

Nonlinear valuation and XVA under credit risk, collateral margins and Funding Costs



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Prof. Damiano Brigo
Chair, Mathematical Finance and Stochastic Analysis Groups
Dept. of Mathematics
<http://www.damianobrigo.it>

Univ. Catholique de Louvain

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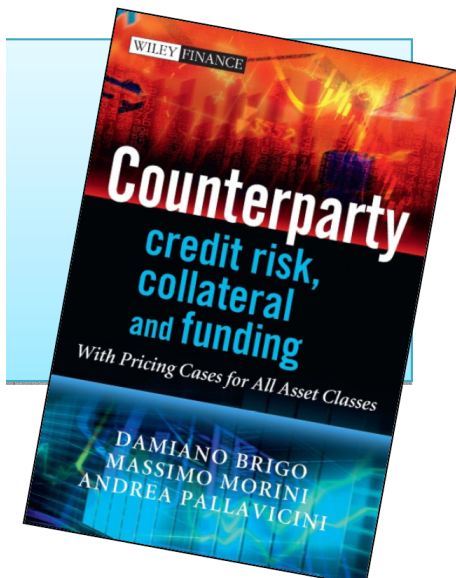
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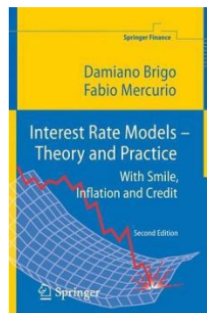
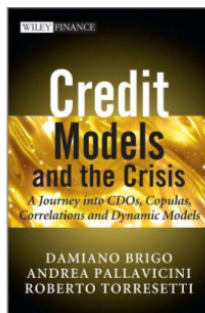
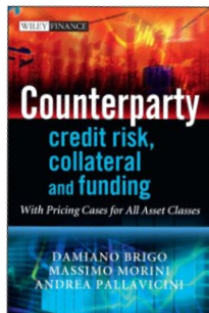
- CVA and FVA Desks: Best Practice

8 References

This course is mostly based on the book:



Check also



PART I. OPTION PRICING AND DERIVATIVES MARKETS

In this quick introductory part we introduce the Black Scholes and Merton result, their precursors (Bachelier, DeFinetti...) and the refinements of their theory (Harrison, Kreps, Pliska....) into no-arbitrage valuation, pointing out its significance, successes and failures.

We also look at the derivatives markets and their significance

This is all well known but sets the stage for the developments we will face later, leading to nonlinear valuation problems pushing the boundaries of the no-arbitrage framework.

The Black Scholes and Merton Analysis

We will follow these steps:

- Arbitrage as self-financing trading strategy with zero initial cost attaining a positive payout at maturity.
- Portfolio replication theory plus Ito's formula to derive the Black and Scholes PDE for the option price under certain assumptions on the dynamics of the underlying stock price.
- The Feynman-Kac theorem to interpret the solution of the Black and Scholes PDE as an expected value of a function of the stock price with a modified dynamics.
- The Girsanov theorem to interpret the modified dynamics of the stock price as a dynamics under a different (martingale) probability measure.
- No-arbitrage theorem (Harrison, Kreps and Pliska): There is no arbitrage opportunity if and only if there exists a martingale measure.

Description of the economy

We consider:

- A probability space with a r.c. filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t : 0 \leq t \leq T), P)$
- (& assume $\mathcal{F}_T = \mathcal{F}$). In the given economy, two securities are traded continuously from time 0 until time T . The first one (a bond) is riskless and its (deterministic) price B_t evolves according to

$$dB_t = B_t r dt, \quad B_0 = 1, \quad (1)$$

which is equivalent to

$$B_t = e^{rt}, \quad (2)$$

where r is a nonnegative number. To state it differently, the short term interest rate is assumed to be constant and equal to r through time.

Description of the economy

- As for the second one, given the (\mathcal{F}_t, P) -Wiener process W_t , consider the following stochastic differential equation

$$dS_t = S_t[\mu dt + \sigma dW_t], \quad 0 \leq t \leq T, \quad (3)$$

with initial condition $S_0 > 0$, and where μ and σ are positive constants. Equation (3) has a unique (strong) solution which is given by

$$S_t = S_0 \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\}, \quad 0 \leq t \leq T. \quad (4)$$

The risky asset, The B e S Assumptions, and Contingent Claims

$$dB_t = B_t r dt, \quad B_0 = 1,$$

$$dS_t = S_t [\mu dt + \sigma dW_t], \quad 0 \leq t \leq T,$$

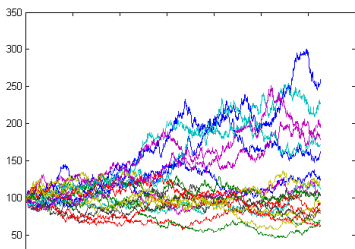
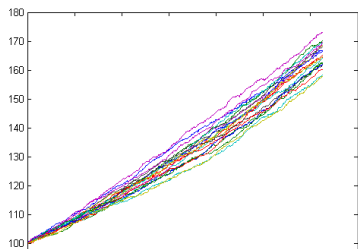
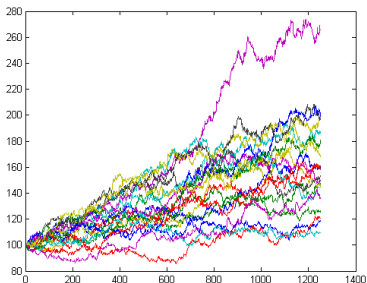
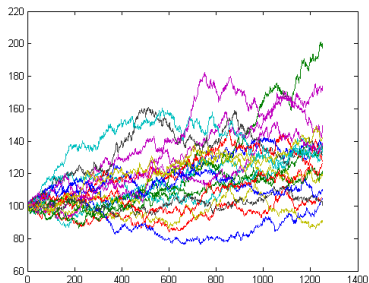
The second asset (a stock) is risky and its price is described by the process S_t . Furthermore, it is assumed that

- (i) there are no transaction costs in trading the stock;
- (ii) the stock pays no dividends or other distributions;
- (iii) shares are infinitely divisible;
- (iv) short selling is allowed without any restriction or penalty.

We refer to these assumptions as to Black and Scholes' *ideal conditions*.

Example of risky asset dynamics over 5 years:

$$S_0 = 100, \quad (\mu, \sigma) = (5\%, 10\%), (10, 10), (10, 1), (1, 20)$$



Contingent claims, pricing problem

A **contingent claim** Y for the maturity T is any square-integrable and positive random variable in $(\Omega, \mathcal{F}_T, P)$, which is in particular \mathcal{F}_T -measurable.

In this introductory part we limit ourselves to *simple* contingent claims, i.e. claims of the form $Y = f(S_T)$.

The idea behind a claim is that it represents an amount that will be paid at maturity to the holder of the contract.

The **Pricing Problem** is giving a fair price to such a contract.

Trading strategies, Value process, gain process I

A **trading strategy** is a pair of stochastic processes $\phi = (\phi^B, \phi^S)$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t : 0 \leq t \leq T), P)$ that are locally bounded and predictable (and, therefore, \mathcal{F}_t -adapted). The pair (ϕ_t^B, ϕ_t^S) represents respectively amounts of bond and stock to be held at time t .

Predictability is assumed to reduce the investor freedom at jump times and assumes that the value ϕ_t will be known immediately before t . However, in our Black Scholes setting, where the paths of the assets are continuous, this issue is not relevant.

The **value process** is the process V describing the value of the portfolio constructed by following the strategy ϕ ,

$$V_t(\phi) = \phi_t^B B_t + \phi_t^S S_t .$$

We will call later $H_t = \phi_t^S S_t$ the risky asset part of the strategy and $F_t = \phi_t^B B_t$ the cash part.

Trading strategies, Value process, gain process II

The **gain process** is defined as

$$G_t(\phi) = \int_0^t \phi_u^B dB_u + \int_0^t \phi_u^S dS_u .$$

and represents the income one obtains thanks to price movements in bond and stock when following the trading strategy ϕ .

The strategy is **self-financing** if $V_t(\phi) \geq 0$ and $V_t(\phi) = V_0(\phi) + G_t(\phi)$, or

$$\phi_t^B B_t + \phi_t^S S_t - (\phi_0^B B_0 + \phi_0^S S_0) = G_t(\phi) ,$$

or, in differential terms, $d V_t(\phi) = d G_t(\phi)$, i.e.

$$d(\phi_t^B B_t + \phi_t^S S_t) = \phi_t^B dB_t + \phi_t^S dS_t . \quad (5)$$

Self-financing strategies, arbitrage

$$d(\phi_t^B B_t + \phi_t^S S_t) = \phi_t^B dB_t + \phi_t^S dS_t .$$

Intuitively, this means that the changes in value of the portfolio described by the strategy ϕ are only due to gains/losses coming from price movements, i.e. to changes in the prices B and S , without any cash inflow and outflow.

An important use of self-financing strategies is in defining arbitrage. An **arbitrage opportunity** is a self-financing strategy ϕ such that (recall $V_t(\phi) \geq 0$)

$$\phi_0^B B_0 + \phi_0^S S_0 = 0 , \quad \mathbb{P}(\phi_T^B B_T + \phi_T^S S_T > 0) > 0 .$$

Basically, an arbitrage opportunity is a strategy which creates a positive cash inflow from nothing with positive probability and never creates a loss or negative flow. It is sometimes called a **free lunch**.

The market **arbitrage-free** if there are no arbitrage opportunities. 



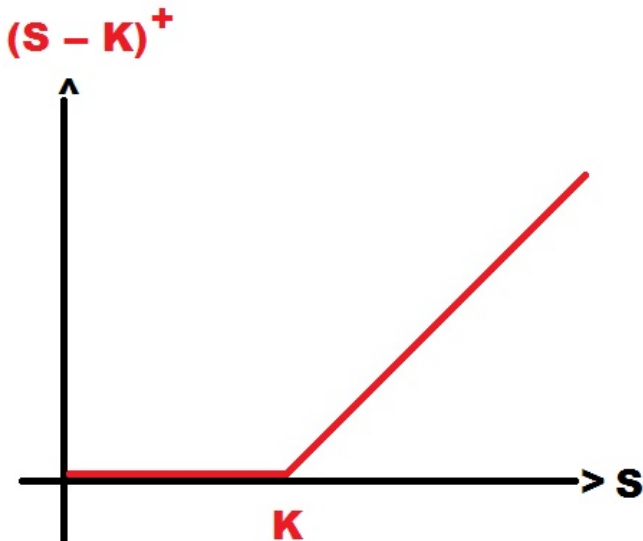
Self-financing strategies, attainable claims, price

Self-financing strategies are important also because they allow us to define **attainable contingent claims**. A contingent claim Y is attainable if \exists self-financing ϕ such that $V_T(\phi) = Y$.

We say that ϕ **generates** Y , & $V_t(\phi)$ is the **price** at time t for Y .

We thus have a **first notion of price** as the value of a self-financing trading strategy attaining (or sometimes we'll say “replicating”) the claim.

Example of Claim: European Call Option



Example of Claim: European Call Option I

Suppose we have to price a simple claim $Y = f(S_T)$ at time t .

We focus on the case of a European call option: Let K be its strike price and T its maturity. The option payoff (to a long position) is represented by $Y = (S_T - K)^+ = \max(S_T - K, 0)$.

This is a contract which at maturity-time T pays nothing if the risky-asset price S_T is smaller than the strike price K , whereas it pays the difference between the two prices in the other case.

An investor who expects the risky-asset value to increase considerably can speculate by buying a call option.

Example of Claim: European Call Option II

Example of use of a call option: Suppose now we are at time 0 and we plan to buy one share (unit) of a certain stock at time T . We wish to pay this stock the same price $K = S_0$ it has now, rather than the price it will have at time T , which could be much higher. What one can do in this situation is to buy a call option on the stock with maturity time T and strike price S_0 .

He then buys the stock at time T paying S_T and receives $(S_T - S_0)^+$ from the option payoff. Clearly, the total amount he pays in T is then $S_T - (S_T - S_0)^+$ which equals S_T if $S_T \leq S_0$ and equals S_0 if $S_T \geq S_0$. Therefore, an European call option can be seen as a contract which locks the stock price at a desired value to be paid at maturity time T . This *locking* has of course a price, which we wish to determine.

We now sketch a derivation of the Black Scholes PDE for the “attainable-claim” price of an option. This is an informal derivation.

The Black and Scholes PDE

Let $V_t = V(t, S_t)$ be the candidate claim (option) value at time t .

Assume the function $V(t, S_t)$ of time t and of the stock price S_t to have regularity $V \in C^{1,2}([0, T] \times \mathbb{R}^+)$.

Apply Ito's Lemma to V so as to obtain

$$dV(t, S_t) = \left(\frac{\partial V}{\partial t}(t, S_t) + \frac{\partial V}{\partial S}(t, S_t)\mu S_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2}(t, S_t)\sigma^2 S_t^2 \right) dt + \frac{\partial V}{\partial S}(t, S_t)\sigma S_t dW_t. \quad (6)$$

Set, for each $0 \leq t \leq T$,

$$\phi_t^S = \frac{\partial V}{\partial S}(t, S_t), \quad \phi_t^B = (V_t - \phi_t^S S_t)/B_t. \quad (7)$$

By construction, the value of this strategy at time t is V itself, since clearly $V(t, S_t) = \phi_t^B B_t + \phi_t^S S_t$.

The Black and Scholes PDE

Now assume ϕ to be self-financing. Since ϕ is self-financing

$$\begin{aligned} dV_t &= \phi_t^B dB_t + \phi_t^S dS_t \\ &= \left[V(t, S_t) - \frac{\partial V}{\partial S}(t, S_t) S_t \right] rdt + \frac{\partial V}{\partial S}(t, S_t) S_t (\mu dt + \sigma dW_t). \end{aligned} \quad (8)$$

Then by equating (6) and (8) (ITO + SELF FINANCING), we obtain that V_t satisfies

$$\frac{\partial V}{\partial t}(t, S_t) + \frac{\partial V}{\partial S}(t, S_t) r S_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2}(t, S_t) \sigma^2 S_t^2 = rV(t, S_t), \quad (9)$$

which is the celebrated Black and Scholes partial differential equation with terminal condition $V_T = (S_T - K)^+$.

Black and Scholes' famous formula

The strategy (ϕ^B, ϕ^S) has final value equal to the claim Y we wish to price (terminal condition of the PDE), and during its life the strategy does not involve cash inflows or outflows (self-financing condition). As a consequence, its initial value V_t at time t must be equal to the unique claim price to avoid arbitrage opportunities.

