operation its place on the list of operations forming the structure does matter. For example, if both μ and ν are binary operations, then (X, μ, ν) is a different structure from (X, ν, μ) unless $\mu = \nu$.

3.4.9.3 The signature of an algebraic structure

We say that an algebraic structure $(X, \mu_1, \ldots, \mu_l)$ has signature $(n_1, \ldots, n_l)$ if μ_i is an n_i-ary operation, $1 \leq i \leq l$. The signature is a sequence of natural numbers.

For example, an algebraic structure of signature

$$\underbrace{(0, \ldots, 0)}_{l \text{ times}}$$

is the same as a set with a *sequence* of l distinguished points (not all necessarily distinct).

3.4.9.4 Morphisms

A morphism between two structures of the *same* signature,

$$(X, \mu_1, \ldots, \mu_l) \longrightarrow (Y, \nu_1, \ldots, \nu_l),$$

is a map $f\colon X \longrightarrow Y$ such that

$$f\colon (X, \mu_i) \longrightarrow (Y, \nu_i)$$

is a homomorphism for each $1 \leq i \leq l$.

In particular, algebraic structures of a given signature and satisfying a given set of identities, form a (unital) category.

3.4.9.5 Substructures of algebraic structures

We say that $(Y, \nu_1, \ldots, \nu_l)$ is a *substructure* of $(X, \mu_1, \ldots, \mu_l)$, if each operation μ_i induces operation ν_i on Y, cf. Section 3.2.2.4. This is frequently if not entirely correctly expressed by saying that Y is closed under each μ_i and that ν_i is the restriction of μ_i to Y.

Note that any identities satisfied by operations $\mu_1, \ldots, \mu_l$ and elements of X are automatically satisfied by operations $\nu_1, \ldots, \nu_l$ and elements of Y.

3.4.9.6

The intersection of any family of substractures of an algebraic structure is a substructure itself. Thus, for any subset $A \subseteq X$, there exists the smallest substructure of $(X, \mu_1, \ldots, \mu_l)$ which contains A. We shall denote it $\langle A \rangle$. If $\langle A \rangle = (X, \mu_1, \ldots, \mu_l)$, we shall say that subset A *generates* structure $(X, \mu_1, \ldots, \mu_l)$.

3.4.9.7

Properties of an algebraic structure that ascertain existence of certain elements can often be expressed as identities, if one introduces appropriate operations.

For example, existence of a left identity for a binary operation μ on a set X can be expressed as a 0-ary operation $e\colon X^0 \longrightarrow X$, i.e., a distinguished element $e \in X$[7] such that

$$\mu(e, x) = x \qquad (x \in X). \tag{3.28}$$

3.4.9.8

Identity (3.28) can be also expressed as commutativity of the following diagram

$$\begin{tikzcd} X^0 \times X \arrow[r, "e \times \mathrm{id}_X"] \arrow[d, leftrightarrow, "(1.9)"'] & X \times X \arrow[d, "\mu"] \\ X \arrow[r, equal] & X \end{tikzcd} \tag{3.29}$$

where the left vertical arrow is the canonical identification of $X^{\emptyset} \times X^{\{1\}}$ with $X^{\emptyset \cup \{1\}}$ which itself is identified with X.

3.4.9.9

Thus, one can define a monoid as a set X equipped with two operations (μ, e), one binary, the other 0-ary, which satisfy two identities: (3.9), (3.28), and the right analog of (3.28)

$$\mu(x, e) = x \qquad (x \in X). \tag{3.30}$$

[7] We identify a map $X^0 \longrightarrow X$ with its single value.

3.4.9.10

In the similar vain, one can define a group as a set X equipped with three operations (μ, e, ι), a binary, 0-ary, and unary, which satisfy identities (3.9), (3.28), (3.30), and the identities

$$\mu(\iota(x), x) = x \qquad (x \in X) \tag{3.31}$$

and

$$\mu(x, \iota(x)) = x \qquad (x \in X),$$

the meaning of which should be obvious.

Exercise 59 *Express identity* (3.31) *as commutativity of a certain diagram.*

3.4.9.11

Existence of a left sink in a binary structure (X, μ) can be expressed as a 0-ary operation $z\colon X^0 \longrightarrow X$ such that

$$\mu(z, x) = z \qquad (x \in X). \tag{3.32}$$

3.4.9.12

Identity (3.32) is expressed also by commutativity of the following diagram

$$\begin{tikzcd} X^0 \times X \arrow[r, "z \times \mathrm{id}_X"] \arrow[d, leftrightarrow, "(1.9)"'] & X \times X \arrow[d, "\mu"] \\ X^0 \arrow[r, "z"] & X \end{tikzcd}$$

3.4.9.13

I will leave it to you to describe semirings with zero and rings as algebraic structures.

3.4.10 Fields

3.4.10.1 Domains

A ring R is a *domain* if the subset of non-zero elements $R \setminus \{0\}$ forms a subsemigroup of $R^{\times}$. This is usually expressed by saying that $R \setminus \{0\}$ is closed under multiplication.

3.4.10.2 Division rings

A unital ring R is a *division ring* if $R \setminus \{0\}$ is a group.

3.4.10.3

A commutative division ring is called a *field*.

3.4.10.4

Domains, division rings, fields are all special kinds of rings. They differ from all the previously encountered kinds of algebraic structures: the property of being a domain, a division ring, or a field cannot be described in terms of a certain number of operations on a set which are supposed to obey a certain number of identities involving arbitrary elements of that set.

3.4.11 Sets with an infinitary operation

3.4.11.1

In Section 3.3.2.4 we saw how iterating a commutative and associative binary operation generates I-ary operations

$$\mu_I \colon X^I \longrightarrow X$$

for arbitrary *finite* nonempty index sets I. This way however one cannot construct I-operations for infinite sets of indices.

3.4.11.2

This leads to an idea of defining an *infinitary* operation on a set X as a system of independent I-ary operations

$$\mu_I\colon X^I \longrightarrow X,$$

where I is an arbitrary *nonempty* set no more assumed to be finite, which satisfy the compatibility conditions expressed through commutativity of diagram (3.14) and of the diagram

$$\begin{tikzcd} X^{I'} \arrow[d, "\rho^*"'] \arrow[dr, "\mu_{I'}"] & \\ X^I \arrow[r, "\mu_I"'] & X \end{tikzcd} \tag{3.33}$$

where $\rho\colon I \longrightarrow I'$ is an arbitrary bijection between indexing sets, cf. diagram (3.2).

3.4.11.3

Any system of operations $\mu_I\colon X^I \longrightarrow X$ for which diagrams (3.14) and (3.33) cummute will be called an *infinitary* operation on a set X.

3.4.11.4

If diagram (3.14) expresses *associativity* of the infinitary operation, then diagram (3.33) expresses its *commutativity*.

3.4.11.5

An infinitary operation on X is not a set since its 'components' are indexed by the class of all nonempty sets. One should think of it as a functor from the category of nonempty sets with morphisms being bijections, to the comma category $\mathrm{Set}^{\to X}$, cf. Section 1.3.4.4,

$$I \longmapsto \left(\mu_I\colon X^I \longrightarrow X\right) \qquad (I \text{ any nonempty set}).$$

3.4.11.6 An infinitary operation with identity

If also $\mu_\emptyset$ is present and diagram (3.14) commutes for arbitrary sets, some of which may be empty, then we obtain a definition of an *infinitary operation with identity*. Note that operation $\mu_\emptyset$ provides a distinguished element $e \in X$ which indeed is an identity element for the binary operation $\mu_{\{1,2\}}$, cf Section 3.4.9.8 and commuting diagram (3.29).

3.4.11.7 An infinitary semigroup

A set equipped with an infinitary operation will be called an *infinitary semigroup*. One should think of it as a commutative semigroup whose operation can be performed on arbitrary families of elements.

3.4.11.8 An infinitary monoid

A set equipped with an infinitary operation with identity will be called an *infinitary monoid*.

3.4.11.9 Variants of the definition: restrict the size of I

Limiting oneself to *finite* indexing sets in the definition of an infinitary operation yields a system of operations induced from a single commutative and associative binary operation

$$\mu := \mu_{\{1,2\}}$$

as described in Section 3.3.2.4. Thus we obtain a structure which is equivalent to the structure of a commutative semigroup.

3.4.11.10 σ-operations

Limiting oneself to *countable* indexing sets in the definition of an infinitary operation, yields the definition of a *σ-operation* on X.

3.4.11.11 A σ-semigroup

A set equipped with a σ-operation will be called σ-semigroup.

3.4.11.12 A σ-monoid

A set equipped with an infinitary operation with identity will be called a *σ-monoid*.

3.4.12 Examples of infinitary semigroups and monoids

3.4.12.1 The semigroup of maps with values in an infinitary semigroup

The set of maps S^X from a set X to an infinitary semigroup S is naturally an infinitary semigroup. If $\mu_I\colon S^I \longrightarrow S$ is the corresponding I-ary operation on S, then

$$\nu_I\colon \left(S^X\right)^I \longrightarrow S^X$$

is the composition of the canonical identifications

$$\left(S^X\right)^I \longleftrightarrow S^{X\times I} \longleftrightarrow \left(S^I\right)^X$$

with operation μ_I performed 'pointwise' at every $x \in X$

$$\left(S^I\right)^X \xrightarrow{\prod_{x\in X}\mu_I} S^X \ .$$

If $e \in S$ is the identity element for S, then the constant map

$$X \longrightarrow S, \qquad x \longmapsto e \qquad (x \in X),$$

is the identity element for S^X.

3.4.12.2 A complete lattice

Exercise 60 *Show that for any family $\mathscr{F} \subseteq \mathscr{P}(L)$ of subsets of a complete lattice $(L, \leq)$ one has*

$$\sup\left\{\sup F \mid F \in \mathscr{F}\right\} = \sup \bigcup \mathscr{F}. \tag{3.34}$$

Given an I-indexed family $\{l_i\}_{i\in I}$ of elements of a complete lattice $(L, \leq)$, let

$$\mu_I\big(\{l_i\}_{i\in I}\big) := \sup\{l_i \mid i \in I\}. \tag{3.35}$$

In view of identity (3.34), the system of maps μ_I defined in (3.35) forms an infinitary operation with identity on L. The identity is the supremum of the empty family of elements of L, i.e., the smallest element of L.