The solution of the above equation is given by

$$V_{BS}(t) = V_{BS}(t, S_t, K, T, \sigma, r) := S_t \Phi(d_1(t)) - Ke^{-r(T-t)} \Phi(d_2(t)), \quad (10)$$

where

$$d_1(t) := \frac{\ln(S_t/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}},$$

$$d_2(t) := d_1(t) - \sigma\sqrt{T - t},$$

and $\Phi(\cdot)$ denotes the cumulative standard normal distribution function.

Black and Scholes' famous formula

Expression (10) is the celebrated Black and Scholes option pricing formula which provides the unique no-arbitrage price for the given European call option.

Notice that the coefficient μ does not appear in (10), indicating that investors, though having different risk preferences or predictions about the future stock price behaviour, must yet agree on this unique option price.

MORE ON THE SIGNIFICANCE OF THIS LATER.

Numerical example

Suppose the current stock value is $S_0 = 100$.

Suppose the risk free interest rate is $r = 2\% = 0.02$.

Suppose that the strike $K = 100$ (at the money option).

Assume the volatility $\sigma = 0.2 = 20\%$.

Take a maturity of $T = 5y$. CALL PRICE IS $V_{BS}(0) = 22.02$.

For example, in Matlab this is obtained through commands

```
S0=100; sig=0.2; r=0.02; K=100; T=5;
d1 = (r + 0.5*sig*sig)*T/(sig*sqrt(T));
d2 = (r - 0.5*sig*sig)*T/(sig*sqrt(T));
V0 = S0*normcdf(d1)-K*exp(-r*T)*normcdf(d2);
```

The same calculation with lower volatility $\sigma = 0.05 = 5\%$ would give

$$V_{BS}(0)|_{\sigma=0.05} = 10.5943, \quad V_{BS}(0)|_{\sigma=0.0001} = 9.52.$$

The last value is very close to the intrinsic value $S_0 - Ke^{-rT}$.

Numerical example

- Acme today is worth $S_0 = 100$.
- The more the value of acme goes up in 5 years, the more we gain as $S_{5y} - S_0$ grows. In a scenario where $S_{5y} = 200$, we gain 100.
- If however Acme goes down instead, $S_{5y} - S_0$ goes negative but the option $(S_{5y} - S_0)^+$ caps it at zero and we lose nothing. For example, in a scenario where Acme goes down to 60, we get $(60 - 100)^+ = (-40)^+ = 0$ ie we lose nothing
- With the original data, entering the gamble costs initially 22 USD out of 100 of stock notional. It is expensive. On the other hand, it is a gamble where we can only win and in principle have scenarios with unlimited profit.
- You will notice that:

$$\uparrow \sigma \Rightarrow V_{CallBS} \uparrow, \quad \uparrow S_0 \Rightarrow V_{CallBS} \uparrow, \quad \downarrow K \Rightarrow V_{CallBS} \uparrow \dots$$

Another numerical example

Take one more example where now the strike K is at the money forward and volatility very low, namely

$S_0=100$; $\sigma=0.0001$; $r=0.02$; $T=5$; $K=S_0 \exp(r \cdot T)$;

Then

$$V_{BS}(0) = 0 \approx S_0 - Ke^{-rT} = S_0 - S_0 = 0.$$

Verifying the Self financing condition

Going back to the general Black Scholes result, we then prove that the strategy

$$\phi_t^S = \frac{\partial V_{BS}}{\partial S}(t, S_t), \quad \phi_t^B = (V_{BS}(t) - \phi_t^S S_t)/B_t$$

$$\left(V_{BS}(t) = V_{BS}(t, S_t, K, T, \sigma, r) := S_t \Phi(d_1(t)) - Ke^{-r(T-t)} \Phi(d_2(t)) \right)$$

is indeed self-financing. By Ito's Lemma, in fact, we have

$$dV_{BS}(t) = \frac{\partial}{\partial t} V_{BS}(t) dt + \frac{\partial}{\partial S} V_{BS}(t) dS_t + \frac{1}{2} \frac{\partial^2}{\partial S^2} V_{BS}(t) \sigma^2 S_t^2 dt. \quad (11)$$

Verifying the Self financing condition

Since straightforward differentiation of V_{BS} expression leads to

$$\frac{\partial}{\partial t} V_{BS}(t) = -\frac{S_t \Phi'(d_1(t)) \sigma}{2\sqrt{T-t}} - rX e^{-r(T-t)} \Phi(d_2(t)),$$

$$\frac{\partial^2}{\partial S^2} V_{BS}(t) = \frac{\Phi'(d_1(t))}{S_t \sigma \sqrt{T-t}},$$

where $\Phi'(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, then it is enough to substitute ϕ^S and ϕ^B expressions given above to obtain from (11) that

$dV_{BS}(t) = \phi_t^S dS_t + \phi_t^B dB_t$, which is the self-financing condition in differential form.

The Feynman Kac theorem for Risk Neutral Valuation

Different interpretation: the Feynman-Kac Theorem allows to interpret the solution of a parabolic PDE such as the Black and Scholes PDE in terms of expected values of a diffusion process. In general, given suitable regularity and integrability conditions, the solution of the PDE

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)b(x) + \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(t, x)\sigma^2(x) = rV(t, x), \quad V(T, x) = f(x), \quad (12)$$

can be expressed as

$$V(t, x) = e^{-r(T-t)} \mathbb{E}_{t,x}^Q \{ f(X_T) | \mathcal{F}_t \} \quad (13)$$

where the diffusion process X has dynamics starting from x at time t

$$dX_s = b(X_s)ds + \sigma(X_s)dW_s^Q, \quad s \geq t, \quad X_t = x \quad (14)$$

under the probability measure \mathbb{Q} under which the expectation $\mathbb{E}_{t,x}^Q \{ \cdot \}$ is taken. The process W^Q is a standard Brownian motion under \mathbb{Q} .

Risk Neutral interpretation of the Black and Scholes formula

By applying this theorem to the Black and Scholes setup, with $b(x) = rx$, $\sigma(x) = \sigma x$ (so that the general PDE of the theorem coincides with the BeS PDE) we obtain:

The unique no-arbitrage price of the integrable contingent claim $Y = (S_T - K)^+$ (European call option) at time t , $0 \leq t \leq T$, is given by

$$V_{BS}(t) = \mathbb{E}^Q \left(e^{-r(T-t)} Y | \mathcal{F}_t \right). \quad (15)$$

The expectation is taken with respect to the so-called martingale measure \mathbb{Q} , i.e. a probability measure $\mathbb{Q} \sim \mathbb{P}$ under which the risky-asset price $S_t/B_t = e^{-rt} S_t$ measured with respect to the risk-free asset price B_t is a martingale, which is equivalent to S having drift rate r under \mathbb{Q} :

$$dS_t = S_t[r dt + \sigma dW_t^Q], \quad 0 \leq t \leq T, \quad (16)$$

An expression for \mathbb{Q} : Girsanov's theorem

We give an informal account. Consider on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ a stochastic differential equation

$$dX_t = b(X_t) dt + v(X_t) dW_t, \quad X_0.$$

Under the relevant technical conditions, define the measure \mathbb{Q} by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \exp \left\{ -\frac{1}{2} \int_0^t \left(\frac{b^{\mathbb{Q}}(X_s) - b(X_s)}{v(X_s)} \right)^2 ds + \int_0^t \frac{b^{\mathbb{Q}}(X_s) - b(X_s)}{v(X_s)} dW_s \right\}$$

Then under $\mathbb{Q} \sim \mathbb{P}$ we have that

$$dW_t^{\mathbb{Q}} = -(b^{\mathbb{Q}}(X_t) - b(X_t))/v(X_t) dt + dW_t$$

is a Brownian motion and

$$dX_t = b^{\mathbb{Q}}(X_t) dt + v(X_t) dW_t^{\mathbb{Q}}, \quad X_0.$$

The Risk Neutral measure via Girsanov's theorem

We apply Girsanov's theorem to move from

$$d S_t = \mu S_t dt + \sigma S_t dW_t$$

to

$$d S_t = r S_t dt + \sigma S_t dW_t^Q$$

We obtain

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left\{ -\frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 T - \frac{\mu - r}{\sigma} W_T \right\}. \quad (17)$$

No arbitrage: Main steps followed so far I

- 1 Self Financing Condition (Portfolio replication theory) plus Ito's formula to derive the Black and Scholes PDE for any attainable payout claim in S_T :

$$d S_t = \mu S_t dt + \sigma S_t dW_t$$

$$\frac{\partial V}{\partial t}(t, S_t) + \frac{\partial V}{\partial S}(t, S_t)rS_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2}(t, S_t)\sigma^2 S_t^2 = rV(t, S_t),$$

$$V_T = \phi(S_T)$$

- 2 If each such claim can be replicated/attained with a unique self financing strategy then there is a unique claim price equal to the initial cost of the strategy (and given by the PDE).

No arbitrage: Main steps followed so far II

- ③ The Feynman-Kac theorem to interpret the price solution of the Black and Scholes PDE as an expected value of a function of the stock price with modified dynamics

$$V(t, S_t) = \mathbb{E}^Q \{ e^{-r(T-t)} \phi(S_T) | \mathcal{F}_t \}$$

$$d S_t = r S_t dt + \sigma S_t dW_t^Q$$

- ④ The Girsanov theorem to interpret the modified dynamics of the stock price as a dynamics under a new (Risk neutral or martingale) probability measure \mathbb{Q} :

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left\{ -\frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 T - \frac{\mu - r}{\sigma} W_T \right\}.$$

No arbitrage: Main steps followed so far III

- 5 Hence the notion of attainable/replication claim price obtained from the PDE (self-financing condition & Ito's formula) coincides with a second notion of price: expectation of the claim payout under a risk neutral measure where the risky asset local mean grows at the risk free rate. This is a second way to express no-arbitrage via the condition that S/B is a martingale (more on martingales in a minute), ie a fair game. Hence no arbitrage will be related to the market for the underlying risky asset S to be a fair game.

Fundamental Theorems I

Pricing, no arbitrage, complete markets

The two approaches attainable claim PDE/ Risk neutral expectation are more generally related by the full theory of Harrison, Kreps and Pliska, and following extensions, and they are equivalent to the absence of arbitrage opportunities as defined earlier. Without specifying fully all the technical details, we report a high level summary:

Theorem. \exists a martingale measure $\mathbb{Q} \iff \nexists$ arbitrage opportunities.

Theorem. \exists a martingale measure $\mathbb{Q} \Rightarrow \exists!$ attainable claim price that can be computed as a \mathbb{Q} expectation of the discounted claim.

Fundamental Theorems II

Pricing, no arbitrage, complete markets

The fundamental result here is that existence of a martingale measure is equivalent to no arbitrage: if $\mathbb{Q} \ni$ there is no arbitrage opportunity, ie there is no self-financing ϕ producing positive wealth with positive probability with zero costs and without losses.

There is a second result related to the uniqueness (rather than existence) of the martingale measure. This is related to complete markets.

A market is **complete** if every contingent claim is attainable.

Theorem. The market is arbitrage free & complete $\iff \exists!$ martingale measure \mathbb{Q}

Fundamental Theorems III

Pricing, no arbitrage, complete markets

If the market is arbitrage free but not complete, the price of any attainable claim is still uniquely given, either as the value of the replicating strategy or as the risk neutral expectation under *any* equivalent martingale measure.

The Black Scholes market (B_t, S_t) we have seen above is arbitrage free and complete.

In reality markets are never complete, and we will explore several sources of incompleteness.

Fundamental Theorems IV

Pricing, no arbitrage, complete markets

The above framework can be applied easily to markets with n diffusive underlying assets S_1, \dots, S_n , each similar to the Black Scholes equity process, and with a bank or cash account B_t . The definitions and results on arbitrage opportunities, attainable claims, price, martingale measure, market completeness extend to the n -dimensional case easily and also to non-simple claims that are path dependent or early exercise.

The idea behind the martingale approach

Why martingales?

A martingale is a stochastic process representing a fair game. Loosely speaking, the above proposition states that in order to price under uncertainty one must price in a world where the probability measure is such that the risky asset evolves as a fair game when expressed in units of the risk-free asset.

Hence in our case S_t/B_t must be a fair game, ie a martingale.

martingales: local mean =0

For regular diffusion processes X_t martingale means "zero-drift", no up or down local direction: $dX_t = 0dt + \sigma(t, X_t)dW_t$.

Indeed, show that the drift of the SDE for $d(S_t/B_t)$ is zero under \mathbb{Q} .

The idea behind the martingale approach

Numeraire

When we consider S_t/B_t we may say that we are looking at S measured with respect to the numeraire B_t .

In general, as we shall see later on, it is possible to adopt any non-dividend paying asset price as numeraire, and price under the particular probability measure associated with that numeraire.

However, the canonical numeraire is the bank account B we have used now and the probability measure associated with the numeraire B is the risk neutral measure \mathbb{Q} .

The idea behind the martingale approach

No need to know the real expected return

We noticed earlier that the coefficient μ does not appear in (10), indicating that investors, though having different risk preferences or predictions about the future stock price behaviour, must yet agree on this unique option price.

This property can also be inferred from (16), since, under \mathbb{Q} , the drift rate of the stock price process equals the risk-free interest rate while the variance rate is unchanged. For this reason the pricing rule (15) is often referred to as **risk-neutral valuation**, and the measure \mathbb{Q} defines what is called **the risk-neutral world**.

Intuitively, in a risk-neutral world the expected rate of return on all securities is the risk-free interest rate, implying that investors do not require any risk premium for trading stocks.

Weak point of the derivation: Uniqueness of ϕ

The above derivation, however, is still not fully satisfactory, since we have implicitly assumed that (ϕ^B, ϕ^S) is the *unique* self-financing strategy replicating the claim with payoff $f(S_T)$. This uniqueness, anyway, can be obtained by applying the more general theory on complete markets, which is beyond the scope of this introduction.

Dynamic Hedging I

In the process of deriving the BS formula, we have also found a way to perfectly hedge the risk embedded in this contract.

Indeed look at the option pricing problem from the following point of view:

- You are the bank and you just sold a call option to the client.
- At the future time T you will have to pay $(S_T - K)^+$ to your client
- Your client pays you V_0 for the option now, at time 0
- Clearly, if the equity goes up a lot in the future, $(S_T - K)^+$ could be very large
- You wish to avoid any risks and decide to hedge away the risk in this contract you sold.
- How should you do that?

Dynamic Hedging II

The answer to this question is in our derivation above.

- You cash in V_0 from the client and use it to buy, at time 0,

$$\frac{\partial V_0}{\partial S_0} = \Phi(d_1(0)) =: \phi_0^S =: \Delta_0 \quad \text{stock and}$$

$$\phi_0^B = (V_0 - \Delta_0 S_0) / B_0 \quad \text{bank account / bond (cash).}$$

- You then implement the self-financing trading strategy, **rebalancing continuously** (hence *dynamic hedging*) your ϕ_t^S, ϕ_t^B amounts of S and B according to

$$\phi_t^S = \frac{\partial V_t}{\partial S_t} = \Phi(d_1(t)) =: \Delta_t \quad \text{stock and}$$

$$\phi_t^B = (V_t - \Delta_t S_t) / B_t \quad \text{bank account / bond (cash).}$$

Dynamic Hedging III

- Because the strategy is self-financing, this rebalancing can be financed thanks to price movements of B and S and you need not add any cash or assets from outside.
- At final maturity we know that the final value will be $V_T = (S_T - K)^+$ as we posed this as boundary condition in our pricing problem.
- Hence by following the above strategy, set up with the initial V_0 and with no subsequent cost, we end up with the payout $(S_T - K)^+$ at maturity.
- We can then deliver this payout to our client and face no risk.
- Basically, our self financing trading strategy in the underlying S , set up with the initial payment V_0 , completely replicated the claim we sold to our client.

Dynamic Hedging IV

- An obvious but often overlooked point is this: If we are perfectly hedged, all the money we received from the client (V_0) is spent to set up the hedge, and we as a bank make no gain.
- That's why in reality only partial hedges are often implemented, in an attempt not to erode all potential profit.

The above framework is called "**delta-hedging**".

Basically one holds an amount of risky asset equal to the sensitivity of the contract price to the risky asset itself (delta).

This strategy is possible only in markets where all risks are directly linked to tradable assets and viceversa (roughly: "complete markets").

Dynamic Hedging V

Metatheorem/folklore: A market is complete if there are as many assets as independent sources of randomness.

In reality markets are incomplete, as there are some risks that are covered by no direct assets, and there are more risks than assets.

This can be partly addressed by including a few derivatives themselves among the basic assets, but it is hard to keep the market complete

For example, in credit risk with intensity models, where the default time is $\tau = \Lambda^{-1}(\xi)$, and Λ is the cumulated instantaneous credit spread and ξ is the jump to default exponential variable, we have that ξ cannot be hedged unless we introduce a credit derivative depending on ξ itself in the pool of our basic assets. And even then the hedge remains partial. We cannot hedge recovery rates, correlations...

Dynamic Hedging VI

A further problem is that continuous rebalancing does not happen. Real hedging happens in discrete time and this will imply a hedging error with respect to the idealized case

In the end hedging is more an art than a science, and it involves many pragmatic choices and rules of thumbs. However, a sound understanding of the idealized case is crucial to appreciate the subtleties in real market applications.

The sensitivities (or greeks) I

When hedging derivatives, often sensitivities (or greeks) are used in practice.

A sensitivity is the partial derivative of the price or of another sensitivity with respect to one of the parameters. It tells us how much a small change of the parameter impacts a change in the price or sensitivity we are examining.

We have already met one of the most important sensitivities, delta.

$$\Delta(t) = \frac{\partial V(t)}{\partial S}$$

which, for a call option price under Black Scholes, is equal to $\Phi(d_1(t))$, as we have seen above. Delta measures how much the option price V changes when there is a small change in the underlying asset price S .

The sensitivities (or greeks) II

In general a large sensitivity with respect to a parameter means that the trade is quite sensitive to that parameter, and the trader may consider trades that reduce the sensitivity if she wishes to be more prudent with respect to that parameter. If the trader is more aggressive she may decide to trade to increase the sensitivity further.

Other sensitivities or greeks are: Time decay or Θ , sensitivity to time,

$$\Theta_t = \frac{\partial V(t)}{\partial t} = -\frac{\partial V(t)}{\partial(T-t)}$$

Gamma, the sensitivity of delta to the underlying:

$$\Gamma_t = \frac{\partial \Delta(t)}{\partial S} = \frac{\partial^2 V(t)}{\partial S^2}$$

The sensitivities (or greeks) III

At this point we may write an equation linking the three sensitivities just introduced. Recall Ito's formula we have seen earlier

$$dV(t, S_t) = \frac{\partial V}{\partial t}(t, S_t)dt + \frac{\partial V}{\partial S}(t, S_t)dS_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2}(t, S_t)\sigma^2 S_t^2 dt.$$

We can rewrite this as

$$dV(t, S_t) = \Theta_t dt + \frac{1}{2} \Gamma_t \sigma^2 S_t^2 dt + \Delta_t dS_t$$

On the other hand we have, from the self financing condition,

$$dV(t, S_t) = \Delta_t dS_t + [(V(t, S_t) - \Delta_t S_t)/B_t] dB_t$$

The sensitivities (or greeks) IV

(the quantity inside square brackets being previously called η_t). By matching the two expressions we have

$$[(V(t, S_t) - \Delta_t S_t)/B_t]dB_t = \Theta_t dt + \frac{1}{2}\Gamma_t \sigma^2 S_t^2 dt$$

or

$$[(V(t, S_t) - \Delta_t S_t)]r_t = \Theta_t dt + \frac{1}{2}\Gamma_t \sigma^2 S_t^2 dt$$

or

$$r_t V(t, S_t) = r_t \Delta_t S_t + \Theta_t dt + \frac{1}{2}\Gamma_t \sigma^2 S_t^2 dt$$

Back to definitions, Vega is the sensitivity to volatility, namely

$$\nu_t = \frac{\partial V(t)}{\partial \sigma}$$

The sensitivities (or greeks) V

ρ is the sensitivity to interest rates r , namely $\rho_t = \frac{\partial V(t)}{\partial r}$.

These greeks can be computed in closed form in Black Scholes for call and put options, for example. There are further higher order greeks Vanna, Charm, Vomma/volga, Veta, Yoghurt, Speed, Zomma, Color Ultima, Tutto... (sounds crazy I know... and one on this list is fake)

The higher the order of the greeks we use, the smoother we are assuming prices to be. For example, Speed = $\partial^3 V / \partial S^3$ requires the price V to be three times differentiable with respect to the underlying S . While this may hold in simple models like Black Scholes for specific payoffs, in general assuming excessive smoothness is not realistic, and therefore using high order greeks has to be done very carefully, especially when the greeks are computed with numerical methods.

What does it all mean

So far we have tried to follow a technical path, but it is time to appreciate the significance of what we have done so far.

We now ask ourselves: What are the implications of what we have calculated on the big picture?

Quantitative Finance deals in large part with Derivatives. So, following our derivation above, why are derivatives so important, so popular and, often, unpopular?

What does it all mean? Call option and Gambling

Assume we wish to enter into a gamble (call option) against a bank, where:

- If the future price of the ACME stock in 1y is larger than the value of ACME today, we receive from the bank the difference between the two prices (on a given notional).
- If the future price of the ACME stock in 1y is smaller or equal than the value of ACME today, nothing happens.

The bank will charge us for entering this wage, since we can only win or get into a draw, whereas the bank can only lose or get to a draw.

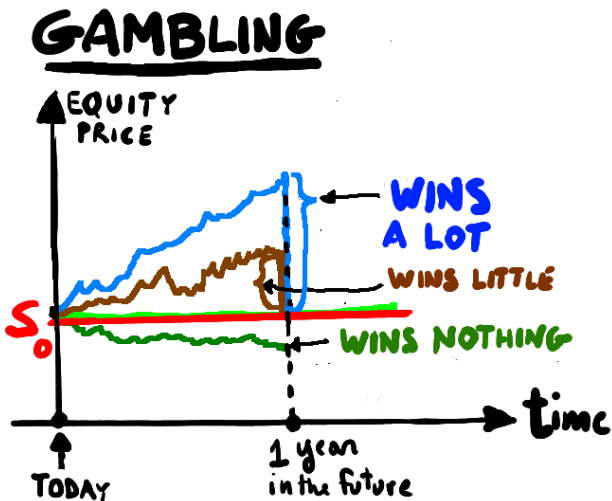


Figure: A one-year maturity Gamble on an equity stock. Call Option.

Call option and Gambling

We have an investor buying a call option on ACME with a 1y maturity.

The Bank's problem is finding the correct price of this option today. This price will be charged to the investor, who may also go to other banks.

This is an option pricing problem.

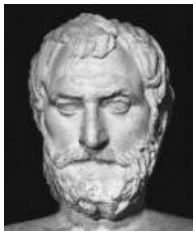
The market introduced options and more generally financial derivatives that may be much more complex than the above example. Such derivatives often work on different sectors: Foreign Exchange Rates, Interest Rates, Default Events, Meteorological events, Energy, etc.

Derivatives can be bought to protect or hedge some risk, but also for speculation or "gambling".

Options and Derivatives

Derivatives outstanding notional as of June 2011 (BIS) is estimated at **708 trillions USD** (US GDP 2011: 15 Trillions; World GDP: 79 Trillions)

708000 billions, 708,000,000,000,000, 7.08×10^{14} USD



How did it start? It has always been there. Around 580 B.C., Thales purchased options on the future use of olive presses and made a fortune when the olives crop was as abundant as he had predicted, and presses were in high demand. (Thales is also considered to be the father of the sciences and of western philosophy, as you know).

Options and Derivatives valuation: precursors



- **Louis Bachelier** (1870 – 1946) (First to introduce Brownian motion W_t in Finance, First in the modern study of Options);
- **Bruno de Finetti** (1906 – 1985) (Father of the subjective interpretation of probability). Shows betting quotients (claim prices?) avoid sure exploitation from gambling broker (market?) if and only if they satisfy axioms of a probability measure. See also Frank Ramsey (1903-1930).

Modern theory follows Nobel awarded **Black, Scholes and Merton** (and then Harrison and Kreps etc) on the correct pricing of options.

Black and Scholes: What does it mean?

We have derived the Black Scholes formula for a call option earlier. Let us recall the key points.

Let S_t be the equity price for ACME at time t .

For the value of the ACME stock S_t let us assume, as before, a SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad \text{or also}$$

$$\underbrace{\frac{dS_t}{S_t}}_{\substack{\text{relative change} \\ \text{in stock ACME} \\ \text{between} \\ t \text{ and } t + dt}} = \underbrace{\mu}_{\substack{\text{instantaneous} \\ \text{"mean" return} \\ \text{of ACME} \\ \text{between } t \text{ and } t + dt}} dt + \underbrace{\sigma}_{\substack{\text{volatility} \\ \text{for ACME}}} \underbrace{dW_t}_{\substack{\text{New} \\ \text{random} \\ \text{shock}}}$$

Black and Scholes: What does it mean?

Then we have seen there exists a formula (Black and Scholes') providing a unique fair price for the above gamble (option) on ACME in one year.

This Black Scholes formula **depends on the volatility** σ of ACME, and from the initial value S_0 of ACME today, but **does NOT depend on the expected return** μ of ACME.

This means that two investors with very different expectations on the future performance of ACME (for example one investor believes ACME will grow, the other one that ACME will go down) will be charged the same price from the bank to enter into the option.



The Gamble price does not depend on the investor perception of future markets. One would think that Red Investor should be willing to pay a higher price for the option with respect to Blue Investor. Instead, both will have to pay the gamble according to the green scenarios, where ACME grows with the same returns as a riskless asset

Derivatives prices independent of expected returns???

This seemingly counterintuitive result renders derivatives pricing independent of the expected returns of their underlying assets.

This makes derivatives valuations quite objective, and has contributed to derivatives growth worldwide.

Today, derivatives are used for several purposes by banks and corporates all over the world

A mathematical result has contributed to create new markets that reached 708 trillions (US GDP: 15 Trillions)

But keep in mind that the derivation of the Black Scholes result holds only under the 4 ideal conditions and actually many more assumptions:

The Black Scholes Merton analysis assumptions

- **Short selling is allowed without restrictions**
- Infinitely divisible shares
- **No transaction costs**
- No dividends in the stock
- **No default risk of the parties in the deal**
- **No funding costs: Cash can be borrowed or lent at the risk free rate r . Remove this and Valuation becomes Nonlinear** (Semi-Linear PDEs, FBSDEs, see several papers B. & Pallavicini 2011-2015)
- **Continuous time and continuous trading/hedging**
- **Perfect market information, Complete markets**
-

Many of the above assumptions are no longer tenable, especially after 2007-2008, but were already unrealistic well before 2008.

Crisis

After Black Scholes 1973...

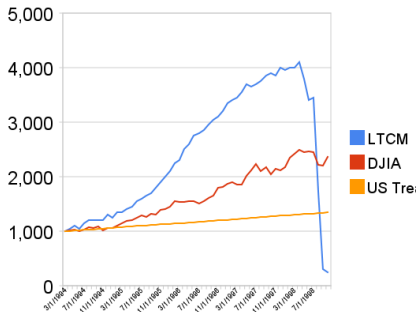
Market players introduced derivatives that may be much more complex functionals of underlying assets and events than the above call option

Gamble/speculate/hedge/protect on anything?

Derivatives on different sectors: Foreign Exchange Rates, Interest Rates, Default Events, Meteorology, Energy, population Longevity...

Aggressive market participants extrapolating the basic theory

The initial Black Scholes theory of 1973 (Nobel award 1997) has often been extrapolated beyond its limits to address new derivatives. One of the most controversial extrapolations is Credit Derivatives and CDOs in particular, which we will address specifically below in the credit part.



Sometimes the timing of the Nobel committee is funny, and we are not talking about the peace Nobel prize. Warning: anecdotal

1997: Nobel award.

1998: the US Long-Term Capital Management hedge fund has to be bailed out after a huge loss. The fund had Merton and Scholes in their board and made high use of leverage (derivatives). This leads us to...

The crisis (2008-current). Multiple curves

Following the 7[8] credit events happening to Financials in one month of 2008,

Fannie Mae, Freddie Mac, Lehman Brothers, Washington Mutual, Landsbanki, Glitnir and Kaupthing [and Merrill Lynch]

the market broke up and interest rates that used to be very close to each other and were used to model risk free rates for different maturities started to diverge.

Multiple curves: LIBOR?

Credit/Default-free interest rates r_t , $L(t, T)$, $F(t, T_{i-1}, T_i)$ etc?

So it is not clear what is the risk free rate r_t anymore, but especially credit/default-free interest rates with finite (rather than infinitesimal) tenor $T - t$ are hard to define: What is the credit/default-free $L(t, T)$, i.e. the simple compounding credit- and liquidity- risk free rate associated with default free bonds? In the classical theory we identify it with LIBOR interbank rates, ie interest rates banks charge each other for lending and borrowing. However, after the credit events above, banks can no longer be considered as default free, so that Interbank rates, and LIBOR rates in particular, are contaminated by counterparty credit risk and liquidity risk.

LIBOR has been also subject to illegal manipulation (see the LIBOR rigging scandal involving a number of major banks), but this is fraud risk and is another story.

Multiple curves: OIS?