3.4.13 Ordered binary structures

3.4.13.1

A binary structure (X, μ) equipped with a partial order $\leq$ is said to be an *ordered binary structure* if the operation respects the order

$$\mu(x,y) \leq \mu(x',y') \qquad \text{whenever} \qquad x \leq x' \quad \text{and} \quad y \leq y'.$$

Ordered binary structures naturally form a category.

Exercise 61 *Let A and B be subsets of an ordered binary structure X. Suppose that* $\sup A$, $\sup B$, *and*

$$\sup\{\mu(a,b) \mid a \in A, b \in B\}$$

exist. Show that

$$\sup\{\mu(a,b) \mid a \in A, b \in B\} \leq \mu(\sup A, \sup B). \tag{3.36}$$

3.4.13.2

In multiplicative notation inequality (3.36) becomes

$$\sup AB \leq (\sup A)(\sup B), \tag{3.37}$$

where

$$AB = \{x \in X \mid x = \mu(a,b) \text{ for some } a \in A \text{ and } b \in B\}, \tag{3.38}$$

cf. Section 3.2.3.6.

3.4.13.3 Distributivity properties of binary structures

We say that an ordered binary structure is *left-distributive* if

$$\sup\{\mu(a,b) \mid b \in B\} = \mu(a, \sup B) \qquad (a \in X; \emptyset \neq B \subseteq X) \tag{3.39}$$

whenever $\sup\{\mu(a,b) \mid b \in B\}$ and $\sup B$ exist.

3.4.13.4

In multiplicative notation identity (3.39) becomes

$$\sup aB = a(\sup B) \qquad (a \in X). \tag{3.40}$$

Exercise 62 *Formulate the definition of a* right-distributive *ordered binary structure.*

3.4.13.5

We shall say that a binary structure is *distributive* if it is left- and right-distributive.

3.4.13.6 An example: $[0,\infty]$

Lemma 3.4.2 *Both* $\big([0,\infty],+,\leq\big)$ *and* $\big([0,\infty],\ \cdot\ ,\leq\big)$ *are distributive ordered monoids.*

Proof. The obvious inequality

$$\sup B \leq \sup(a+B) \qquad (a \in [0,\infty])$$

shows that $\sup(a+B) = \infty$ if $\sup B = \infty$.

Assuming $\beta = \sup B < \infty$ we note that, for any $0 \leq \epsilon \leq \beta$, there exists $b \in B$ such that $\beta - \epsilon \leq b$, and therefore

$$a + (\beta - \epsilon) \leq a + b \leq \sup(a+B).$$

It follows that

$$a + \sup B = \sup\{a + (\beta - \epsilon) \mid 0 \leq \epsilon \leq \beta\} \leq \sup(a+B)$$

which completes the proof of distributivity of $([0,\infty],+,\leq)$.

Turning to $\big([0,\infty],\ \cdot\ ,\leq\big)$, we note that, for $a \in (0,\infty)$, inequality (3.37) yields the following inequality

$$\sup B = \sup\left(\frac{1}{a}\cdot aB\right) \leq \frac{1}{a}\sup aB.$$

After multiplying both sides by a we obtain the desired inequality.

For $a = \infty$, we note that

$$\sup \infty B = \begin{cases} 0 & \text{if } B = \{0\} \\ \infty & \text{if } B \text{ contains } b > 0 \end{cases}$$

and equality (3.40) holds in either case. For $a = 0$ equality (3.40) is trivially satisfied. □

Exercise 63 *Show that the product of any family of distributive ordered binary structures* $(X_i, \mu_i, \leq_i)$ *is distributive.*

3.4.13.7

In particular, the set of maps X^Y from *any* set Y to a distributive ordered binary structure is distributive when equipped with the 'pointwise' multiplication and pointwise order.

3.4.13.8

If we limit identity (3.39) to *finite* (respectively, *countable*) nonempty subsets $B \subseteq X$, then we obtain the definition of a *finitely* (respectively, *countably*) left-distributive ordered binary structure.

3.4.13.9

We shall say that a binary structure is *finitely distributive* (respectively, *countably distributive*) if it is finitely left- and right-distributive, (respectively, countably left- and right-distributive).[8]

Lemma 3.4.3 *Suppose an ordered binary structure* $(X, \mu, \leq)$ *be distributive, and the ordered set* $(X, \leq)$ *be a complete lattice. Then inequality* (3.36) *becomes equality*

$$\sup\{\mu(a,b) \mid a \in A, b \in B\} = \mu(\sup A, \sup B) \qquad (\emptyset \neq A, B \subseteq X), \tag{3.41}$$

valid for any pair of nonempty subsets of X.

[8] We could also refer to distributive binary structures as *completely* distributive, and use the term *distributive* without any adjective to describe *finitely* distributive binary structures. The latter would be in exact agreement with terminology prevalent in Lattice Theory.

3.4.13.10

In multiplicative notation identity (3.41) becomes

$$\sup AB = (\sup A)(\sup B) \qquad (\varnothing \neq A, B \subseteq X). \tag{3.42}$$

Proof. To ease comprehension, we will employ multiplicative notation in the proof. Note that

$$AB = \bigcup_{a \in A} aB$$

and therefore

$$\sup AB = \sup\left(\bigcup\nolimits_{a \in A} aB\right) = \sup\{\sup aB \mid a \in A\} \tag{3.43}$$

by identity (3.34). Denote $\sup B$ by $\bar{b}$. Complete left-distributivity identity (3.40) yields

$$\{\sup aB \mid a \in A\} = \{a \sup B \mid a \in A\} = \{a\bar{b} \mid a \in A\} = A\bar{b}. \tag{3.44}$$

By combining (3.43) with (3.44) we obtain

$$\sup AB = \sup A\bar{b} = (\sup A)\,\bar{b} = (\sup A)(\sup B)$$

aided by right-distributivity of $(X, \mu, \leq)$. ◻

3.4.13.11

Note that identity (3.41) holds when $B = \varnothing$ if and only if $(X, \leq)$ has the smallest element and that element is a left sink of binary structure (X, μ).

3.4.13.12

By replacing two sets in identity (3.42) by n sets, we obtain the following corollary of Lemma 3.4.3.

Corollary 3.4.4 *Under hypotheses of Lemma 3.4.3 one has*

$$\sup(A_1 \cdots A_n) = (\sup A_1) \cdots (\sup A_n) \qquad (\varnothing \neq A_1, \ldots, A_n \subseteq X).$$

3.4.13.13 A positively ordered binary structure

An ordered binary structure $(X, \mu, \leq)$ is said to be *positively ordered*, if it also satisfies the following inequality

$$x \leq \mu(x, y) \qquad \text{and} \qquad y \leq \mu(x, y) \qquad (x, y \in X).$$

3.4.13.14 Example: $[0, \infty]$

Additive monoid $\big([0, \infty], +\big)$ is positively ordered while multiplicative monoid $\big([0, \infty], \cdot\big)$ is not. The latter, however, contains two positively ordered submonoids: $[1, \infty]$ and $[0, 1]$ except that the latter is positively ordered with respect to the *reverse* order $\leq^{\mathrm{rev}}$.

Exercise 64 *Show that if e is a left identity element in a positively ordered binary structure $(X, \mu, \leq)$, then e is the smallest element of X.*

Exercise 65 *Let (X, μ) be a binary structure. Show that $\mathscr{P}(X)$ is a positively ordered binary structure when equipped with the induced multiplication, cf.* (3.38) *above, and ordered by $\subseteq$.*

3.4.13.15 A construction of an infinitary operation

We shall now extend the binary operation on a commutative positively ordered semigroup to an infinitary operation if the former is a complete lattice under the partial order. The exposition will be easier to follow if we adopt additive notation for the binary operation and for operations μ_I.

Suppose $(S, +, \leq)$ is a commutative positively ordered semigroup and assume that $(S, \leq)$ is a complete lattice. The iterated operations $\sum_{i \in I}$, introduced for fininite nonempty sets I in Section 3.3.2.4, satisfy the following formula

$$\sum_{i \in I} s_i = \sup_{I' \subseteq I} \sum_{i \in I'} s_i \qquad (I \text{ finite nonempty}). \tag{3.45}$$

Exercise 66 *Let $(S, +, \leq)$ be any commutative positively ordered semigroup (not necessarily a complete lattice when viewed as a partially ordered set). Demonstrate identity* (3.45)

3.4.13.16

Formula (3.45) suggests a natural method to extend $\sum_{i\in I}$ from finite to arbitrary nonempty sets of indices:

$$\sum_{i\in I} s_i := \sup_{\substack{I'\subseteq I \\ I' \text{ finite}}} \sum_{i\in I'} s_i. \tag{3.46}$$

Exercise 67 *Given two I-indexed families $\{s_i\}_{i\in I}$ and $\{t_i\}_{i\in I}$ of elements of S show that*

$$\sum_{i\in I} s_i \leq \sum_{i\in I} t_i$$

whenever $s_i \leq t_i$ for all $i \in I$.

Lemma 3.4.5 *Operations $\sum_{i\in I}$ defined in* (3.46) *satisfy inequality*

$$\sum_{l\in L} s_l \leq \sum_{j\in J} \sum_{i_j\in I_j} s_{i_j} \qquad \textit{where} \qquad L = \coprod_{j\in J} I_j. \tag{3.47}$$

Proof. Any subset L' of the disjoint union of family of indexing sets $\{I_j\}_{j\in J}$ is the disjoint union

$$L' = \coprod_{j\in J} I'_j$$

of sets $I'_j = \{i_j \in I_j \mid (j, i_j) \in L'\}$ consisting of elements contributed by I_j to L'.

Let $J' \subseteq J$ be the set of $j \in J$ such that $I'_j \neq \emptyset$. If L' is finite, then each I'_j is finite, J' is finite, and

$$\sum_{l\in L'} s_l = \sum_{j\in J'} \sum_{i_j\in I'_j} s_{i_j} \leq \sum_{j\in J'} \sum_{i_j\in I_j} s_{i_j} \leq \sum_{j\in J} \sum_{i_j\in I_j} s_{i_j}. \tag{3.48}$$

Inequality (3.48) holds for any finite subset $L' \subseteq L$, hence it holds if we replace the sum over L' by

$$\sum_{l\in L} s_i = \sup_{\substack{L'\subseteq L \\ L' \text{ finite}}} \sum_{l\in L'} s_l.$$

$\square$

Lemma 3.4.6 *Inequality in* (3.47) *becomes equality if ordered semigroup* $(S, +, \leq)$ *is distributive. In particular, operations defined in* (3.46) *transform* S *into an infinitary semigroup.*

Proof. Denote $\sum_{l \in L} s_l$ by u. For any $J' \subseteq J$ and any $I'_j \subseteq I_j$ let

$$L' := \coprod_{j \in J} I'_j.$$

Associativity and commutativity of addition in S mean that, when L' is a finite nonempty set, then

$$\sum_{j \in J'} \sum_{i_j \in I'_j} s_{i_j} = \sum_{l \in L'} s_l \leq u. \tag{3.49}$$

Let A_j be the set formed by the sums

$$\sum_{i_j \in I'_j} s_{i_j}$$

where I'_j are arbitrary finite nonempty subsets of I_j. Then the set $\sum_{j \in J'} A_j$ is formed by the sums

$$\sum_{j \in J'} \sum_{i_j \in I'_j} s_{i_j}.$$

where finite set J' is fixed and I'_j are arbitrary finite nonempty subsets of I_j.

Distributivity of $(S, +, \leq)$ yields with help of Corollary 3.4.4

$$\sum_{j \in J'} \sum_{i_j \in I_j} s_{i_j} = \sum_{j \in J'} \sup A_j = \sup\Big(\textstyle\sum_{j \in J'} A_j\Big)$$

where

$$\textstyle\sum_{j \in J'} A_j = \Big\{\sum_{j \in J'} \sum_{i_j \in I'_j} s_{i_j} \mid I'_j \subseteq I_j \text{ finite nonempty}\Big\}.$$

Inequality (3.49) implies that

$$\sup\Big(\textstyle\sum_{j \in J'} A_j\Big) \leq u \tag{3.50}$$

which, combined with (3.50), implies

$$\sum_{j\in J'}\sum_{i_j\in I_j} s_{i_j} \leq u$$

for all finite nonempty $J' \subseteq J$. Passing to the supremum over J' we obtain the inequality

$$\sum_{j\in J}\sum_{i_j\in I'_j} s_{i_j} = \sum_{l\in L} s_l.$$

The reverse inequality holds without any distributivity hypotheses, cf. Lemma 3.4.5. □

3.4.14 Example: $[0,\infty]^X$

The additive monoid of semiring $[0,\infty]^X$ is distributive and positively ordered. According to Lemma 3.4.6 it becomes an infinitary monoid.

Exercise 68 *Show that if* $\sum_{i\in I} a_i < \infty$*, then the* support *of family* $\{a_i\}_{i\in I}$*,*

$$\operatorname{supp}\{a_i\}_{i\in I} := \{i \in I \mid \alpha_i \neq 0\}$$

is countable.

Exercise 69 *Show that, for a sequence* $\{a_n\}_{n\in\mathbf{N}}$*,*

$$\sum_{n\in\mathbf{N}} a_n = \sum_{n=0}^{\infty} a_n.$$

In other words, show that the sum of elements of family $\{a_n\}_{n\in\mathbf{N}}$ *coincides with the value of the corresponding infinite series.*

3.4.14.1 A variant: construction of a σ-operation

By replacing *arbitrary* sets of indices in (3.46) by *countable* ones we obtain a variant of the previous construction which produces a σ-operation.

Countable distributivity of an ordered semigroup $(S,+,\leq)$ guarantees that identities S becomes a σ-semigroup. The proof of this fact is word by word the same

3.4.15 Infinitary semirings

3.4.15.1

3.5 Sets with an action

3.5.1 Sets with an action of another set

3.5.1.1

We say that a set G *acts* on a set X if we associate with every element $g \in G$, a selfmap $\lambda_g \colon X \longrightarrow X$. The family of selfmaps $\{\lambda_g\}_{g \in G}$ is a map

$$\lambda \colon G \longrightarrow X^X, \qquad g \longmapsto \lambda_g \qquad (g \in G). \tag{3.51}$$

3.5.1.2

The action of G on X can be also given in the form of a pairing

$$\tilde{\lambda} \colon G \times X \longrightarrow X,$$

where $\tilde{\lambda}$ and λ are linked by the identity

$$\tilde{\lambda}(g, x) = \lambda_g(x) \qquad (g \in G; x \in X). \tag{3.52}$$

Using identity (3.52) one can recover λ from $\tilde{\lambda}$.

3.5.1.3 Simplified notation

A common practice is to denote $\lambda_g(x)$ by gx as if we are multiplying x by g on the left.

3.5.1.4 The category of G-sets

Sets equipped with an action of a given set G naturally form a category. Morphisms $(X, \lambda) \longrightarrow (Y, \mu)$ are maps $f \colon X \longrightarrow Y$ which are compatible with G-action. This translates into f satisfying the equalities

$$f \circ \lambda_g = \mu_g \circ f \qquad (g \in G).$$

Explicitly,

$$f\left(\lambda_g(x)\right) = \mu_g\left(f(x)\right) \qquad (g \in G; x \in X) \tag{3.53}$$

or, in simplified notation,

$$f(gx) = gf(x) \qquad (g \in G; x \in X).$$

The category of G-sets will be denoted $G\text{-Set}$.

3.5.1.5

In the language of commuting diagrams identity (3.53) is equivalent to commutativity of the square diagram

$$\begin{tikzcd} G \times X \arrow[r, "\tilde{\lambda}"] \arrow[d, "\mathrm{id}_G \times f"'] & X \arrow[d, "f"] \\ G \times Y \arrow[r, "\tilde{\mu}"'] & Y \end{tikzcd}$$

3.5.1.6 Equivariant maps

Traditionally, morphisms in the category of G-sets are called *equivariant maps*.

3.5.2 Objects with an action of a set

3.5.2.1

This is an obvious generalization of the previous structure. We say that a set G *acts* on an object a of a category $\mathcal{C}$ if we associate with every element $g \in G$, an endomorphism $\lambda_g \colon a \longrightarrow a$. The family of endomorphisms $\{\lambda_g\}_{g \in G}$ is a map

$$\lambda \colon G \longrightarrow \mathrm{End}_{\mathcal{C}}(a), \qquad g \longmapsto \lambda_g \qquad (g \in G). \tag{3.54}$$

3.5.2.2 The category of G-objects

Objects of a category $\mathcal{C}$ equipped with an action of a given set G form a category. Morphisms $(a, \lambda) \longrightarrow (b, \mu)$ are morphisms $\alpha \colon a \longrightarrow b$ which are compatible with G-action. This translates into α satisfying the identity

$$\alpha \circ \lambda_g = \mu_g \circ \alpha \qquad (g \in G)$$

The category of G-objects for a category $\mathcal{C}$ will be denoted $G\text{-}\mathcal{C}$.

3.5.3 Sets with an action of a semigroup

3.5.3.1

When a set G that acts on a set X, is equipped with a binary operation, it is natural to require that the operation and the action are compatible. This translates into saying that map (3.51) should be a homomorphism,

$$\lambda_{gh} = \lambda_g \circ \lambda_h \qquad (g, h \in G)$$

or, using simplified notation, that the identity

$$(gh)x = g(hx) \qquad (g, h \in G; x \in X). \tag{3.55}$$

3.5.3.2

Noting that (3.55) closely resembles the associativity condition, it is not surprising that this definition is particularly well suited to the case when multiplication in G is associative, i.e., when G is a semigroup.

3.5.3.3

In the context of semigroups, the phrase 'a G-set' always means

> *a set equipped with an action of set G such that the structural map,* (3.51)*, is a homomorphism of semigroups.*

In this restricted sense, G-sets form a full subcategory of the category of G-sets where G is simply considered to be a set.

3.5.3.4

Similarly, we say that a semigroup G acts on an object a of a category $\mathcal{C}$, if a *homomorphism* of semigroups (3.54) is given.

3.5.3.5 Notation

Notation G-Set and G-$\mathcal{C}$ will be used to denote the corresponding categories of G-sets and G-objects.

3.5.3.6 The case of a monoid

We have seen above that a homomorphism of semigroups does not preserve the identity elements, in general. A homomorphism of monoids is explicitly required to preserve the identity elements.

Note that X^X and $\mathrm{End}_{\mathcal{C}}(a)$ are monoids. If G is a monoid G, we require that the structural maps, (3.51) and (3.54) are homomorphisms of monoids. In other words, we require them to be homomorphisms of the corresponding binary operations and, additionally to preserve the identity:

$$\lambda_e = \mathrm{id}_X \qquad \text{(in the case of an action on a set } X\text{)}$$

and

$$\lambda_e = \mathrm{id}_a \qquad \text{(in the case of an action on an object } a\text{).}$$

3.5.3.7 Group actions

This case is of particular importance. The role played by groups in Mathematics and its applications to Physics, Chemistry, and Engineering, is primarily as groups of symmetries of various objects.

3.5.4 Semimodules

3.5.4.1

The set of endomorphisms $\mathrm{End}_{\mathrm{Semigrp}}(A)$ of a commutative semigroup A is a unital semiring, cf. Section 3.4.3.7. We will say that a semiring R

acts on a commutative group A if a homomorphism of semirings

$$\lambda\colon R \longrightarrow \mathrm{End}_{\mathrm{Semigrp}}(A) \tag{3.56}$$

is given.

3.5.4.2 The action of a semiring analyzed

Let us translate into concrete identities the fact that (3.56) defines an action of semiring R on semigroup A. We will be using simplified notation throuout: $ra := \lambda_r(a)$.