Besides LIBOR, other rates have been considered as default/credit risk free rates in the past. One of the most popular is the overnight rate. This is an interest rate $O(t_{i-1}, t_i)$ applied at time t_{i-1} to a loan that is closed one or two days later at t_i . Hence the credit risk embedded in the overnight rate is only on one day and is limited. Furthermore, overnight rates are harder to manipulate illegally (some are quoted by central banks).

There are *swaps* built on overnight rates, and they are called Overnight Indexed Swaps (OIS).

Multiple curves: OIS?

OIS have been introduced back in the mid nineties. The maturities T of OISs range from 1 week to 2 years or longer.

Overnight swaps

At maturity T , the swap parties calculate the final payment as a difference between the accrued interest of the fixed rate K and the geometric average $L^O(0, T)$ of the floating index rates $O(t_{i-1}, t_i)$ on the swap notional for t_i ranging from the initial time $t_{\text{first}} = 0$ to the swap maturity $t_{\text{last}} = T$. Since the net difference is exchanged, rather than swapping the actual rates, OISs have little counterparty credit risk.

Overnight swaps vs LIBOR indexed swaps: Counterparty risk

In a LIBOR based swap where we pay L and receive K , if our counterparty defaults (say with zero recovery) we still pay L and we lose the whole K . If the net rate were exchanged as in OIS, at default we would only lose $K - L$ if positive.



Figure: Spread between 3 months Libor and 3 months ONIA (OIS) swaps. Plotting $t \mapsto L(t, t + 3m) - L^O(t, t + 3m)$ (proxy of credit and liquidity risk). From Economic Synopses 2008, Number 25, FRB of St Louis

The crisis (2008-current). Multiple curves

At the moment it is no longer realistic to neglect credit risk and liquidity effects in interest rate modeling, pretending there is a risk free rate that is governing the LIBOR and interbank markets.

The OIS rate partly solves the problem as it is a best proxy for a default- and liquidity-free interest rate. Residual credit risk is still present and liquidity effects may still be visible, especially under strong stress scenarios.

These days one tends to use overnight swap rates as proxies for the risk free rates, whereas LIBOR and LIBOR-based swap rates have to be managed more carefully. **There are multiple curves that can be built for discounting, some LIBOR based, other OIS based, and yet other different ones.**

The following table is taken by a presentation of Marco Bianchetti (2011)

The crisis (2008-current). Multiple curves

	Libor	Euribor	Eonia	Eurepo
Definition	London InterBank Offered Rate	Euro InterBank Offered Rate	Euro OverNight Index Average	Euro Repurchase Agreement rate
Market	London Interbank	Euro Interbank	Euro Interbank	Euro Interbank
Side	Offer	Offer	Offer	Offer
Rate quotation specs	EURLibor = Euribor, Other currencies: minor differences (e.g. act/365, T+0, London calendar for GBPLibor).	TARGET calendar, settlement T+2, act/360, three decimal places, modified following, end of month, tenor variable.	TARGET calendar, settlement T+1, act/360, three decimal places, tenor 1d.	As Euribor
Maturities	1d-12m	1w, 2w, 3w, 1m, ..., 12m	1d	T/N-12m
Publication time	12.30 CET	11:00 am CET	6:45-7:00 pm CET	As Euribor
Panel banks	8-16 banks (London based) per currency	42 banks from 15 EU countries + 4 international banks	Same as Euribor	34 EU banks plus some large international bank from non-EU countries
Calculation agent	Reuters	Reuters	European Central Bank	Reuters
Transactions based	No	No	Yes	No
Collateral	No (unsecured)	No (unsecured)	No (unsecured)	Yes (secured)
Counterparty risk	Yes	Yes	Low	Negligible
Liquidity risk	Yes	Yes	Low	Negligible
Tenor basis	Yes	Yes	No	No

The crisis (2008-current). Multiple curves

The uncertainty on which rate could be considered as a natural discounting rate is pushing banks to use multiple curves, trying to patch them together, at times in inconsistent ways.

Much work needs to be done to include consistently credit and liquidity effects in interest rate theory from the start, thus avoiding the confusion of unexplained multiple curves. The industry is looking at this now.

Multiple curves explained as synthesis of more fundamental Credit, Liquidity and Funding effects

Multiple curves explained as synthesis of more fundamental Credit, Liquidity and Funding effects.

Rather than taking the curves as fundamental objects, we need to interpret them as incorporating fundamental effects that need to be modeled first.

These effects are Credit Risk and Liquidity Funding Risk.

We face this challenge later. To do this, we need to look at credit risk modeling and products. We do this now.

Intro to Basic Credit Risk Products and Models

Before dealing with the current topical issues of Counterparty Credit Risk, CVA, DVA and Funding, we need to introduce some basic elements of Credit Risk Products and Credit Risk Modelling.

We now briefly look at:

- Products: Credit Default Swaps (CDS) and Defaultable Bonds
- Payoffs and prices of such products
- Market implied \mathbb{Q} probabilities of default defined by such models
- Intensity models and probabilities of defaults as credit spreads
- Credit spreads as possibly constant, curved or even stochastic
- Credit spread volatility (stochastic credit spreads)
- Firm value models: Merton, Black Cox and AT1P
- Multi-name Credit derivatives
- Copula Models, implied correlation and the CDO crisis
- The CDO crisis in the media
- Dynamic Loss models... Where now in credit risk?

Defaultable (corporate) zero coupon bonds

We started this course by defining the zero coupon bond price $P(t, T)$. Similarly to $P(t, T)$ being one of the possible fundamental quantities for describing the interest-rate curve, we now consider a defaultable bond $\bar{P}(t, T)$ as a possible fundamental variable for describing the defaultable market.

DEFAULT FREE		with DEFAULT	
time t		time t	time T :
:		:	NO DEFAULT: 1
$P(t, T)$	←	$\bar{P}(t, T)$	DEFAULT: 0
			1

When considering default, we have a random time τ representing the time at which the bond issuer defaults. τ : Default time of the issuer

Defaultable (corporate) zero coupon bonds I

The value of a bond issued by the company and promising the payment of 1 at time T , as seen from time t , is the risk neutral expectation of the discounted payoff

BondPrice = Expectation[Discount x Payoff]

$$P(t, T) = \mathbb{E}\{D(t, T) \mathbf{1}_{\{\tau > t\}} | \mathcal{F}_t\}, \quad \bar{P}(t, T) := \mathbb{E}\{D(t, T) \mathbf{1}_{\{\tau > T\}} | \mathcal{G}_t\}$$

where \mathcal{G}_t represents the flow of information on whether default occurred before t and if so at what time exactly, and on the default free market variables (like for example the risk free rate r_t) up to t . The filtration of default-free market variables is denoted by \mathcal{F}_t . Formally, we assume

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau \leq u\}, 0 \leq u \leq t).$$

D is the stochastic discount factor between two dates, depending on interest rates, and represents discounting.

Defaultable (corporate) zero coupon bonds II

The “indicator” function $\mathbf{1}_{\text{condition}}$ is 1 if “condition” is satisfied and 0 otherwise. In particular, $\mathbf{1}_{\{\tau > T\}}$ reads 1 if default τ did not occur before T , and 0 in the other case.

We understand then that (ignoring recovery) $\mathbf{1}_{\{\tau > T\}}$ is the correct payoff for a corporate bond at time T : the contract pays 1 if the company has not defaulted, and 0 if it defaulted before T , according to our earlier stylized description.

Defaultable (corporate) zero coupon bonds

If we include a recovery amount REC to be paid at default τ in case of early default, we have as discounted payoff at time t

$$D(t, T)\mathbf{1}_{\{\tau > T\}} + \text{REC}D(t, \tau)\mathbf{1}_{\{\tau \leq T\}}$$

If we include a recovery amount REC paid at maturity T , we have as discounted payoff

$$D(t, T)\mathbf{1}_{\{\tau > T\}} + \text{REC}D(t, T)\mathbf{1}_{\{\tau \leq T\}}$$

Taking $\mathbb{E}[\cdot | \mathcal{G}_t]$ on the above expressions gives the price of the bond.



A Lehman bond price example, maturity June 2046, Default on Sep 14, 2008 with indicative recovery 7.625 at the time (auction for CDS will give 8.62%, see below)

Fundamental Credit Derivatives: Credit Default Swaps

Credit Default Swaps are basic protection contracts that became quite liquid on a large number of entities after their introduction in 1991-94.

CDS reached a notional of \$3.7 trillion in 2003, \$62.2 trillion in 2007, \$38.6 trillion in 2008, \$25 tr 2012 (ISDA).

CDS's are now actively traded and are a sort of basic product of the credit derivatives area, analogously to interest-rate swaps and FRA's being basic products in the interest-rate derivatives world.

Then we don't need a model to value CDS's, but rather we need a model that can be *calibrated* to CDS's, i.e. to take CDS's as model inputs (rather than outputs), in order to price more complex derivatives.

As for options, single name CDS options have never been liquid, as there is more liquidity in the CDS index options. We may expect models will have to incorporate CDS index options quotes rather than price them, similarly to what happened to CDS themselves.

Fundamental Credit Derivatives: CDS's

A CDS contract ensures protection against default. Two companies "A" (Protection buyer) and "B" (Protection seller) agree on the following. If a third company "C" (Reference Credit) defaults at time τ , with $T_a < \tau < T_b$, "B" pays to "A" a certain (deterministic) cash amount L_{GD} . In turn, "A" pays to "B" a rate R at times T_{a+1}, \dots, T_b or until default. Set $\alpha_j = T_j - T_{j-1}$ and $T_0 = 0$.

Protection
Seller B

→ protection L_{GD} at default τ_C if $T_a < \tau_C \leq T_b$ →
← rate R at T_{a+1}, \dots, T_b or until default τ_C ←

Protection
Buyer A

(protection leg and premium leg respectively). The cash amount L_{GD} is a *protection* for "A" in case "C" defaults. Typically $L_{GD} = \text{notional}$, or "notional - recovery" = $1 - R_{EC}$.

Fundamental Credit Derivatives: CDS's

A typical stylized case occurs when "A" has bought a corporate bond issued by "C" and is waiting for the coupons and final notional payment from "C": If "C" defaults before the corporate bond maturity, "A" does not receive such payments. "A" then goes to "B" and buys some protection against this risk, asking "B" a payment that roughly amounts to the loss on the bond (e.g. notional minus deterministic recovery) that A would face in case "C" defaults.

Or again "A" has a portfolio of several instruments with a large exposure to counterparty "C". To partly hedge such exposure, "A" enters into a CDS where it buys protection from a bank "B" against the default of "C".

Fundamental Credit Derivatives: CDS's I

What counts as a credit event triggering τ_C ?

- Bankruptcy of “C”
- Failure to pay of “C”;
- Obligation acceleration, when “C” is requested to pay debt ahead of schedule because “C” didn't meet the terms of the loan
- Restructuring, when “C” undergoes reorganization to consolidate its debt (there are several types of restructuring and definitions may be different in Europe and US)

Fundamental Credit Derivatives: CDS's II

- What happens in a CDS contract at default of “C”?
- Cash Settlement. Protection seller pays to the buyer the loss in value of the referenced instruments (e.g. “C” issued bonds) following the credit event. Bonds or loans are not transferred. When more instruments can be referenced the cheapest to deliver price variation is used (see below).
- Physical Settlement. The protection buyer receives a cash payment, typically the “insured” face value, from the protection seller, and the seller takes possession of the defaulted loan instrument or bonds for an equivalent notional amount.
- Physical S.: most CDS allow the protection buyer to choose deliverables from a pool of defaulted bonds with equal seniority. Cheapest to deliver bond is typically chosen (different value in a reorganization, higher accrued interest...)

Fundamental Credit Derivatives: CDS's III

- Physical S: Auction. If there are not enough bonds to match the insured face value, a credit event auction occurs, and the payment received is usually substantially less than the face value of the loan.
- Recovery rate REC is implicitly defined by these procedures and by market value decline after credit event and is very hard to estimate a priori.

Fundamental Credit Derivatives: CDS's IV

Recovery Rate?

- Prior to 2007 assume REC = 40% in most cases and 50% for financials. Lehman REC in immediate auction was 8.625%!^a
- Lehman asset liquidations is still ongoing. Recovery has led to legal battles. The final recovery might exceed 40%.
- ISDA 2009 big bang recommends REC = 20% or REC = 40%
- Analysis is mostly possible in aggregate on large pools of bonds or loans with similar ratings
- Only few studies available. In aggregate, an inverse relationship between Recovery rates and credit risk/spread or default rates.
- Postulate inverse relationships between spreads and recoveries, but no consensus on how this should be shaped precisely.

^aA final value of 8.625% was set on the bonds of Lehman Brothers [...], in an auction intended to cash-settle credit default swap (CDS) trades linked to the toppled dealer. Over 350 firms participated in the auction protocol, according to ISDA. The final price is about four points lower than that for Lehman's actual defaulted debt, according to Morgan Stanley. It means protection sellers will pay 91.375% of par to settle defaulted CDSs (Risk Magazine)

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Journal of Business

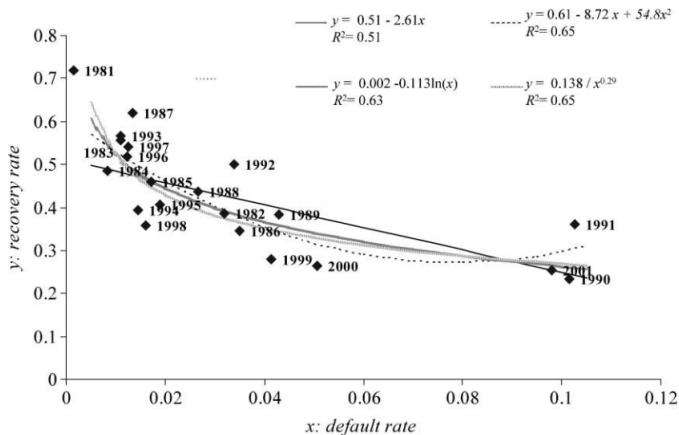


FIG. 1.—Univariate models. Results of a set of univariate regressions carried out between the recovery rate (BRR) or its natural log (BLRR) and the default rate (BDR) or its natural log (BLDR). See table 2 for more details.

Based on Corporate Bond data 1982-2001, Altman, Brady, Resti, Sironi, Journal of Business, 2005, vol. 78, no. 6, 2005

Fundamental Credit Derivatives: CDS's

Protection
Seller B

→ protection LGD at default τ_C if $T_a < \tau_C \leq T_b$ →
← rate R at T_{a+1}, \dots, T_b or until default τ_C ←

Protection
Buyer A

Formally we may write the (Running) CDS discounted payoff to “B” at time $t < T_a$ as

$$\Pi_{\text{RCDS}_{a,b}}(t) := D(t, \tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a < \tau < T_b\}} + \sum_{i=a+1}^b D(t, T_i)\alpha_i R\mathbf{1}_{\{\tau > T_i\}} - \mathbf{1}_{\{T_a < \tau \leq T_b\}} D(t, \tau) \text{LGD}$$

where $T_{\beta(\tau)}$ is the first of the T_i 's following τ .