3.5.4.3

Let us begin from the fact that, for each $r \in R$, map λ_r is supposed to be an endomorphism of semigroup A. This is expressed by the identity

$$r(a+b) = ra + rb \qquad (r \in R; a, b \in A). \tag{3.57}$$

3.5.4.4

Map (3.56) is supposed to be a homomorphism of the additive semigroup R^+ of R into the addititive semigroup of semiring $\mathrm{End}_{\mathrm{Semigrp}}(A)$. This is expressed by the identity

$$(r+s)a = ra + sa \qquad (r, s \in R; a \in A). \tag{3.58}$$

3.5.4.5

Finally, map (3.56) is supposed to be also a homomorphism of the *multiplicative* semigroup $R^\times$ of R into the multiplicative semigroup of semiring $\mathrm{End}_{\mathrm{Semigrp}}(A)$. This is expressed by the identity

$$(rs)a = r(sa) \qquad (r, s \in R; a \in A). \tag{3.59}$$

3.5.4.6

If we diregard the fact that the left multiplier in the expression *ra* belongs to R while the right multiplier belongs to A, then we can interpret identity (3.57) as left-distributivity of multiplication by elements of R over addition in A.

Similarly, identity (3.58) can be interpreted as right-distributivity of multiplication by elements of A over addition in R.

Finally, identity (3.59) looks like associativity of multiplication, except that the left-hand-side of (3.59) involves two diferent 'multiplications': of two elements of R, and of an element of R and an element of A.

3.5.4.7 R-semimodules

A short name for a semiring R acting on a commutative semigroup is an R-*semimodule* or, to be precise, a *left* R-semimodule—since there is a version of the semimodule definition in which R acts from the right. An alternative way to say the same: a *semimodule over* R.

3.5.4.8 The category of R-semimodules

Given two R-semimodules, a morphism $A \longrightarrow B$ is a morphism of semigroups $f \colon A \longrightarrow B$, i.e., an additive map, which is compatible with actions of R on A and B. This last requirement is expressed as the identity

$$f(ra) = rf(a) \qquad (r \in R; a \in A). \tag{3.60}$$

Maps between commutative semigroups $f \colon A \longrightarrow B$ which satisfy identity (3.60) are said to be *homogeneous* (of degree 1).

Thus, morphisms between R-semimodules are maps that are additive and homogeneous of degree 1. Maps with these two properties are also called R-*linear* or, simply, linear, when the semiring of coefficients is clear from the context.

3.5.4.9 Terminology

Given an R-semimodule A, if we need to refer to the underlying structure of a semigroup forgetting the action of R, then we call it the *additive semigroup* of A.

If we need to refer to R, we call it the *semiring of coefficients*, or the *ground semiring*.

3.5.4.10 Subsemimodules

Let us look at the additive semigroup A^+ of an R-semimodule A. Suppose that a subsemigroup B be of A^+ atisfies the property

$$rb \in B \quad \textit{for any} \quad r \in R; \textit{ and } ; b \in B.$$

Then one can consider B, equipped with the R-action induced from A, as an R-semimodule. Such a semimodule is called a *subsemimodule* of A.

3.5.4.11

Exercise 70 *For any subset X of an R-semimodule A, let RX be the set formed by sums in A*

$$\xi = \sum_{x \in X'} r_x x$$

where X' is any finite nonempty *subset of X and $\{r_x\}_{x \in X'}$ is any family of elements of R, indexed by set X'. Show that RX is a subsemimodule of A.*

3.5.4.12

Note that $R\emptyset = \emptyset$.

Exercise 71 *Show that the intersection of any family $\{B_i\}_{i \in I}$ of subsemimodules of A is a subsemimodule.*

Exercise 72 *Show that RX coincides with the intersection of the family of subsemimodules of A which contain subset X.*

3.5.4.13 The subsemimodule generated by a subset

Subsemimodule RX is the smallest subsemimodule of A which contains subset X. We shall refer to it as the subsemimodule *generated* by $X \subseteq A$.

3.5.4.14 Sets of generators

If $RX = A$, we say that X is a *set of generators* for R-semimodule A.

Exercise 73 *Suppose that $z \in R$ is a right zero, i.e.,*

$$r + z = r \qquad \textit{and} \qquad rz = z \qquad (r \in R).$$

Show that the set

$$zA := \{za \mid a \in A\}$$

is a subsemimodule of A and every element in zA is an additive idempotent

$$b + b = b \qquad (b \in zA).$$

3.5.4.15 Semimodules over a semiring with zero

When A is a commutative monoid, then $\mathrm{End}_{\mathrm{Mon}}(A)$ is a semiring with zero: the constant map $A \longrightarrow A$ which sends every element of A to $0 \in A$ playing the role of the zero element.

If the ground semiring itself has zero, then in the definition of a semimodule we additionally request that $0 \in R$ acts on elements of A via the zero map

$$0_R \cdot a = 0_A \qquad (a \in A)$$

or, in simplified notation,

$$0a = 0 \qquad (a \in A).$$

3.5.4.16 The category of semimodules over a semiring with zero

This is a subcategory of the category of semimodules whose objects are commutative monoids instead of commutative semigroups, and morphisms are supposed to be homomorphisms of commutative monoids, i.e., be additive maps and additionally send 0 to 0.

This is an example of a not full subcategory.

3.5.4.17 Unitary semimodules

Suppose that the ground ring is unital. If $1 \in R$ acts on A as the identity endomorphism, then A is said to be a *unitary* R-semimodule.

3.5.4.18 Example: unitary $\mathbf{Z}_+$-semimodules

Consider the set of positive integers

$$\mathbf{Z}_+ := \{1, 2, \dots\}$$

equipped with usual addition and multiplication. It is a unital semiring.

For any unitary semimodule over $\mathbf{Z}_+$, one has

$$na = \underbrace{(1 + \cdots + 1)}_{n \text{ times}} a = \underbrace{a + \cdots + a}_{n \text{ times}} \qquad (a \in A) \tag{3.61}$$

which means that a structure of a unitary $\mathbf{Z}_+$-semimodule on a semigroup A is completely determined by the structure of A as a semigroup. In particular, there is only one structure of a unitary $\mathbf{Z}_+$-semimodule on any given commutative semigroup.

Vice-versa, for any commutative semigroup A, formula (3.61) defines an action of the semiring of positive integers on A making it a unitary $\mathbf{Z}_+$-semimodule.

Exercise 74 *Show that any homomorphism of commutative semigroups is automatically a homomorphism of $\mathbf{Z}_+$-semimodules.*

3.5.4.19

It follows that the category of unitary $\mathbf{Z}_+$-semimodules is isomorphic to the category of commutative semigroups.

3.5.4.20 Example: unitary N-semimodules with zero

Consider the set of natural numbers

$$\mathbf{N} := \{0, 1, 2, \dots\}$$

equipped with usual addition and multiplication. It is a unital semiring with zero which contains $\mathbf{Z}_+$ as a subsemiring.

Let A be a unitary $\mathbf{N}$-semimodule with zero. Thus, $0 \in \mathbf{N}$ acts by sending any element $a \in A$ to $0 \in A$ while any positive integer acts on A by formula (3.61). Accordingly, a structure of a unitary $\mathbf{N}$-semimodule with zero on a monoid A is completely determined by the structure of A as a monoid. In particular, there is only one structure of a unitary $\mathbf{N}$-semimodule on any given commutative monoid.

Vice-versa, for any commutative monoid A, formulae (3.61) and

$$0a = 0$$

define an action of the semiring of natural numbers on A making it a unitary $\mathbf{N}$-semimodule with zero.

Any homomorphism of commutative monoids is automatically a homomorphism of $\mathbf{N}$-semimodules. It follows that the category of unitary $\mathbf{N}$-semimodules with zero is isomorphic to the category of commutative monoids.

3.5.4.21 Modules over a ring

When A is an abelian group, then $\mathrm{End}_{\mathrm{Grp}}(A)$ is a unital ring. If the ground semiring is a ring, then any homomorphism of semirings (3.56) is automatically also a homomorphism of rings (rings form a full subcategory in the category of semirings). In this case we speak of R-modules, or modules over R, rather than semimodules.

3.5.4.22 The category of R-modules

The category of R-modules is a full subcategory of the category of R-semimodules.

3.5.4.23 The category of unitary R-modules

The most frequently encountered is the category of unitary modules over a unital ring R. It is this category that is usually denoted $R\text{-mod}$.

3.5.4.24 Vector spaces

Unitary modules over a field F are called *F-vector spaces* or *vector spaces over F*.

3.5.4.25

As we noted above, a semimodule A over a ring R is a module precisely when the additive semigroup of A is a group. When the ground ring is unital and A is a unitary R-semimodule, then the additive semigroup of A is forced to be a group.

Observation 3.5.1 *Any unitary semimodule is automatically a module. More precisely, for any $a \in A$, the element $(-1)a$ is the additive inverse to a.*

Indeed, for any $a \in A$, one has

$$0 = 0a = (1 + (-1))\, a = 1a + (-1)a = a + (-1)a,$$

i.e., $(-1)a$ is the right inverse of a in the additive semigroup of A (it is automatically a two-sided inverse since addition in A is commutative).

Exercise 75 *Let A be an R-semimodule over a nonunital ring R. Show that, for any $a \in A$, the following subset of A*

$$Ra := \{ra \mid r \in R\}$$

is a subgroup of the aditive group of A.

3.5.4.26 Example: unitary Z-modules

Consider the set of integers

$$\mathbf{N} := \{0, \pm 1, \pm 2, \dots\}$$

equipped with usual addition and multiplication. It is a unital ring which contains $\mathbf{N}$ as a subsemiring with zero.

Let A be a unitary $\mathbf{Z}$-module. Action of positive integers is governed by identity (3.61).

3.5.5 Semialgebras

3.5.5.1

We defined semirings as commutative semigroups equipped with a bi-additive binary operation. It happens very often that the semigroup is a semimodule over certain semiring, and that the operation is *bilinear*.

3.5.5.2 Bilinear pairings

Suppose that R-semimodules A, B, and C be given. A biadditive pairing

$$\mu\colon A \times B \longrightarrow C$$

is said to be R-*bilinear*, or a R-*biadditive pairing*, if it is homogeneous of (degree 1) in each argument:

$$\mu(ra, b) = r\mu(a, b) \qquad (r \in R; a \in A; b \in B),$$

and

$$\mu(a, rb) = r\mu(a, b) \qquad (r \in R; a \in A; b \in B).$$

3.5.5.3

The notion of of a bilinear pairing makes sense for semimodules over any semiring. When R is not commutative, however, its usefulness is limited. A proper context for 'bilinear' and, more generally, multilinear maps requires replacing semimodules by *semibimodules*. This will not be discussed here, so from now on we shall assume that the ground ring is *commutative*.

3.5.5.4 Semialgebras: terminology and notation

A semimodule equipped with a bilinear multiplication

$$\mu\colon A \times A \longrightarrow A$$

is called an *algebra*. There is a tradition to denote by k the ground semiring which, as you remember, is assumed to be commutative. If one needs to be more specific, terms like 'a k-semialgebra' or 'a semialgebra over k' are used as well.

3.5.5.5

All the terms applicable to semirings continue to be applicable to semialgebras: 'associative', 'commutative', 'with zero', 'unital', etc.

3.5.5.6 Example: semirings as $\mathbf{Z}_+$-semialgebras

Every semiring is automatically a semialgebra over $\mathbf{Z}_+$, cf. The category of semirings and the category of $\mathbf{Z}_+$-semialgebras are isomorphic.

3.5.5.7 Example: k^X

The set of maps $X \longrightarrow k$ with values in a commutative semiring, with pointwise addition and multiplication is naturally a k-semialgebra.

3.5.5.8 Example: semirings with zero as $\mathbf{N}$-semialgebras

Every semiring with zero is automatically a semialgebra over $\mathbf{N}$, cf. The category of semirings with zero and the category of $\mathbf{N}$-semialgebras are isomorphic.

3.5.5.9 Algebras

'Semialgebras' are called *algebras*, if k is a ring and A is a k-module.

3.5.5.10 Morphisms

Morphisms between semialgebras $A \longrightarrow B$ are maps $f\colon A \longrightarrow B$ which are simultaneously homomorphisms of the corresponding additive semimodules and of the multiplicative binary structures. Like for other algebraic structures, they are usually called *homomorphisms*.

3.5.5.11

Morphisms $A \longrightarrow B$ between semialgebras with zero are of course expected to send 0_A to 0_B.

3.5.5.12

We said it already twice before: associativity is of such importance that it became a standard practice to tacitly assume associativity when talking about semialgebras and algebras.

The category of associative semialgebras over k will be denoted $k\text{-semialg}$, and the category of associative algebras will be denoted $k\text{-alg}$.

3.5.5.13 Terminology: a warning

The term ‘algebra’ is used in Mathematics in at least two different ways: as a special kind of algebraic structure, and as a branch of Mathematics. In the latter sense I suggest to always capitalize it: Algebra.

Term ‘algebra’ can be also used in a loose sense of anything that involves extensive manipulations of symbolic expressions.

You have to be aware that various structures, not necessarily strictly algebraic, were designated with term ‘algebra’ before the latter became attached to that particular algebraic structure we call now *an algebra*.

For example, neither Boolean algebras nor σ-algebras are algebras in the sense given above. Both, however, are semirings of special kind.

3.5.5.14 Example: $\mathscr{P}(X)$ as an $\mathbf{F}_2$-algebra

The set of all subsets of any set X is only a semiring when considered with operations $\cup$ and $\cap$. If one replaces *union* by *disjoint union*,

$$A \mid B := (A \cup B) \setminus (A \cap B),$$

then $(\mathscr{P}(X), |, \cup)$ becomes a commutative unital algebra over the field with two elements $\mathbf{F}_2 = \{0, 1\}$.

Exercise 76 *Show that $\cap$ distributes over $|$.*

Chapter 4

Some universal constructions

4.1 Universal properties

4.1.1 Product

4.1.1.1

Let

$$\{a_i\}_{i \in I} \tag{4.1}$$

be a family of objects in a category $\mathcal{C}$. Given a morphism $\alpha \colon x \longrightarrow y$, and a family of morphisms $g_i \colon y \longrightarrow a_i$, we can form the family of morphisms

$$f_i = g_i \circ \alpha \qquad (i \in I). \tag{4.2}$$

The family $\{f_i\}_{i \in I}$ is said to be *induced by morphism* α *from family* $\{f_i\}_{i \in I}$.

4.1.1.2 A universal family

We say that an object $p \in \operatorname{Ob} \mathcal{C}$ equipped with a family of morphisms $\{\pi_i \colon p \longrightarrow a_i\}_{i \in I}$, is a *product* of family (4.1) if any family of morphisms (4.2) can be induced from family $\{\pi_i\}_{i \in I}$ by a *unique* morphism $\alpha \colon x \longrightarrow p$.

Exercise 77 *Show that if* $\{\pi_i \colon p \longrightarrow a_i\}_{i \in I}$ *and* $\{\pi'_i \colon p' \longrightarrow a_i\}_{i \in I}$ *are products of family* $\{a_i\}_{i \in I}$*, then* p *and* p' *they are isomorphic.*

4.1.1.3 Terminology and notation

Morphisms $\pi_i \colon p \longrightarrow a_i$ are referred to as the *canonical projections*. Even though the term 'product' is often applied just to object p, the canonical projections form a part of the product structure.

4.1.1.4 Example: the category of fields

A product may fail to exist. This happens, in particular, when for a given family of objects (4.1) there is no object $x \in \operatorname{Ob} \mathcal{C}$ such that

$$\operatorname{Hom}_{\mathcal{C}}(x, a_i) \neq \emptyset.$$

This situation may occur, e.g., in the category of fields where any morphism $E \longrightarrow F$ is an injective map between the underlying sets. Thus, a product of $\mathbf{F}_2$ and $\mathbf{F}_3 = \{0, 1, -1\}$ does not exist.

Exercise 78 *Show that* $\mathbf{F}_2$ *is a product of* $\mathbf{F}_2$ *and* $\mathbf{F}_2$ *in the category of fields, with the 'canonical projections' being the identity maps* $\mathbf{F}_2 \longrightarrow \mathbf{F}_2$*.*

4.1.1.5 Example: a partially ordered set viewed as a category

Here, a product of family (4.1) exists if and only if the *set*

$$\{a_i \mid i \in I\}$$

has infimum. In this case product is *unique*, namely

$$\inf \{a_i \mid i \in I\}.$$

Exercise 79 *Prove the above two assertions.*

4.1.1.6

When a product of (4.1) exists it is not unique if there exists $p' \in \operatorname{Ob} \mathcal{C}$ and a non-identity isomorphism $p \simeq p'$. However, any two solutions to the problem of existence of a product of family $\{a_i\}_{i \in I}$ are isomorphic, and there exists only one such isomorphism which is compatible with all the projection morphisms.

4.1.1.7 Functorial products

When a product exists for any family of objects in a category $\mathcal{C}$, it frequently happens, that there is a *functorial* solution to the problem of existence of a product. This means that there exists a functor, denoted $\prod$, from the category of I-indexed families $\{a_i\}_{i\in I}$ of objects in $\mathcal{C}$ to the category of I-indexed families of morphisms $\{f_i\colon x\longrightarrow a_i\}_{i\in I}$ whose 'values' are products of the corresponding families.

There is no need to say more about it now. We will signal such functorial constructions of products when we encounter them.

4.1.1.8 Product of a family of sets

For a family of sets $\{X_i\}_{i\in I}$, let

$$X = \bigcup_{i\in I} X_i,$$

and let

$$\prod_{i\in I} X_i := \{\xi\colon I\longrightarrow X \mid \xi(i) \in X_i\}. \tag{4.3}$$

Set (4.3) is called the Cartesian product of family $\{X_i\}_{i\in I}$. Usual interpretation of elements of the Cartesian product is as families $\{x_i\}_{i\in I}$ of elements of X such that $x_i \in X_i$. The maps that forget all but one component of $\{x_i\}_{i\in I}$ are the canonical projections.

4.1.1.9

Given any family of maps $f_i\colon W\longrightarrow X_i$, define

$$\tilde{f}\colon W\longrightarrow \prod_{i\in I} X_i \qquad w \longmapsto \{f_i(x)\}_{i\in I}\,.$$

4.1.1.10 Functoriality of the Cartesian product

A morphism $\{X_i\}_{i\in I}\longrightarrow\{Y_i\}_{i\in I}$ is a family of maps $\{f_i\colon X_i\longrightarrow Y_i\}$. It induces the map

$$\prod_{i\in I} f_i\colon \prod_{i\in I} X_i\longrightarrow \prod_{i\in I} Y_i, \qquad \{x_i\}_{i\in I} \longmapsto \{f_i(x)_i\}_{i\in I}.$$

which is compatible with the composition of morphisms of families of sets.

4.1.1.11 Alternative notation

If I is a finite linearly ordered set like $\{1, \ldots, n\}$, an alternative notation is frequently used

$$X_1 \times \cdots \times X_n \qquad \text{and} \qquad f_1 \times \cdots \times f_n$$

instead of

$$\prod_{i=1}^{n} X_i \qquad \text{and} \qquad \prod_{i=1}^{n} f_i.$$

4.1.1.12 Product of a family of algebraic structures

Consider a family of algebraic structures

$$\{(X_i, \mu_{i1}, \ldots, \mu_{il})\}_{i \in I}$$

of signature $(n_1, \ldots, n_l)$. We shall equip the Cartesian product of sets $\prod_{i \in I} X_i$ with the structure of the same signature by applying corresponding operations *componentwise*:

$$\nu_j\left(\{x_i\}_{i \in I}, \{y_i\}_{i \in I}\right) := \left\{\mu_{ij}(x_i, y_i)\right\}_{i \in I} \qquad (j = 1, \ldots, l).$$

This construction depends functorially on the family of structures and one can easily verify that the

4.1.1.13 Product of a family of topological spaces

4.1.1.14 Product of a family of measurable spaces

Part II

Theory of Measure and Integration

Chapter 5

Integration on measurable spaces

5.1 A preintegral

5.1.1

5.1.1.1

Let L be a complete, countably distributive lattice, and $\mathcal{S} \subseteq L^X$ be a subset closed under formation of infima of finite nonempty sets, i.e., $\mathcal{S}$ is a $\wedge$-subsemilattice of L^X.