CDS payout to Protection seller (receiver CDS)

The 3 terms in the payout are as follows (they are seen from the protection seller, receiver CDS):

- Discounted Accrued rate at default : This is supposed to compensate the protection seller for the protection he provided from the last T_i before default until default τ :

$$D(t, \tau)(\tau - T_{\beta(\tau)-1})R1_{\{T_a < \tau < T_b\}}$$

- CDS Rate premium payments if no default: This is the premium received by the protection seller for the protection being provided

$$\sum_{i=a+1}^b D(t, T_i)\alpha_i R1_{\{\tau > T_i\}}$$

- Payment of protection at default if this happens before final T_b

$$-1_{\{T_a < \tau \leq T_b\}} D(t, \tau) L_{GD}$$

These are random discounted cash flows, not yet the CDS price.

CDS's: Risk Neutral Valuation Formula

Denote by $CDS_{a,b}(t, R, L_{GD})$ the time t *price* of the above Running standard CDS's *payoffs*.

As usual, the price associated to a discounted payoff is its *risk neutral expectation*.

The resulting pricing formula depends on the assumptions on interest-rate dynamics and on the default time τ (reduced form models, structural models...).

CDS's: Risk Neutral Valuation

In general by risk-neutral valuation we can compute the CDS price at time 0 (or at any other time similarly):

$$\text{CDS}_{a,b}(0, R, L_{\text{GD}}) = \mathbb{E}\{\Pi_{\text{RCDS}_{a,b}}(0)\},$$

with the CDS discounted payoffs defined earlier. As usual, \mathbb{E} denotes the risk-neutral expectation, the related measure being denoted by \mathbb{Q} .

However, we will not use the formulas resulting from this approach to price CDS that are already quoted in the market. *Rather, we will invert these formulas in correspondence of market CDS quotes to calibrate our models to the CDS quotes themselves. We will give examples of this later.*

Now let us have a look at some particular formulas resulting from the general risk neutral approach through some simplifying assumptions.

CDS Model-independent formulas

Assume the stochastic discount factors $D(s, t)$ to be independent of the default time τ for all possible $0 < s < t$. The price of the premium leg of the CDS at time 0 is:

$$\begin{aligned} \text{PremiumLeg}_{a,b}(R) &= \mathbb{E}[D(0, \tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a < \tau < T_b\}}] + \\ &\quad + \sum_{i=a+1}^b \mathbb{E}[D(0, T_i)\alpha_i R\mathbf{1}_{\{\tau \geq T_i\}}] \\ &= \mathbb{E} \left[\int_{t=0}^{\infty} D(0, t)(t - T_{\beta(t)-1})R\mathbf{1}_{\{T_a < t < T_b\}} \mathbf{1}_{\{\tau \in [t, t+dt]\}} \right] \\ &\quad + \sum_{i=a+1}^b \mathbb{E}[D(0, T_i)]\alpha_i R \mathbb{E}[\mathbf{1}_{\{\tau \geq T_i\}}] = \end{aligned}$$

For those who know the theory of distributions (Dirac's delta etc), read $\mathbf{1}_{\{\tau \in [t, t+dt]\}} = \delta_{\tau}(t)dt$.

CDS Model-independent formulas

$$\begin{aligned}
 \text{PremiumLeg}_{a,b}(R) &= \int_{t=T_a}^{T_b} \mathbb{E}[D(0,t)(t - T_{\beta(t)-1})R \mathbf{1}_{\{\tau \in [t,t+dt]\}}] + \\
 &\quad + \sum_{i=a+1}^b P(0, T_i)\alpha_i R \mathbb{Q}(\tau \geq T_i) = \\
 &= \int_{t=T_a}^{T_b} \mathbb{E}[D(0,t)](t - T_{\beta(t)-1})R \mathbb{E}[\mathbf{1}_{\{\tau \in [t,t+dt]\}}] + \sum_{i=a+1}^b P(0, T_i)\alpha_i R \mathbb{Q}(\tau \geq T_i) \\
 &= R \int_{t=T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1})\mathbb{Q}(\tau \in [t, t + dt]) + \\
 &\quad + R \sum_{i=a+1}^b P(0, T_i)\alpha_i \mathbb{Q}(\tau \geq T_i),
 \end{aligned}$$

where we have used independence in factoring terms.

CDS Model-independent formulas

We have thus, by rearranging terms and introducing a “unit-premium” premium leg (sometimes called “DV01”, “Risky duration” or “annuity”):

$$\text{PremiumLeg}_{a,b}(R; P(0, \cdot), \mathbb{Q}(\tau > \cdot)) = R \text{PremiumLeg1}_{a,b}(P(0, \cdot), \mathbb{Q}(\tau > \cdot)),$$

$$\begin{aligned} \text{PremiumLeg1}_{a,b}(P(0, \cdot), \mathbb{Q}(\tau > \cdot)) &:= \int_{T_a}^{T_b} P(0, t)(t - T_{\beta(t)-1}) dt \boxed{\mathbb{Q}(\tau \leq t)} \\ &+ \sum_{i=a+1}^b P(0, T_i) \alpha_i \boxed{\mathbb{Q}(\tau \geq T_i)} \end{aligned}$$

This model independent formula uses the initial market zero coupon curve (bonds) at time 0 (i.e. $P(0, \cdot)$) and the survival probabilities $\mathbb{Q}(\tau \geq \cdot)$ at time 0 (terms in the boxes).

A similar formula holds for the protection leg, again under independence between default τ and interest rates.

CDS Model-independent formulas

$$\begin{aligned}
 \text{ ProtecLeg}_{a,b}(\text{LGD}) &= \mathbb{E}[\mathbf{1}_{\{T_a < \tau \leq T_b\}} D(0, \tau) \text{LGD}] \\
 &= \text{LGD} \mathbb{E} \left[\int_{t=0}^{\infty} \mathbf{1}_{\{T_a < t \leq T_b\}} D(0, t) \mathbf{1}_{\{\tau \in [t, t+dt]\}} \right] \\
 &= \text{LGD} \left[\int_{t=T_a}^{T_b} \mathbb{E}[D(0, t) \mathbf{1}_{\{\tau \in [t, t+dt]\}}] \right] \\
 &= \text{LGD} \int_{t=T_a}^{T_b} \mathbb{E}[D(0, t)] \mathbb{E}[\mathbf{1}_{\{\tau \in [t, t+dt]\}}] \\
 &= \text{LGD} \int_{t=T_a}^{T_b} P(0, t) \mathbb{Q}(\tau \in [t, t + dt]) \\
 &= \text{LGD} \int_{t=T_a}^{T_b} P(0, t) dt \mathbb{Q}(\tau \leq t)
 \end{aligned}$$

(again interpret $\mathbf{1}_{\{\tau \in [t, t+dt]\}} = \delta_{\tau}(t) dt$)

CDS Model-independent formulas

so that we have, by introducing a “unit-notional” protection leg:

$$\text{ ProtecLeg}_{a,b}(\text{LGD}; P(0, \cdot), \mathbb{Q}(\tau > \cdot)) = \text{LGD} \text{ ProtecLeg1}_{a,b}(P(0, \cdot), \mathbb{Q}(\tau > \cdot)),$$

$$\text{ ProtecLeg1}_{a,b}(P(0, \cdot), \mathbb{Q}(\tau > \cdot)) := \int_{T_a}^{T_b} P(0, t) d_t \boxed{\mathbb{Q}(\tau \leq t)}$$

This formula too is model independent given the initial zero coupon curve (bonds) at time 0 observed in the market and given the survival probabilities at time 0 (term in the box).

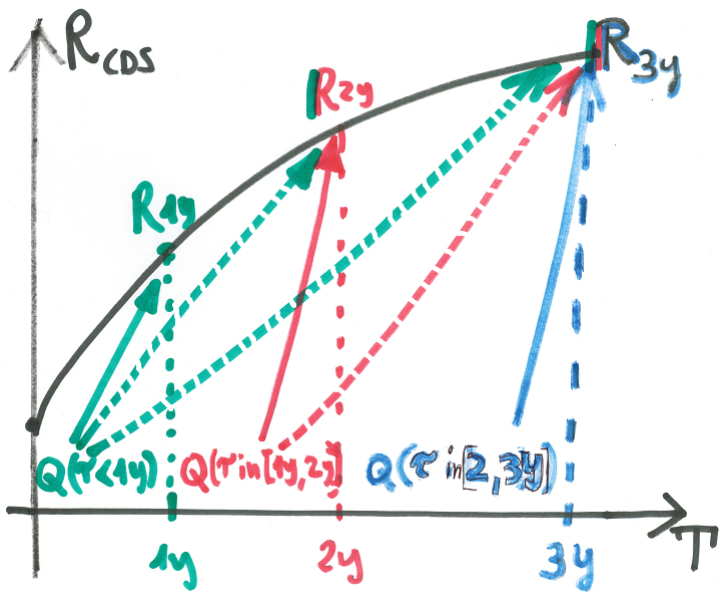
(Receiver) CDS Model-independent formulas

We have

$$\text{CDS}_{a,b}(t, R, \text{LGD}; \mathbb{Q}(\tau \leq \cdot)) = -\text{LGD} \left[\int_{T_a}^{T_b} P(0, t) d_t \mathbb{Q}(\tau \leq t) \right] +$$

$$R \left[\int_{T_a}^{T_b} P(0, t)(t - T_{\beta(t)-1}) d_t \mathbb{Q}(\tau \leq t) + \sum_{i=a+1}^b P(0, T_i) \alpha_i \mathbb{Q}(\tau \geq T_i) \right]$$

The integrals in the survival probabilities given in the above formulas can be valued as Stieltjes integrals in the survival probabilities themselves, and can easily be approximated numerically by summations through Riemann-Stieltjes sums, considering a low enough discretization time step.



CDS Model-independent formulas

The market quotes, at time 0, the fair $R = R_{0,b}^{\text{mkt MID}}(0)$ coming from bid and ask quotes for this fair R .

This fair R equates the two legs for a set of CDS with initial protection time $T_a = 0$ and final protection time

$T_b \in \{1y, 2y, 3y, 4y, 5y, 6y, 7y, 8y, 9y, 10y\}$, although often only a subset of the maturities $\{1y, 3y, 5y, 7y, 10y\}$ is available.

Solve then

$$\text{CDS}_{0,b}(t, R_{0,b}^{\text{mkt MID}}(0), L_{\text{GD}}; \mathbb{Q}(\tau > \cdot)) = 0$$

in portions of $\mathbb{Q}(\tau > \cdot)$ starting from $T_b = 1y$, finding the market implied survival $\{\mathbb{Q}(\tau \geq t), t \leq 1y\}$; plugging this into the $T_b = 2y$ CDS legs formulas, and then solving the same equation with $T_b = 2y$, we find the market implied survival $\{\mathbb{Q}(\tau \geq t), t \in (1y, 2y]\}$, and so on up to $T_b = 10y$.

CDS Model-independent formulas

This is a way to strip survival (or equivalently default) probabilities from CDS quotes in a model independent way. No need to assume an intensity or a structural model for default here.

However, the market in doing the above stripping typically resorts to intensities (also called hazard rates), assuming existence of intensities associated with the default time.

We will refer to the method just highlighted as **”CDS stripping”**.

CDS and Defaultable Bonds: Intensity Models

In intensity models the random default time τ is assumed to be exponentially distributed.

A strictly positive stochastic process $t \mapsto \lambda_t$ called *default intensity* (or hazard rate) is given for the bond issuer or the CDS reference name.

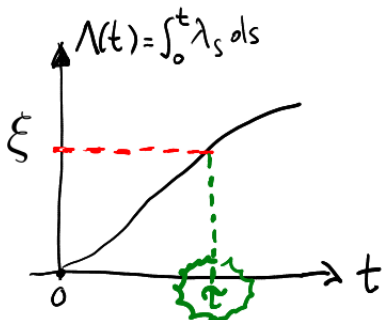
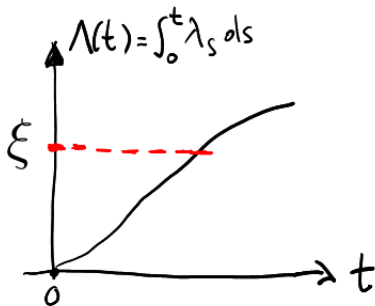
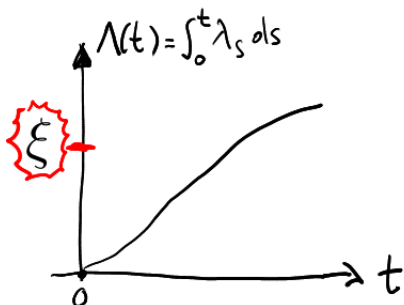
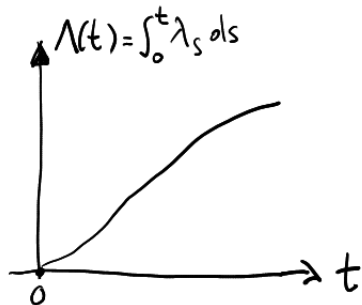
The *cumulated intensity* (or hazard function) is the process $t \mapsto \int_0^t \lambda_s ds =: \Lambda_t$. Since λ is positive, Λ is increasing in time.

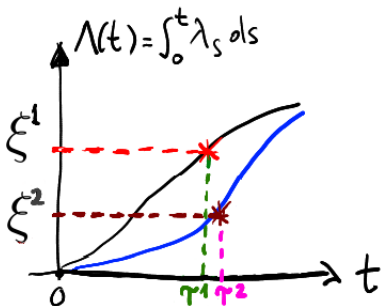
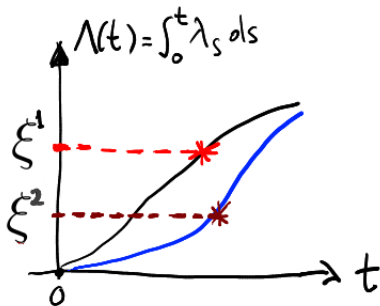
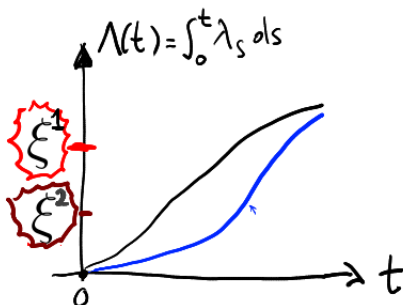
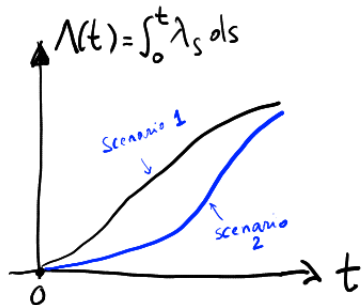
The default time is defined as the inverse of the cumulative intensity on an exponential random variable ξ with mean 1 and independent of λ

$$\tau = \Lambda^{-1}(\xi).$$

Recall that

$$\mathbb{Q}(\xi > u) = e^{-u}, \quad \mathbb{Q}(\xi < u) = 1 - e^{-u}, \quad \mathbb{E}(\xi) = 1.$$





CDS and Defaultable Bonds: Intensity Models

A few calculations: Probability of surviving time t :

$$\mathbb{Q}(\tau > t) = \mathbb{Q}(\Lambda^{-1}(\xi) > t) = \mathbb{Q}(\xi > \Lambda(t)) \Rightarrow$$

Let's use the tower property of conditional expectation and the fact that Λ is independent of ξ :

$$\Rightarrow \mathbb{E}[\mathbb{Q}(\xi > \Lambda(t) | \Lambda(t))] = \mathbb{E}[e^{-\Lambda(t)}] = \mathbb{E}[e^{-\int_0^t \lambda_s ds}]$$

This looks exactly like a bond price if we replace r by λ !