Lemma 5.1.1 *Let $\Lambda\colon \mathcal{S} \longrightarrow L$ be map with the property that, for any nondecreasing sequence $\{s_i\}_{i \in \mathbf{N}}$ in $\mathcal{S}$ whose supremum belongs to $\mathcal{S}$, one has*

$$\sup \Lambda(s_i) = \Lambda(\sup s_i).$$

Then, for any pair of nondecreasing sequences in $\mathcal{S}$,

$$\sup \Lambda(g_i) = \sup \Lambda(h_i)$$

whenever

$$\sup g_i = \sup h_i. \tag{5.1}$$

Proof. Let us denote the common value of (5.1) by f. For a given $i \in \mathbf{N}$, the sequence

$$\{g_i \wedge h_j\}_{j \in \mathbf{N}}$$

is nondecreasing and, in view of countable distributivity of L^X, its supremum equals

$$g_i \wedge \sup\{h_j \mid j \in \mathbf{N}\} = g_i \wedge f = g_i.$$

The last equality follows from the inequality $g_i \leq f$. Hence

$$\sup\{\Lambda(g_i \wedge h_j) \mid j \in \mathbf{N}\} = \sup g_i \qquad (i \in \mathbf{N}), \tag{5.2}$$

and, likewise,

$$\sup\{\Lambda(g_i \wedge h_j) \mid i \in \mathbf{N}\} = \sup h_j \qquad (j \in \mathbf{N}). \tag{5.3}$$

Let us consider two sequences of subsets of L:

$$A_i := \{\Lambda(g_i \wedge h_j) \mid n \in \mathbf{N}\} \quad \text{and} \quad B_j := \{\Lambda(g_i \wedge h_j) \mid m \in \mathbf{N}\}.$$

Noting that

$$\bigcup_{i \in \mathbf{N}} A_i = \{\Lambda(g_i \wedge h_j) \mid i, j \in \mathbf{N}\} = \bigcup_{j \in \mathbf{N}} B_j,$$

we deduce that

$$\sup\{\sup A_i \mid i \in \mathbf{N}\} = \sup\{\Lambda(g_i \wedge h_j) \mid i, j \in \mathbf{N}\} = \sup\{\sup B_j \mid j \in \mathbf{N}\}.$$

According to (5.2), the left-hand-side equals

$$\sup\{\Lambda(g_i) \mid i \in \mathbf{N}\}$$

while the right-hand-side, according to (5.3), equals

$$\sup\{\Lambda(h_j) \mid j \in \mathbf{N}\}.$$

□

5.1.2

We say that a map $\Lambda\colon \mathcal{S} \longrightarrow L$ is a *preintegral*

5.2 Measure

5.2.1 Definitions

5.2.1.1

Let $\mathfrak{M}$ be a σ-algebra on a set X. We shall refer to members of $\mathfrak{M}$ as *measurable* sets.

5.2.1.2

A function
$$\mu\colon \mathfrak{M} \longrightarrow [0,\infty] \tag{5.4}$$
is said to be *σ-additive* if it is additive on any countable family $\mathscr{A} \subseteq \mathfrak{M}$ of *disjoint* subsets of $\mathfrak{M}$:
$$\mu\left(\bigcup_{A\in\mathscr{A}} A\right) = \sum_{A\in\mathscr{A}} \mu(A).$$
A σ-additive function (5.4) is called a *measure* on measurable space $(X,\mathfrak{M})$.

5.2.2 Properties

5.2.2.1 Monotonicity

If $A \subseteq B$ and both belong to $\mathfrak{M}$, then
$$\mu(A) \le \mu(A) + \mu(B \setminus A) = \mu(B).$$
This shows that a measure is a *monotonic* function:
$$\text{if } \; A \subseteq B \; \text{, then } \; \mu(A) \le \mu(B).$$

5.2.2.2

According to the definition we gave, it is possible for $\mu(\emptyset)$ not to be zero.

Exercise 80 *Show that, if there exists a nonempty subset $A \subseteq \mathfrak{M}$ with $\mu(A) < \infty$, then $\mu(\emptyset) = 0$.*

5.2.2.3 Continuity on nondecreasing sequences of sets

If
$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$$
is a nondecreasing sequence of measurable sets, then
$$\mu\left(\bigcup_{i\in\mathbf{N}} A_i\right) = \sup_{n\in\mathbf{N}} \mu\left(A_n\right).$$

Indeed,

$$\begin{aligned}\mu\left(\bigcup_{i\in\mathbf{N}} A_i\right) &= \mu\left(\bigcup_{i\in\mathbf{N}} (A_{i+1}\setminus A_i)\right) = \sum_{i=0}^{\infty}\mu\left(A_{i+1}\setminus A_i\right)\\ &= \lim_{n\longrightarrow\infty}\sum_{i=0}^{n}\mu\left(A_{i+1}\setminus A_i\right) = \lim_{n\longrightarrow\infty}\mu\left(A_n\right).\end{aligned}$$

5.2.2.4 Continuity on nonincreasing sequences of sets

If

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$$

is a nonincreasing sequence of measurable sets, then

$$\mu\left(\bigcup_{i\in\mathbf{N}} A_i\right) = \inf_{n\in\mathbf{N}}\mu\left(A_n\right)$$

provided

$$\inf_{n\in\mathbf{N}}\mu\left(A_n\right) < \infty. \tag{5.5}$$

Indeed, if (5.5) is satisfied, then $\mu\left(A_l\right) < \infty$ for some $l\in\mathbf{N}$ and

$$\mu\left(A_l\setminus\bigcap_{i\in\mathbf{N}} A_{i+l}\right) = \mu\left(\bigcup_{i\in\mathbf{N}}\left(A_l\setminus A_{i+l}\right)\right) = \sup_{n\in\mathbf{N}}\mu\left(A_l\setminus A_{n+l}\right).$$

In view of additivity of μ and the fact that $\mu\left(A_l\right) < \infty$, the left-hand-side equals

$$\mu\left(A_l\right) - \mu\left(\bigcap_{i\in\mathbf{N}} A_i\right)$$

while the right-hand-side equals

$$\sup_{n\in\mathbf{N}}\left(\mu\left(A_l\right) - \mu\left(A_{n+l}\right)\right) = \mu\left(A_l\right) + \sup_{n\in\mathbf{N}}\left(-\left(\mu(A_n)\right) = \mu\left(A_l\right) - \inf_{n\in\mathbf{N}}\mu\left(A_n\right).$$

Exercise 81 *Show that a function $\mu\colon\mathfrak{M}\longrightarrow[0,\infty]$ is a measure if it is continuous on nondecreasing sequences of sets, cf. 5.2.2.3, and is* finitely *additive, i.e.,*

$$\mu\left(\bigcup_{i\in I} A_i\right) = \sum_{i\in I}\mu\left(A_i\right)$$

for any finite family of disjoint sets $A_i\in\mathfrak{M}$.

5.2.3 Some special measures

5.2.3.1 A measure associated with a function $\phi\colon X \longrightarrow [0,\infty]$

The formula
$$\mu_\phi(A) := \sum_{x\in A} \phi(x)$$
defines a measure on measurable space $(X, \mathscr{P}(X))$.

5.2.3.2

For $\phi = \chi_E$ being the characteristic function of a subset $E \subseteq X$, cf. (1.3), we have
$$\mu_{\chi_E}(A) = \begin{cases} |A\cap E| & \text{if } A\cap E \text{ is finite} \\ \infty & \text{otherwise} \end{cases}. \tag{5.6}$$

5.2.3.3 Probabilistic measures

If $\mu(X) = 1$, then μ is called a *probabilistic measure* on $(X, \mathfrak{M})$.

5.2.3.4 0-1 measures

In the special case of a singleton set E, measure (5.6) takes just two values: 0 and 1.

Suppose that μ is a measure on $(X, \mathscr{P}(X))$ which takes exactly two values, 0 and 1. Thus, μ is the characteristic function $\mathscr{P}(X) \longrightarrow \{0,1\}$ of the family
$$\mathscr{F} := \{A \in \mathscr{P}(X) \mid \mu(A) = 1\}$$
viewed as a subset of $\mathscr{F}$.

Exercise 82 *Show that $\mathscr{F}$ satisfies the following properties*

(**F₁**) *for any* countable *family $\{F_i\}_{i\in I}$ of elements of $\mathscr{F}$, one has $\bigcap_{i\in I} F_i \in \mathscr{F}$.*

(**F₂**) *for any $F \in \mathscr{F}$ and $A \subseteq X$, if $F \subseteq A$, then $A \in \mathscr{F}$*

(**F₃**) $\emptyset \notin \mathscr{F}$

(**F₄**) *for any $A \subseteq X$, either $A \in \mathscr{F}$ or $A^c \in \mathscr{F}$.*

Exercise 83 *Show that the characteristic function* $\chi_{\mathscr{F}} : \mathscr{P}(X) \longrightarrow \{0,1\}$ *of a family* $\mathscr{F} \subseteq \mathscr{P}(X)$ *which satisfies conditions* $(\mathbf{F_1})$–$(\mathbf{F_4})$ *is a measure.*

5.2.3.5 Ultrafilters

If we replace 'countable' by 'finite' in condition $(\mathbf{F_1})$, then we obtain the definition of an *ultrafilter* on a set X. For this reason, we shall call a family $\mathscr{F} \subseteq \mathscr{P}(X)$ which satisfies conditions $(\mathbf{F_1})$–$(\mathbf{F_4})$, a σ-ultrafilter.

5.2.3.6 βX

The set of ultrafilters on any set X possesses a natural topology in which it is a Hausdorff compact space, denoted βX.

5.2.3.7 Principal ultrafilters

For any element $x \in X$, the family of subsets containing x,

$$\mathscr{F}_x := \{A \in \mathscr{P}(X) \mid x \in A\},$$

is called a *principal* ultrafilter, and provides an example of a σ-altrafilter. The corresponding measure, $\chi_{\mathscr{F}}$, coincides with μ_E for the singleton set $E = \{x\}$.