CDS and Defaultable Bonds: Intensity Models

Let's price a defaultable zero coupon bond with zero recovery. Assume that ξ is also independent of r .

$$\begin{aligned}
 \bar{P}(0, T) &= \mathbb{E}[D(0, T)1_{\{\tau > T\}}] = \mathbb{E}[e^{-\int_0^T r_s ds} 1_{\{\Lambda^{-1}(\xi) > T\}}] = \\
 &= \mathbb{E}[e^{-\int_0^T r_s ds} 1_{\{\xi > \Lambda(T)\}}] = \mathbb{E}[\mathbb{E}\{e^{-\int_0^T r_s ds} 1_{\{\xi > \Lambda(T)\}} | \Lambda, r\}] \\
 &= \mathbb{E}[e^{-\int_0^T r_s ds} \mathbb{E}\{1_{\{\xi > \Lambda(T)\}} | \Lambda, r\}] \\
 &= \mathbb{E}[e^{-\int_0^T r_s ds} \mathbb{Q}\{\xi > \Lambda(T) | \Lambda\}] = \mathbb{E}[e^{-\int_0^T r_s ds} e^{-\Lambda(T)}] = \\
 &= \mathbb{E}[e^{-\int_0^T r_s ds - \int_0^T \lambda_s ds} = \mathbb{E}[e^{-\int_0^T (r_s + \lambda_s) ds}]
 \end{aligned}$$

So the price of a defaultable bond is like the price of a default-free bond *where the risk free discount short rate r has been replaced by r plus a spread λ .*

CDS and Defaultable Bonds: Intensity Models

This is why in intensity models, the intensity is interpreted as a credit spread.

Because of properties of the exponential random variable, one can also prove that

$$\mathbb{Q}(\tau \in [t, t + dt] | \tau > t, " \lambda[0, t] ") = \lambda_t dt$$

and the intensity $\lambda_t dt$ is also a local probability of defaulting around t .

So:

λ is an instantaneous credit spread or local default probability

ξ is pure jump to default risk

Intensity models and Interest Rate Models

As is now clear, the exponential structure of τ in intensity models makes the modeling of credit risk very similar to interest rate models.

The spread/intensity λ behaves exactly like an interest rate in discounting

Then it is possible to use a lot of techniques from interest rate modeling (short rate models for r , first choice seen earlier) for credit as well.

Intensity: Constant, time dependent or stochastic

- Constant λ_t : in this case $\lambda_t = \gamma$ for a deterministic constant credit spread (intensity);
- Time dependent deterministic intensity λ_t : in this case $\lambda_t = \gamma(t)$ for a deterministic curve in time $\gamma(t)$. This is a model with a term structure of credit spreads but without credit spread volatility.
- Time dependent and stochastic intensity λ_t : in this case λ_t is a full stochastic process. This allows us to model the term structure of credit spreads but also their volatility.

The case with constant intensity $\lambda_t = \gamma$: CDS

Assume as an approximation that the CDS premium leg pays continuously.

Instead of paying $(T_i - T_{i-1})R$ at T_i as the standard CDS, given that there has been no default before T_i , we approximate this premium leg by assuming that it pays " $dt R$ " in $[t, t + dt)$ if there has been no default before $t + dt$.

We also use $\mathbb{Q}(\tau \leq t) = 1 - \mathbb{Q}(\tau > t)$ and
 $d_t \mathbb{Q}(\tau \leq t) = -d_t \mathbb{Q}(\tau > t)$

The case with constant intensity $\lambda_t = \gamma$: CDS

This amounts to replace the original pricing formula of a CDS (receiver case, spot CDS with $T_a = 0 = \text{today}$)

$$\begin{aligned} \text{CDS}_{0,b}(0, R, \text{LGD}; \mathbb{Q}(\tau > \cdot)) &= R \left[- \int_0^{T_b} P(0, t)(t - T_{\beta(t)-1}) dt \mathbb{Q}(\tau \geq t) \right. \\ &\quad \left. + \sum_{i=1}^b P(0, T_i) \alpha_i \mathbb{Q}(\tau \geq T_i) \right] + \text{LGD} \left[\int_0^{T_b} P(0, t) dt \mathbb{Q}(\tau \geq t) \right] \end{aligned}$$

with (accrual term vanishes because payments continuous now)

$$R \int_0^{T_b} P(0, t) \mathbb{Q}(\tau \geq t) dt + \text{LGD} \int_0^{T_b} P(0, t) dt \mathbb{Q}(\tau \geq t)$$

The case with constant intensity $\lambda_t = \gamma$: CDS

If the intensity is a constant γ we have

$$\mathbb{Q}(\tau > t) = e^{-\gamma t}, \quad d_t \mathbb{Q}(\tau > t) = -\gamma e^{-\gamma t} dt = -\gamma \mathbb{Q}(\tau > t) dt,$$

and the receiver CDS price we have seen earlier becomes

$$\begin{aligned} \text{CDS}_{0,b}(t, R, \text{LGD}; \mathbb{Q}(\tau > \cdot)) &= -\text{LGD} \left[\int_0^{T_b} P(0, t) \gamma \mathbb{Q}(\tau \geq t) dt \right] \\ &+ R \left[\int_0^{T_b} P(0, t) \mathbb{Q}(\tau \geq t) dt \right] \end{aligned}$$

If we insert the market CDS rate $R = R_{0,b}^{\text{mkt MID}}(0)$ in the premium leg, then the CDS present value should be zero. Solve

$$\text{CDS}_{a,b}(t, R, \text{LGD}; \mathbb{Q}(\tau > \cdot)) = 0 \quad \text{in } R$$

to obtain
$$\gamma = \frac{R_{0,b}^{\text{mkt MID}}(0)}{\text{LGD}}$$

The case with constant intensity $\lambda_t = \gamma$: CDS

from which we see that also **the CDS premium rate R is indeed a sort of CREDIT SPREAD, or INTENSITY.**

We can play with this formula with a few examples.

CDS of FIAT trades at 300bps for 5y, with recovery 0.3

What is a quick rough calcul for the risk neutral probability that FIAT survives 10 years?

$$\gamma = \frac{R_{0,b}^{\text{mkt FIAT}}(0)}{\text{LGD}} = \frac{300/10000}{1 - 0.3} = 4.29\%$$

The case with constant intensity $\lambda_t = \gamma$: CDS

Survive 10 years:

$$\mathbb{Q}(\tau > 10y) = \exp(-\gamma 10) = \exp(-0.0429 * 10) = 65.1\%$$

Default between 3 and 5 years:

$$\begin{aligned}\mathbb{Q}(\tau > 3y) - \mathbb{Q}(\tau > 5y) &= \exp(-\gamma 3) - \exp(-\gamma 5) \\ &= \exp(-0.0429 * 3) - \exp(-0.0429 * 5) = 7.2\%\end{aligned}$$

If R_{CDS} goes up and REC remains the same, γ goes up and survival probabilities go down (default probs go up)

If REC goes up and R_{CDS} remains the same, LGD goes down and γ goes up - default probabilities go up

The case with time dependent intensity $\lambda_t = \gamma(t)$: CDS

We consider now **deterministic time-varying** intensity $\gamma(t)$, which we assume to be a positive and piecewise continuous function. We define

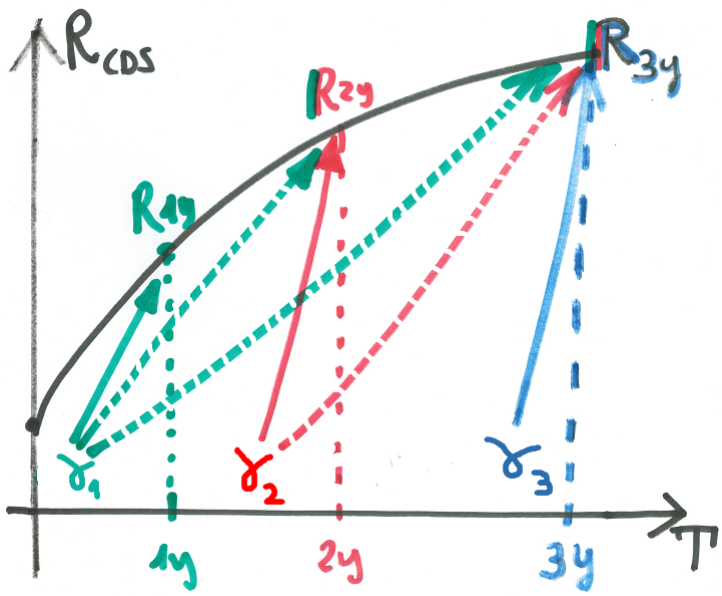
$$\Gamma(t) := \int_0^t \gamma(u) du,$$

the **cumulated intensity, cumulated hazard rate**, or also **Hazard function**.

From the exponential assumption, we have easily

$$\begin{aligned} \mathbb{Q}\{s < \tau \leq t\} &= \mathbb{Q}\{s < \Gamma^{-1}(\xi) \leq t\} = \mathbb{Q}\{\Gamma(s) < \xi \leq \Gamma(t)\} = \\ &= \mathbb{Q}\{\xi > \Gamma(s)\} - \mathbb{Q}\{\xi > \Gamma(t)\} = \exp(-\Gamma(s)) - \exp(-\Gamma(t)) \text{ i.e.} \end{aligned}$$

“prob of default between s and t is “ $e^{-\int_0^s \gamma(u) du} - e^{-\int_0^t \gamma(u) du} \approx \int_s^t \gamma(u) du$ ”
(where the final approximation is good ONLY for small exponents).



CDS Calibration and Implied Hazard Rates/Intensities

Reduced form models are the models that are most commonly used in the market to infer implied default probabilities from market quotes.

Market instruments from which these probabilities are drawn are especially CDS and Bonds.

We just implement the stripping algorithm sketched earlier for "CDS stripping", but now taking into account that the probabilities are expressed as exponentials of the deterministic intensity γ , that is assumed to be piecewise constant.

By adding iteratively CDS with longer and longer maturities, at each step we will strip the new part of the intensity $\gamma(t)$ associated with the last added CDS, while keeping the previous values of γ , for earlier times, that were used to fit CDS with shorter maturities.

A Case Study of CDS stripping: Lehman Brothers

Here we show an intensity model with piecewise constant λ obtained by CDS stripping.

We also show the AT1P structural / firm value model by Brigo et al (2004-2010). This will not be subject for this course, but in case of interest, for details on AT1P see

<http://arxiv.org/abs/0912.3028>

<http://arxiv.org/abs/0912.3031>

<http://arxiv.org/abs/0912.4404>

Otherwise ignore the AT1P and σ_i parts of the tables.

- **August 23, 2007:** Lehman announces that it is going to shut one of its home lending units (*BNC Mortgage*) and lay off 1,200 employees. The bank says it would take a \$52 million charge to third-quarter earnings.
- **March 18, 2008:** Lehman announces better than expected first-quarter results (but profits have more than halved).
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We are in the middle of the crisis. We see that the CDS spreads R_i have increased with respect to the previous case, but are not very high, indicating the fact that the market is aware of the difficulties suffered by Lehman but thinks that it can come out of the crisis. Notice that now the term structure of both R and *intensities* is inverted. This is typical of names in crisis

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Until the end, rating agencies maintained a good rating for Lehman¹

In our [S&P] view, Lehman [...] had adequate liquidity relative to reasonably severe and foreseeable temporary stresses. [...] We believe the downfall of Lehman reflected escalating fears that led to a loss of confidence – ultimately becoming a real threat to Lehman's viability in a way that fundamental credit analysis could not have anticipated with greater levels of certainty

If we check from published transition matrices what has been the probability that a S&P "A" rated name defaulted we have

$\mathbb{P}(A \text{ rated name defaults within } 1y) = 0 \text{ in } 2005/6/7, \text{ and } 0.38\% \text{ in } 2008$

Compare with CDS market: \mathbb{Q} default prob in 1y: 21%.

0.38% vs 21%. **Huge risk premium between \mathbb{P} and \mathbb{Q} for Lehman.**

This is a regular feature: market implied \mathbb{Q} default probabilities are always larger than fundamental history-based ones under \mathbb{P} .

¹<http://ww2.cfo.com/banking-capital-markets/2008/09/rating-itself-sp-defends-lehmans-a/> accessed Sept 9 2014

CDS-Bond Basis (funding liquidity proxy)

The above stripping method could allow us to obtain intensities both from bonds (Z-spread) $\gamma^b(t)$ and CDS (fair spread) $\gamma^c(t)$.

Since Bonds are funded instruments and CDS are not, the CDS-bond basis is considered to be an indicator of funding liquidity

$$\ell(t) = \gamma^c(t) - \gamma^b(t).$$

The basis has been both “+” and “-” through history. Traders may set up basis trades if convinced arbitrage opportunities are showing up.

- Bond funding cost: $\ell \downarrow$
- CDS counterparty risk: $\ell \downarrow$
- Shorting credit: Easier buying CDS protection than shorting bonds. CDS more attractive and default leg more expensive $\ell \uparrow$.
- CDS protect from more general defaults than bonds and have cheapest do deliver advantages when buying protection, as one delivers a less valuable bond in exchange for face value: $\ell \uparrow$.

Stochastic Intensity. The CIR++ model

We have seen in detail CDS calibration in presence of **deterministic** and **time varying** intensity or hazard rates, $\gamma(t)dt = \mathbb{Q}\{\tau \in dt | \tau > t\}$

As explained, this accounts for credit spread structure but not for **volatility**.