Exercise 84 *Show that* $\mathscr{F}_x = \mathscr{F}_y$ *if and only if* $x = y$*.*

5.2.3.8

The correspondence $x \longmapsto \mathscr{F}_x$ embeds set X onto a discrete and dense subset of βX.

5.2.3.9

One can prove that if there exists a *non*-principal σ-filter on a set X, then X is uncountable and the cardinality of X is *strongly inaccessible*. The latter means, that for any set Y of cardinality *smaller* than the cardinality of X, also $\mathscr{P}(Y)$ has smaller cardinality.

Any infite countable set has strongly inaccessible cardinality since any set of smaller cardinality is finite, and the set of all subsets of a finite set is finite.

5.2.3.10

Do uncountable sets with strongly inaccessible cardinality exist? One can prove that the hypothesis to the effect that 'there are no such sets,' is consistent with classical axioms of Set Theory.

5.2.3.11 Measurable cardinals

We say that a set X has *measurable cardinality* (or, that the cardinality of X is a *measurable cardinal*), if there exists a 0-1 measure on $(X, \mathscr{P}(X))$ such that the family of subsets of measure 1 is *not a principal* ultrafilter.

5.2.3.12 The question of existence

Even though it was proven already in the 1930-ies that one cannot prove the existence of a set whose cardinality is a measurable cardinal, nobody was so far able to prove that such 'large' sets *do not* exist.

There are mathematicians who believe that the reverse hypothesis, namely that such sets exist, is consistent with classical Set Theory.

5.3 Integral

5.3.1 Construction of integral

5.3.1.1 Posing the problem

The starting point is the following diagram

$$\begin{tikzcd}
\mathfrak{M} \arrow[r, "\mu"] \arrow[dr, hook, "\chi"'] & {[0,\infty]} \\
& {[0,\infty]^X}
\end{tikzcd}$$

where map χ sends a measurable subset A to the associated characteristic function $\chi_A \colon X \longrightarrow [0,\infty]$,

$$\mathfrak{M} \ni A \longmapsto \chi_A, \qquad \chi_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}.$$

The characteristic-function map, χ, is injective which allows us to identify $\mathfrak{M}$ with a subset of the set of $[0,\infty]$-valued functions on X.

5.3.1.2

The problem is to extend μ from the set of characteristic functions of measurable sets to the set of all $[0,\infty]$-valued functions on X, i.e., to find a map

$$\int \colon [0,\infty]^X \longrightarrow [0,\infty]$$

which makes the following diagram commute

$$\begin{tikzcd} \mathfrak{M} \arrow[r, "\mu"] \arrow[dr, hook, "\chi"'] & {[0,\infty]} \\ & {[0,\infty]^X} \arrow[u, dashed, "\int"'] \end{tikzcd}$$

and which is *monotonic*,

$$\int f \leq \int g \qquad \text{if} \qquad f \leq g \qquad \left(f, g \in [0,\infty]^X\right), \tag{5.7}$$

additive,

$$\int (f+g) = \int f + \int g \qquad \left(f, g \in [0,\infty]^X\right), \tag{5.8}$$

and *homogeneous* (of degree 1)

$$\int (cf) = c \int f \qquad \left(c \in [0,\infty]; f \in [0,\infty]^X\right). \tag{5.9}$$

5.3.1.3

Conditions (5.8)–(5.9) together mean that the extension is meant to be a morphism of $[0,\infty]$-semimodules.

5.3.1.4

By the universal property of free semimodules, there exists a unique extension of map $\mu\colon \mathfrak{M} \longrightarrow [0,\infty]$ to a $[0,\infty]$-linear map from $[0,\infty]\mathfrak{M}$, the free $[0,\infty]$-semimodule generated by set $\mathfrak{M}$, to $[0,\infty]$, making the diagram

$$\begin{tikzcd}
\mathfrak{M} \arrow[r, "\mu"] \arrow[d, hook, "\iota"'] \arrow[dr, hook, "\chi"] & {[0,\infty]} \\
{[0,\infty]\mathfrak{M}} \arrow[ur, "\tilde{\mu}"] & {[0,\infty]^X}
\end{tikzcd}$$

commute. Here $\iota\colon \mathfrak{M} \hookrightarrow [0,\infty]\mathfrak{M}$ denotes the canonical embedding of $\mathfrak{M}$ into $[0,\infty]\mathfrak{M}$, and the extension is given by the following formula

$$\tilde{\mu}\left(\sum_{A\in\mathfrak{M}} c_A A\right) := \sum_{A\in\mathfrak{M}} c_A \mu(A)$$

cf. (REFERENCE TO THE SECTION ON FREE SEMIMODULES STILL TO BE WRITTEN).

5.3.1.5

Since

$$\chi_{A\cap B} = \chi_A \chi_B \qquad (A, B \in \mathfrak{M}),$$

and χ_X is the constant function 1, map $\chi\colon \mathfrak{M} \longrightarrow [0,\infty]^X$ is a homomorphism of the monoid $(\mathfrak{M}, \cap)$ into the *multiplicative* monoid of the unital semialgebra of $[0,\infty]$-valued functions on X.

Operation $\cap$ induces $[0,\infty]$-bilinear multiplication on $[0,\infty]\mathfrak{M}$ making into the $[0,\infty]$-semialgebra of monoid $(\mathfrak{M}, \cap)$.

5.3.1.6

By the universal property of monoid semialgebras, the homomorphism of monoids

$$\chi\colon (\mathfrak{M}, \cap) \longrightarrow \left([0,\infty]^X\right)^\times$$

extends to a unique homomorphism of unital $[0,\infty]$-semialgebras with zero,

$$\tilde{\chi}\colon [0,\infty] \longrightarrow [0,\infty]^X, \tag{5.10}$$

and we obtain the following commuting diagram

$$\begin{tikzcd}
\mathfrak{M} \arrow[r, "\mu"] \arrow[d, "\iota"'] \arrow[dr, hook, "\chi"] & {[0,\infty]} \\
{[0,\infty]\mathfrak{M}} \arrow[ur, "\tilde{\mu}"] \arrow[r, "\tilde{\chi}"] & {[0,\infty]^X}
\end{tikzcd}$$

The extension is given by the following formula

$$\tilde{\chi}\left(\sum_{A\in\mathfrak{M}} c_A A\right) := \sum_{A\in\mathfrak{M}} c_A \chi_A$$

cf. (REFERENCE TO THE SECTION ON MONOID SEMIALGEBRAS STILL TO BE WRITTEN).

5.3.2 Simple functions

5.3.2.1

Homomorphism (5.10) is not surjective: let us denote its image by $\mathcal{S}_{\mathfrak{M}}^{[0,\infty]}$. It is a $[0,\infty]$-subalgebra of $[0,\infty]^X$ formed by functions $s\colon X \longrightarrow [0,\infty]$ which take only finitely many values (functions with this property will be called *simple*).

Exercise 85 *For functions f and g from X to $[0,\infty]$, let $h = f + g$. Show that if both $f(X)$ and $g(X)$ are finite, then*

$$|h(X)| \leq |f(X)| \cdot |g(X)|.$$

Exercise 86 *Show that the function*

$$s = \sum_{A \in \mathfrak{M}} c_A \chi_A$$

takes no more than 2^ν values where ν is the cardinality of the support of the family of coefficients $\{c_A\}_{A \in \mathfrak{M}}$,

$$\nu = |\{A \in \mathfrak{M} \mid c_A \neq 0\}|.$$

5.3.2.2 Canonical representation of a simple function

The last exercise implies that $\mathcal{S}_{\mathfrak{M}}^{[0,\infty]}$ consists of simple measurable functions $X \longrightarrow [0,\infty]$. Every measurable function $f \in [0,\infty]^X$ is in $\mathcal{S}_{\mathfrak{M}}^{[0,\infty]}$.

Indeed, such a function can be represented as

$$f = \sum_{a \in f(X)} a \chi_{f^{-1}(a)}. \tag{5.11}$$

We shall refer to (5.11) as the *canonical representation* of a simple function f.

5.3.2.3

Note that $\mathcal{S}_{\mathfrak{M}}^{[0,\infty]}$ is the subsemialgebra of $[0,\infty]^X$ which is generated by the image of $\mathfrak{M}$ in $[0,\infty]^X$.

5.3.2.4

Let us represent $\tilde{\chi}$ as the homomorphism of $[0,\infty]$-semialgebra $[0,\infty]\mathfrak{M}$ *onto* $\mathcal{S}_{\mathfrak{M}}^{[0,\infty]}$ followed by the inclusion map

$$\begin{tikzcd} {[0,\infty]\mathfrak{M}} \arrow[rr, "\tilde{\chi}"] \arrow[dr, two heads, "\pi"'] & & {[0,\infty]^X} \\ & \mathcal{S}_{\mathfrak{M}}^{[0,\infty]} \arrow[ur, hook, "j"'] & \end{tikzcd}$$

Lemma 5.3.1 *One has*

$$\sum_{A\in\mathfrak{M}} b_A\mu(A) = \sum_{A\in\mathfrak{M}} c_A\mu(A) \qquad \textit{if and only if} \qquad \sum_{A\in\mathfrak{M}} b_A\chi_A = \sum_{A\in\mathfrak{M}} c_A\chi_A.$$

5.3.2.5

In equivalent formulation, Lemma 5.3.1 asserts that that, for elements $\beta,\gamma \in [0,\infty]\mathfrak{M}$, one has

$$\tilde{\mu}(\beta) = \tilde{\mu}(\gamma) \qquad \text{if and only if} \qquad \tilde{\chi}(\beta) = \tilde{\chi}(\gamma).$$

It follows that $\tilde{\mu}$ factors through $\mathcal{S}_{\mathfrak{M}}^{[0,\infty]}$ producing a map $\mathcal{S}_{\mathfrak{M}}^{[0,\infty]} \longrightarrow [0,\infty]$ which will be denoted $\int$. The latter is automatically a homomorphism of $[0,\infty]$-semimodules with zero as the following exercise demonstrates.