The latter is obtained moving to stochastic intensity (Cox process). The deterministic function $t \mapsto \gamma(t)$ is replaced by a stochastic process $t \mapsto \lambda(t) = \lambda_t$. The Hazard function $\Gamma(t) = \int_0^t \gamma(u)du$ is replaced by the Hazard process (or cumulated intensity) $\Lambda(t) = \int_0^t \lambda(u)du$.

CIR++ stochastic intensity λ

We model the stochastic intensity as follows: consider

$$\lambda_t = y_t + \psi(t; \beta), \quad t \geq 0,$$

where the intensity has a random component y and a deterministic component ψ to fit the CDS term structure. For y we take a Jump-CIR model

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dZ_t + dJ_t, \quad \beta = (\kappa, \mu, \nu, y_0), \quad 2\kappa\mu > \nu^2.$$

Jumps are taken themselves independent of anything else, with exponential arrival times with intensity η and exponential jump size with a given parameter.

In this course we will focus on the case with no jumps J , see B and El-Bachir (2006) or B and M (2006) for the case with jumps.

CIR++ stochastic intensity λ .

Calibrating Implied Default Probabilities

With no jumps, y follows a noncentral chi-square distribution; Very important: $y > 0$ as must be for an intensity model (Vasicek would not work). This is the CIR++ model we have seen earlier for interest rates.

About the parameters of CIR:

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dZ_t$$

κ : speed of mean reversion

μ : long term mean reversion level

ν : volatility.

CIR++ stochastic intensity λ . I

Calibrating Implied Default Probabilities

$$E[y_t] = y_0 e^{-\kappa t} + \mu(1 - e^{-\kappa t})$$

$$\text{VAR}(y_t) = \lambda_0 \frac{\nu^2}{\kappa} (e^{-\kappa t} - e^{-2\kappa t}) + \mu \frac{\nu^2}{2\kappa} (1 - e^{-\kappa t})^2$$

After a long time the process reaches (asymptotically) a stationary distribution around the mean μ and with a corridor of variance $\mu\nu^2/2\kappa$. The largest κ , the fastest the process converges to the stationary state. So, ceteris paribus, increasing κ kills the volatility of the credit spread. The largest μ , the highest the long term mean, so the model will tend to higher spreads in the future in average. The largest ν , the largest the volatility. Notice however that κ and ν fight each other as far as the influence on volatility is concerned.

CIR++ stochastic intensity λ . II

Calibrating Implied Default Probabilities

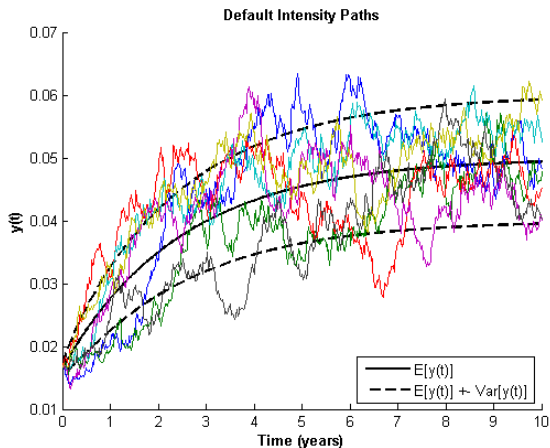


Figure: $y_0 = 0.0165$, $\kappa = 0.4$, $\mu = 0.05$, $\nu = 0.04$

CIR++ stochastic intensity λ . I

Calibrating Implied Default Probabilities

For restrictions on the β 's that keep ψ and hence λ positive, **as is required in intensity models**, we may use the results in B. and M. (2001) or (2006). We will often use the hazard process $\Lambda(t) = \int_0^t \lambda_s ds$, and also $Y(t) = \int_0^t y_s ds$ and $\Psi(t, \beta) = \int_0^t \psi(s, \beta) ds$.

If we can read from the market some implied risk-neutral default probabilities, and associate to them implied hazard functions Γ^{Mkt} (as we have done in the Lehman example), we may wish our stochastic intensity model to agree with them. By recalling that survival probabilities look exactly like bonds formulas in short rate models for r , we see that our model agrees with the market if

$$\exp(-\Gamma^{\text{Mkt}}(t)) = \exp(-\Psi(t, \beta)) \mathbb{E}[e^{-\int_0^t y_s ds}]$$

CIR++ stochastic intensity λ . II

Calibrating Implied Default Probabilities

IMPORTANT 1: This is possible only if λ is strictly positive;
IMPORTANT 2: It is fundamental, if we aim at calibrating default probabilities, that the last expected value can be computed analytically.
The only known diffusion model used in interest rates satisfying both constraints is CIR++

CIR++ stochastic intensity λ

Calibrating Implied Default Probabilities

$$\exp(-\Gamma^{\text{Mkt}}(t)) = \mathbb{Q}\{\tau > t\} = \exp(-\Psi(t, \beta)) \mathbb{E}[e^{-\int_0^t y_s ds}]$$

Now notice that $\mathbb{E}[e^{-\int_0^t y_s ds}]$ is simply the bond price for a CIR interest rate model with short rate given by y , so that it is known analytically. We denote it by $P^y(0, t, y_0; \beta)$.

Similarly to the interest-rate case, λ is calibrated to the market implied hazard function Γ^{Mkt} if we set

$$\Psi(t, \beta) := \Gamma^{\text{Mkt}}(t) + \ln(P^y(0, t, y_0; \beta))$$

where we choose the parameters β in order to have a positive function ψ , by resorting to the condition seen earlier.

Firm Value (structural) models

The other family of important credit models, other than the **intensity/reduced form** models, is the family of

Firm Value (or Structural) Models.

These models have more economics content and are more link to traditional economics/finance.

Indeed with intensity models we have defined $\tau = \Lambda^{-1}(\xi)$ but we have no answer to the question "what are the economic causes of default and how do they manifest themselves in ξ and λ_t "? Hence the name "reduced form" for intensity models.

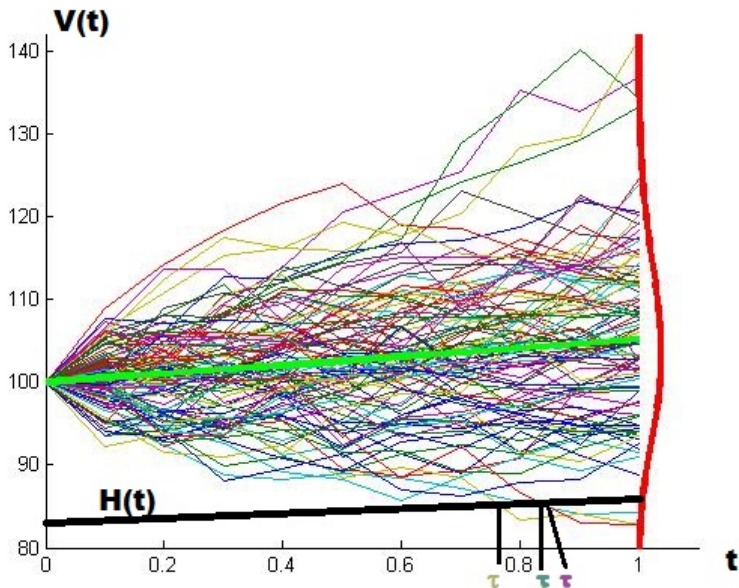
It will be possible to answer a question like this in firm value models. Hence the name "structural models".

The Basic Idea of Structural Models I

The stylized structure of the firm economy is modeled through:

- $V(t)$: **stochastic process for the value of the firm**
- $t \mapsto H(t)$ **barrier representing debt** and safety covenants. This is often estimated based on balance sheet data: Short term debt, long term debt, etc.
- τ : **The default time is the first time instant where the value of the firm V touches the safety barrier H .**

Important difference with Intensity: In basic structural models there is nothing external to the basic market information in the default process, nothing like ξ . Default is induced by a completely observable (? More on this later) variable, the firm value.



Intensity and Structural Models: Different Uses I

- Intensity models suited to model credit spreads; easier to calibrate to corporate bond or CDS data;
- Suited to refined relative value pricing (CDS options, CMCDS, bonds with optionalities, etc);
- Structural models easier to use in situations where we need to model also equity;
- More suited to "fundamental pricing/risk analysis" than to relative value pricing, even if often used for the latter through some tricks; difficult to calibrate with precision to CDS or Bonds;
- Cases include equity return swaps with counterparty risk, total rate of return swaps, and counterparty risk in any equity product, and Equity Default Swaps
- more naturally extended to multi-name situation (no "out of the blue" copula) than stochastic intensity models;

Structural Models: Merton's Model I

The first Structural Model is due to Merton (1974). The **value of the firm V is assumed to follow a Geometric Brownian Motion.**

$$dV(t) = mV(t)dt + \sigma V(t)dW(t), \quad m = \mu \text{ under } \mathbb{P}, \quad m = r - k \text{ under } \mathbb{Q}$$

(μ is return under historical measure, r is the risk free rate, k is the payout ratio and σ is the volatility, all constant).

This dynamics is lognormal; Crouhy et al. notice that “this assumption [lognormal V] is quite robust and, according to KMVs own empirical studies, actual data conform quite well to this hypothesis.”.

V is seen as composed by the equity part S and the debt part D ,

$$\text{Firm Value} = \text{Debt Value} + \text{Equity}, \quad V(t) = D(t) + S(t)$$

Structural Models: Merton's Model II

Debt structure simple: zero coupon debt maturity \bar{T} & face value L .

Default linked to capability of firm to pay back all debt issued.

If at maturity \bar{T} the firm value V is greater than L , then all the debt is paid back and the firm survives; if V is smaller than L then the company is not able to pay the bondholders and then there is the default. **Default can happen only at the debt maturity \bar{T} .** Quite restrictive assumption. We will see later how it can be relaxed.

Some basic calculation in Merton's Model I

$$\text{Prob}\{V_T < L\} = \Phi\left(\frac{\log \frac{L}{V(0)} - (m - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) = \Phi(-d_2)$$

(Black Scholes term) where Φ is the CDF of standard normal random variable, is the default probability. Hazard rate:

$$\lim_{T \downarrow 0} \frac{\text{Prob}\{\tau \leq T\}}{T} = 0$$

(see for example B. Morini Pallavicini (2013)). Compare with a standard constant hazard rate / intensity model, where

$$\mathbb{Q}(\tau \leq T) = 1 - e^{-\gamma T} \Rightarrow \lim_{T \downarrow 0} \frac{\mathbb{Q}\{\tau \leq T\}}{T} = \gamma > 0.$$

Some basic calculation in Merton's Model II

This is an important difference: basic structural models like Merton have no short term credit spreads (the limit is zero). Intensity models instead have non-zero short term credit spread. This is a modeling advantage of intensity models. It means that for very short maturities the Merton model will have great difficulties in fitting nonzero spreads, whereas the intensity model will have no problem.

Keep this in mind.

Debt and equity in Merton's model I

The value of debt at maturity is hence $D_{\bar{T}} = \min(V_{\bar{T}}, L)$, from which, since we assume $V(t) = D(t) + S(t)$, we have equity as

$$\begin{aligned} S(t) &= V(t) - D(t) = \\ &= \text{Call}(t, \bar{T}; V(t), \sigma^2, L) = V(t)\Phi(d_1) - P(t, \bar{T})L\Phi(d_2) \end{aligned} \quad (19)$$

(Black Scholes) $d_1 = \frac{\ln\left(\frac{V(t)}{L}\right) + \left(r - k + \frac{\sigma^2}{2}\right)(\bar{T} - t)}{\sigma\sqrt{\bar{T} - t}}$ and $d_2 = d_1 - \sigma\sqrt{\bar{T} - t}$

Then, as is well known, **in Merton's model the equity can be interpreted as a call option on the value of the firm.**

If we have estimates for the drift and volatilities we can evaluate particular payoff depending on V or just default probabilities. While r and k can be simple to estimate, σ is not: V is not directly observable.

Debt and equity in Merton's model II

A possible way to estimate σ is **to link it to the equity volatility**, available by market data. To avoid confusion: σ_V is **the firm value volatility**, σ_S is the **equity volatility**. Compute

$$dS(t) = d\text{call}(V(t)) = (\dots)dt + \frac{\partial \text{call}}{\partial V} \sigma_V V(t) dW(t).$$

Comparing with an hypothetical dynamics

$dS(t) = (r - q)S(t)dt + \sigma_S(t)S(t)dW(t)$ we immediately find the following important relation between σ_V and σ_S

$$\boxed{\sigma_S = \sigma_V \Delta_{\text{call}} \frac{V}{S}}, \quad \Delta_{\text{call}} = \frac{\partial \text{call}}{\partial V} = \Phi(d_1) \quad (20)$$

Debt and equity in Merton's model III

We may now derive $V > 0$ and σ_V from equity data: given equity data S_0, σ_S , solve the following system in V_0 and σ_V (this is partly the KMV methodology)

$$\begin{cases} S_0 = V(0)\Phi(d_1(V_0, \sigma_V)) - P(0, \bar{T})L\Phi(d_2(V_0, \sigma_V)) \\ \sigma_S = \sigma_V\Phi(d_1(V_0, \sigma_V))\frac{V_0}{S_0} \end{cases}$$

However equity gives information on default only when it is near zero. Otherwise equity implied vols are too far from the critical zone $S = 0$ to provide reliable credit information via the above strategy.

Rating agencies occasionally use Equity implied ratings and other firms use equity information to deduce default probabilities, but this is dubious.

Structural Models: AT1P I

Drawbacks of Merton: default only at the debt maturity \bar{T} .
Unsatisfactory: there could be scenarios in which default happens before \bar{T} , related to problems of optimal capital structure and stockholders decisions to reorganize the firm.