Exercise 87 *Consider the composite of two maps between semimodules over a semiring R*

$$A \xrightarrow{g} \mathrel{\mkern-14mu}\rightarrow B \xrightarrow{f} C$$

Show that, if g is R-linear and surjective, then f is R-linear if and only if $f\circ g$ is R-linear.

5.3.3 The final step: extending $\int$ from $\mathcal{S}_{\mathfrak{M}}^{[0,\infty]}$ to $[0,\infty]^X$

5.3.3.1

Let us take a look at the commutative diagram

$$\begin{tikzcd}
& \mathfrak{M} \arrow[r, "\mu"] \arrow[dl, "\iota"'] \arrow[drr, hook, "\chi"] & {[0,\infty]} & \\
{[0,\infty]\mathfrak{M}} \arrow[urr, "\tilde{\mu}"] \arrow[rrr, "\tilde{\chi}"] \arrow[drr, two heads, "\pi"'] & & & {[0,\infty]^X} \\
& & \mathcal{S}_{\mathfrak{M}}^{[0,\infty]} \arrow[uu, dashed, "\int"] \arrow[ur, hook, "j"'] &
\end{tikzcd}$$

We have partially solved the original problem by extending μ to a subsemialgebra of $[0,\infty]^X$ which is the smallest subsemialgebra of $[0,\infty]$ which contains the image of $\mathfrak{M}$ under the characteristic map.

The obtained extension by design satisfies all three properties (5.8)–(5.9).

5.3.3.2 Lower and upper integrals

For any $f \in [0,\infty]^X$, we produce two numbers

$$\int_* f := \sup\left\{s \in \mathcal{S}_{\mathfrak{M}}^{[0,\infty]} \mid s \le f\right\} \tag{5.12}$$

and

$$\int^* f := \inf\left\{s \in \mathcal{S}_{\mathfrak{M}}^{[0,\infty]} \mid f \le s\right\}. \tag{5.13}$$

We will call (5.12) the *lower integral* of f and (5.13) the *upper integral* of f.

Exercise 88 *Show that*

$$\int_* s = \int s = \int^* s \qquad \left(s \in \mathcal{S}_{\mathfrak{M}}^{[0,\infty]}\right).$$

5.3.3.3

It follows that both

$$f \longmapsto \int_* f \qquad \left(f \in [0,\infty]^X\right),$$

and

$$f \longmapsto \int_* f \qquad \left(f \in [0,\infty]^X\right),$$

provide extensions of map $\int$ to $[0,\infty]^X$.

Exercise 89 *Show that*

$$\int_* f \ \le \ \int^* f \qquad \left(f \in [0,\infty]^X\right).$$

Exercise 90 *Show that both* $\int_*$ *and* $\int^*$ *satisfy property* (5.7).

Exercise 91 *Show that both* $\int_*$ *and* $\int^*$ *satisfy property* (5.9).

Exercise 92 *Show that* $\int_*$ *is* ***superadditive***

$$\int_* f + \int_* g \;\leq\; \int_* (f+g) \qquad \left(f, g \in [0,\infty]^X\right).$$

Exercise 93 *Show that* $\int^*$ *is* ***subadditive***

$$\int^* (f+g) \;\leq\; \int^* f + \int^* g \qquad \left(f, g \in [0,\infty]^X\right).$$

5.3.3.4

Lower and upper integral solve our original task only partially: the latter is subadditive while the former is superadditive. This immediately leads to the conclusion to the effect that the subset of $[0,\infty]^X$ on which $\int_*$ and $\int^*$ coincide is a natural domain for integral.

5.3.4 Integrable functions

5.3.4.1

A function $f \in [0,\infty]^X$ is said to be *integrable* (*with respect to measure* μ), if the values of the lower and the upper integral of f coincide:

$$\int_* f = \int^* f. \tag{5.14}$$

The common value (5.14) will be denoted

$$\int_X f d\mu \qquad \text{or} \qquad \int_X f(x) d\mu(x).$$

This notation pays tribute to the traditional notation for the integral of a function, introduced in 17th century.

Let us denote the set of μ-integrable functions by $\mathcal{I}_\mu^{[0,\infty]}$:

$$\mathcal{I}_\mu^{[0,\infty]} := \left\{ f \in [0,\infty]^X \text{ such that } \int_* f = \int^* f \right\}.$$

Proposition 5.3.2 *The set of μ-integrable functions is a $[0,\infty]$-subsemimodule of $[0,\infty]^X$ and integral is additive on it.*

Proof. The following triple inequality

$$\int_* f + \int_* g \ \leq\ \int_* (f+g) \leq \int^* (f+g)\ \leq\ \int^* f + \int^* g \qquad \left(f, g \in [0,\infty]^X\right),$$

combined with the equality of the leftmost and the rightmost terms for $f, g \in \mathcal{I}_\mu^{[0,\infty]}$, shows that $f+g$ is integrable and that the integral of the sum is the sum of the integrals.

Integrability of cf for $c \in [0,\infty]$ and $f \in \mathcal{I}_\mu^{[0,\infty]}$ is an immediate consequence of homogeneity of $\int_*$ and $\int^*$. □

5.3.4.2 Comment

Nowhere did we use the fact that μ is σ-additive. This will be used to show that every emasurable function $f\colon X \longrightarrow [0,\infty]$ is integrable. We used only finite additivity of measure μ, and this only once—in the proof of Lemma 5.3.1.

5.3.4.3

We conclude the construction of integral by providing the final form of the diagram that was used to construct it.

$$
\begin{tikzcd}
 & \mathfrak{M} \arrow[rr, "\mu"] \arrow[dl, "\iota"'] \arrow[drrr, "\chi"] & & {[0,\infty]} & \\
{[0,\infty]}\mathfrak{M} \arrow[urrr, "\tilde{\mu}"] \arrow[rrrr, "\tilde{\chi}"'] \arrow[drr, "\pi"'] & & & & \mathcal{I}_{\mu}^{[0,\infty]} \arrow[ul, "\int"'] \\
 & & \mathcal{S}_{\mathfrak{M}}^{[0,\infty]} \arrow[uur, "\int"] \arrow[urr, "j"'] & &
\end{tikzcd}
$$

All triangles in this diagram commute.

5.4 Properties of integral

5.4.1 Observations about $\int : \mathcal{S}_{\mathfrak{M}}^{[0,\infty]} \longrightarrow [0,\infty]$

5.4.1.1

5.4.2 Properties of lower and upper integrals

5.4.2.1 Lower and upper 'measure'

The characteristic-function map, χ, embeds the whole $\mathscr{P}(X)$ into $[0,\infty]^X$, not just $\mathfrak{M}$. We can now extend μ to any subset of X by using either the lower or upper integrals,

$$\mu_*(E) := \int_* \chi_E \qquad (E \subseteq X),$$

and

$$\mu^*(E) := \int^* \chi_E \qquad (E \subseteq X).$$

Exercise 94 *Show that, for any $E \subseteq X$, one has*

$$\mu_*(E) = \sup\{\mu(A) \mid A \in \mathfrak{M} \text{ and } A \subseteq E\}$$

and

$$\mu^*(E) = \inf\{\mu(B) \mid B \in \mathfrak{M} \text{ and } E \subseteq B\}.$$

5.4.2.2

For any function $f\colon X \longrightarrow [0,\infty]$ and any measurable subset $A \subseteq f^{-1}(\infty)$, one has the following two obvious inequalities

$$\infty\chi_A \leq f \qquad \text{and} \qquad \infty\mu(A) = \int (\infty\chi_A) \leq \int_* f.$$

Noting that $\infty a < \infty$ implies $a = 0$, we deduce that $\mu(A) = 0$ for any measurable subset $A \subseteq f^{-1}(\infty)$. In view of the definition of μ_*, we therefore make the following observation

$$\text{if} \quad \int_* f < \infty, \quad \text{then} \quad \mu_*\left(f^{-1}(\infty)\right) = 0.$$

5.4.2.3 Disjoint additivity

Lower and upper integrals are usually not additive on $[0,\infty]^X$ unless $\mathfrak{M} = \mathscr{P}(X)$. They enjoy, however, a restricted notion of additivity which closely reflects additivity properties of the measure itself.

5.4.2.4 Measurably disjoint functions

We shall say that functions $f, g \in [0,\infty]^X$ are $\mathfrak{M}$-*disjoint* or, when the σ-algebra is clear from the context, *measurably disjoint*, if there exists $A \in \mathfrak{M}$ such that

$$f_{|A} = 0 \qquad \text{and} \qquad g_{|A^c} = 0. \tag{5.15}$$

Lemma 5.4.1 *For any pair of measurably disjoint functions $f, g \in [0,\infty]^X$, one has*

$$\int_* (f+g) = \int_* f + \int_* g,$$

and similarly for $\int^*$.

Proof. Conditions (5.15) are equivalent to

$$(f+g)\chi_A = g \qquad \text{and} \qquad (f+g)\chi_{A^c} = f.$$

For any $s \in \left\{ s \in \mathcal{S}_{\mathfrak{M}}^{[0,\infty]} \mid s \leq f+g \right\}$ one has

$$s\chi_A \leq (f+g)\chi_A = g \qquad \text{and} \qquad s\chi_{A^c} \leq (f+g)\chi_{A^c} = f$$

and both $s\chi_A$ and $s\chi_{A^c}$ belong to $\mathcal{S}_{\mathfrak{M}}^{[0,\infty]}$. Hence, by using additivity of $\int$ on $\mathcal{S}_{\mathfrak{M}}^{[0,\infty]}$, we obtain

$$\int s = \int s(\chi_{A^c} + \chi_A) = \int s\chi_{A^c} + \int s\chi_A \leq \int_* f + \int_* g. \tag{5.16}$$

In particular, the supremum of the left-hand-side of (5.16) over s does not exceed the right-hand-side of (5.16). This shows that

$$\int_* (f+g) \leq \int_* f + \int_* g.$$

The reverse inequality holds without any hypotheses on f and g, cf. Exercise 92. $\square$

Exercise 95 *Prove the assertion of Lemma 5.4.1 for upper integral.*