Black and Cox (1976) assume a barrier representing safety covenants for the firm. Default is triggered by the firm value V hitting this barrier from above. At default the firm reimburses the debt-holders. However, too few parameters. We generalize BC to an Analytically Tractable 1st Passage model (AT1P)

Structural Models: CDS Calibration? AT1P Model

Our strategy: (CDS Calibration?)

$$\left. \begin{array}{l} R_{0,1y}^{\text{MktCDS}} \\ R_{0,2y}^{\text{MktCDS}} \\ \vdots \\ R_{0,10y}^{\text{MktCDS}} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} dV(t) = (r - k)V(t)dt + \boxed{\sigma_V(t)} V(t)dW(t) \\ H(t) = \dots \\ \text{model parameters: } t \mapsto \sigma_V(t), \quad t \mapsto H(t) \end{array} \right.$$

Now we would have infinite parameters (all the values of $\sigma(t)$, for example) to account for 10 CDS quotes. The problem is: can we insert a time-dependent V dynamics **and** preserve barrier-like analytical formulas for survival probabilities $\mathbb{Q}(\tau > t)$ (and thus CDS etc?)? This works if the barrier has a special but reasonable shape (AT1P model)

$$\hat{H}(t) = \frac{H}{V_0} \mathbb{E}[V_t] e^{-B \int_0^t \sigma_s^2 ds}$$

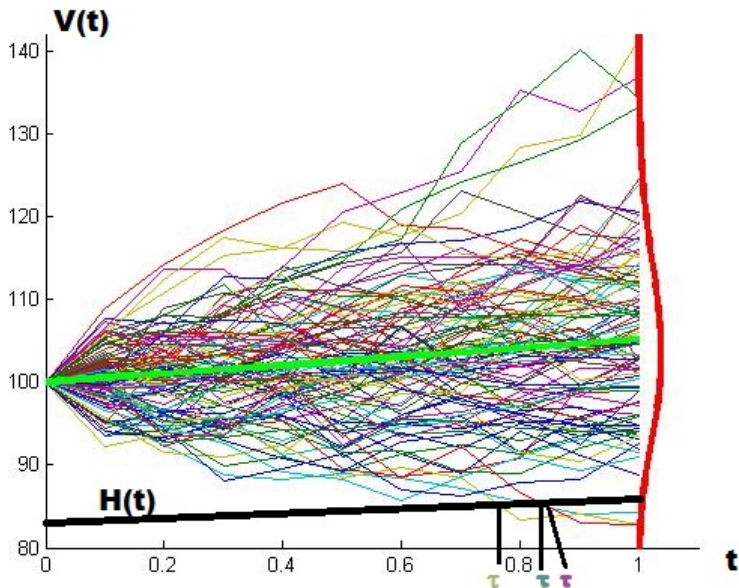
CDS Calibration with the structural model I

AT1P can calibrate exactly CDS market quotes, and the survival probabilities obtained are in accordance with those obtained using an intensity model (as should be, as they are model independent)

We now go back to the Lehman example and look at the last two columns.

We also find:

- *Scarce relevance of the barrier in calibration: the barrier parameter H has been fixed before calibration and everything is left to the volatility calibration;*
- *High discrepancy between first volatility bucket and the following values (related to no short term credit spreads).*



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Recovery at 40%! Actual one will be $\approx 8\%$.

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Comments and Generalizations I

In AT1P, H is known. To take market uncertainty into account, H can be replaced by a random variable assuming different values in different scenarios (SBTV model).

This has been used successfully and with more stable outputs than in AT1P for Parmalat and Lehman for example. See

<http://arxiv.org/abs/0912.4404>

also in: Brigo, D., Morini, M., and Tarengi, M. (2011). Credit Calibration with Structural Models and Equity Return Swap valuation under Counterparty Risk. In: Bielecki, Brigo and Patras (Editors), Credit Risk Frontiers: Sub- prime crisis, Pricing and Hedging, CVA, MBS, Ratings and Liquidity, Wiley.

We now present a brief introduction to the problems caused in the market by multi-name credit products in 2007-2008.

The Credit Crisis: Is this Mathematics fault?

Quantitative Analysts ("quants") and Academics guilty?

Recent past: articles disputed role of Mathematics in Finance, especially in relationship with Counterparty Credit Risk and Credit Derivatives (esp. CDOs). Quants accused of being unaware of models limitations & of providing the market with false sense of security.

- "The formula that killed Wall Street"²
- "The formula that fell Wall Street"³
- "Wall Street Math Wizards forgot a few variables"⁴
- "Misplaced reliance on sophisticated (mathematical) models"⁵
- BUT WHAT ARE THIS FORMULA and CDOs PRECISELY?

²Recipe for disaster. Wired Magazine, 17.03.

³The Financial Times, Jones, S. (2009). April 24 2009.

⁴Lohr (2009), New York Times of September 12.

⁵Turner, J.A. (2009). The Turner Review. 03/2009. FSA, UK.

CDS Index and tranches (iTraxx, CDX...) I

DJ-iTRAXX is a family of CDS indices, which spans the main credit market in Europe.

This family was created with the purpose to standardize market quotes, and also to create a reference liquid multi-name credit derivative. These indices constitute now the most liquid quotations in the credit-derivatives market.

Since the quotation paradigm is standardized (we see the details below) the index tranche reference quotes are practically the only safe source of market cross-sectional default correlation.

There are also indices for different areas. We can find indices relative to Europe, the US (CDX), Japan, Asia, Australia, high yields and emerging markets.

CDS Index and tranches (iTraxx, CDX...) II

The credit indices are constructed in order to provide exposure to the most liquid segments of the credit markets. This is achieved by **selecting the most liquid CDS** in the market and **equally weighting them in the index**

Each index is subject to regular rebalancing every 6 months in March and September. Rebalancing follows the same rules as the initial composition of the indices.

Here we focus on the main index and its tranches.

- **DJ iTraxx Benchmark (“main index”)** - Top 125 European names in terms of the CDS volumes traded by the market makers in the past six months. Dealer liquidity poll every six months. Sectorial diversification; 3,5,7 and 10y maturities

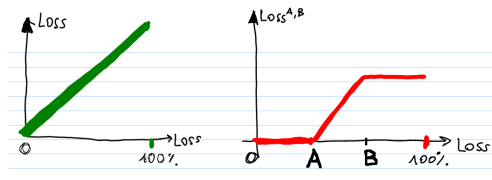
CDS Index and tranches (iTraxx, CDX...) III

DJ iTraxx Europe can easily be used as a simple and cheap instrument to trade the general direction of credit spreads. It has a number of advantages:

- Immediate diversification. DJ iTraxx Europe enables the investor to gain immediate diversification in a single liquid transaction.
- Accurate market tracking. The inclusion of only the most liquid names and the fact that these are updated every six months, ensures that DJ iTraxx Europe accurately reflects the composition of the European credit market.
- Low bid/offer spread compared to single names.
- High liquidity. The large number of market makers ensures that investors can trade large sizes without affecting the market.

CDS Index and tranches (iTraxx, CDX...) IV

The index can be traded also in terms of tranches (red graph).



DJ-iTraxx Europe: **Equity tranche**, responsible for all losses between $A = 0\%$ and $B = 3\%$, then other **mezzanine** and **senior** tranches

$A - B$: 3% – 6%, 6% – 9%, 9% – 12%, 12% – 22%, 22% – 100%.

For the main US index, the DJ CDX NA the tranche sizes are different:

0% – 3%, 3% – 7%, 7% – 10%, 10% – 15% 15% – 30%, 30% – 100%.

During the crisis, the need to hedge systemic risk brought about the supersenior tranche 60%-100%. It has traded at 24bps.

CDS index (CDO) tranche trade example I

Example: Investor sells EUR 10mn protection on the 3%-6% tranche. We assume a credit spread $R^{A,B}$ of 135bp. Therefore, the market maker pays the investor 135bp per annum quarterly on a notional of EUR 10mn. We assume (!) $R_{EC} = 40\%$ for all names ($L_{GD} = 0.6$).

- Each single name in the portfolio has a credit position in the index of $1/125 = 0.8\%$ and participates to the aggregate loss in terms of $0.8\% \times L_{GD} = 0.8\% \times 0.6 = 0.48\%$.
- This means that each default corresponds to a loss of 0.48% in the global portfolio.
- After 6 defaults, the total loss in the portfolio is EUR $0.48\% \times 6 = 2.88\%$, and the tranche buyer is still protected.
- When the 7th name in the pool defaults the total loss amounts to 3.36% and the lower attachment point of the tranche is reached.

CDS index (CDO) tranche trade example II

- To compute loss of the tranche we have to normalize total loss wrt the tranche size: The net loss is then $(3.36\% - 3\%) / 3\% \times 10\text{mn} = \text{EUR } 1.2\text{mn}$ which is immediately paid by the protection seller.
- The notional on which the premium is paid reduces to $10\text{mn} - 1.2\text{mn} = \text{EUR } 8.8\text{mn}$, and the investor receives monthly a 135bp premium on EUR 8.8mn until maturity or the next default.
- Each following default leads to change in the tranche loss (paid by the protection seller) of $0.48\% / 3\% \times 10\text{mn} = \text{EUR } 1.6\text{mn}$, and the tranche notional decreases correspondingly.
- After the 13th default the total loss exceeds 6% ($13 \times 0.48\% = 6.24\%$) and the tranche is completely wiped out.
- In this case one last payment is made of $(6\% - 5.76\%) / 3\% \times 10\text{mn} = \text{EUR } 0.8\text{mn}$ to the protection buyer, which in turn stops paying the premium since the outstanding notional has reduced to zero.

CDS Indices: DJ-iTRAXX tranches. Quotes examples

3y protection maturity

Date	Index	0-3% (upfront)	3%-6%	6%-9%
Oct-05	21	4.8%	20	7
May-06	18	0.5%	3.5	1.1

5y protection maturity

Date	Index	0-3% (upfront)	3%-6%	6%-9%	9%-12%	12%-22%
Oct-05	36	28%	91	28	12	6.5
May-06	32	24%	69	21	9	4.3

10y protection maturity

Date	Index	0-3% (upfront)	3%-6%	6%-9%	9%-12%	12%-22%
Oct-05	56	57%	483	102	49	21
May-06	54	52%	553	131	62	24

Quotes in bps;

0-3% equity tranche is quoted upfront + 500bps running

CDOs: The standard synthetic case I

- Let's look at this more carefully. Portfolio of say $N = 125$ names (eg iTraxx Europe or CDX US). Names may default at random times τ_1, \dots, τ_N , generating losses.
- A tranche is a portion of the loss between two percentages. The 3% – 6% tranche focuses on the losses between $A = 3\%$ (attachment point) and $B = 6\%$ (detachment point).
- The CDO protection seller agrees to pay to the buyer all notional default losses (minus the recoveries) in the portfolio whenever they occur due to one or more defaults, within $A = 3\%$ and $B = 6\%$ of the total pool loss.
- In exchange for this, the buyer pays the seller a periodic fee on the notional given by the portion of the tranche that is still “alive” in each relevant period.
- Valuation problem: What is the fair price of this “insurance”?

CDOs: Copula I

$$\text{Loss}(T) = \frac{1}{N} \sum_{i=1}^N (1 - \text{REC}_i) 1_{\{\tau_i \leq T\}},$$

$$\text{Loss}_{A,B}^{\text{tr}}(t) := \frac{1}{B-A} \left[(\text{Loss}(t) - A) 1_{\{A < \text{Loss}(t) \leq B\}} + (B - A) 1_{\{\text{Loss}(t) > B\}} \right].$$

(single name default times τ_i enter **index** & **tranche** payoff). Industry:

$$\tau_1 = \Lambda_1^{-1}(-\ln(1 - \Phi(X_1))), \dots, \tau_N = \Lambda_N^{-1}(-\ln(1 - \Phi(X_N))).$$

where intensities Λ_i are DETERMINISTIC (no spread volatility...BAD!), X_i 's are correlated std Gaussians (we are under pricing measure \mathbb{Q})

$$X_i = \sqrt{\rho_i} M + \sqrt{1 - \rho_i} Y_i, \quad (21)$$

with M common systemic factor, M, Y_i i.i.d standard Gaussian. Industry assumes all ρ are the same across the pool.

$$\rho_i = \rho \geq 0 \text{ for all } i \Rightarrow \text{Corr}(X_i, X_j) = \rho \text{ for all } i, j$$

Tranches and Correlations

The dependence of the tranche on “correlation” is crucial. The market assumes a Gaussian Copula connecting the defaults of the 125 names, parametrized by a correlation matrix with $125 \times 124 / 2 = 7750$ entries. However, when looking at a tranche:

7750 parameters \longrightarrow 1 parameter.

How does the loss change with correlation?

Tranches and Correlations

Example. Take 125 names with same spread $100\text{bps} = 1\%$, assume recovery is 0.4. We now show the two loss distributions in 5 years for the two extreme values of ρ : 0 and 1.

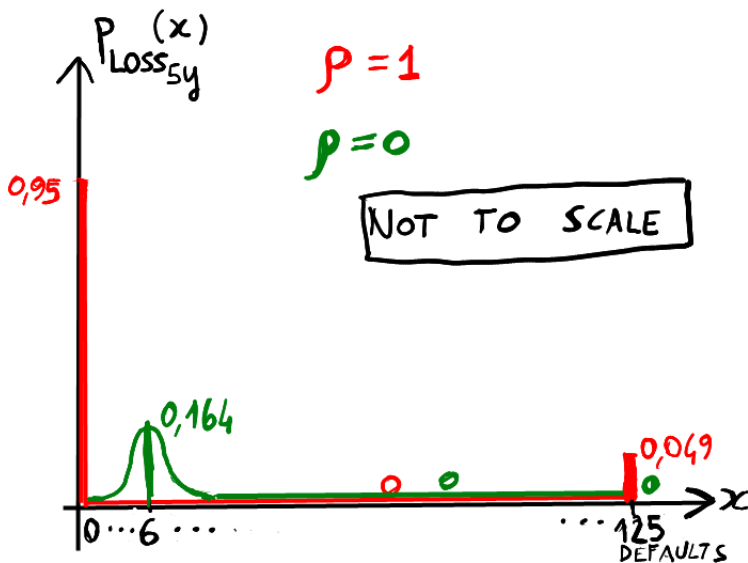
$$\rho = 1 \Rightarrow \mathbb{Q}(\text{Loss}_{5y} = 0\%) = 0.951, \quad \mathbb{Q}(\text{Loss}_{5y} = 100\%) = 0.049,$$

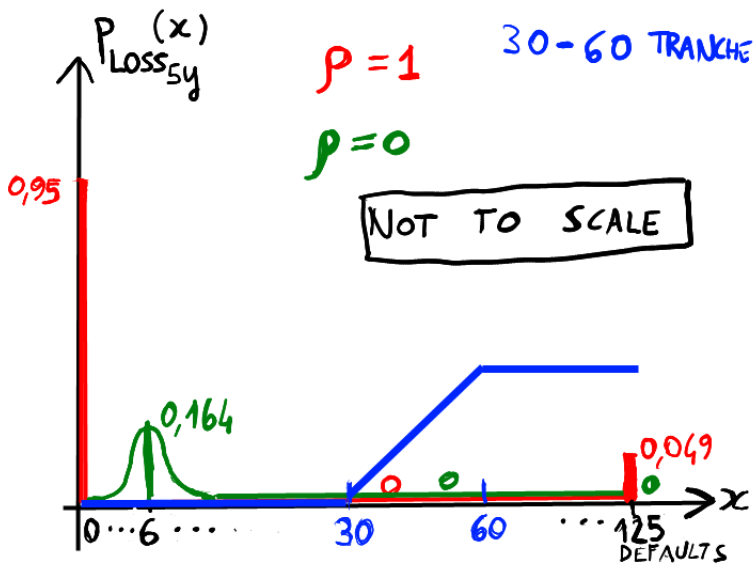
and all other loss values have zero probability. Extreme bimodal case: either all default, or no one does. No other case.

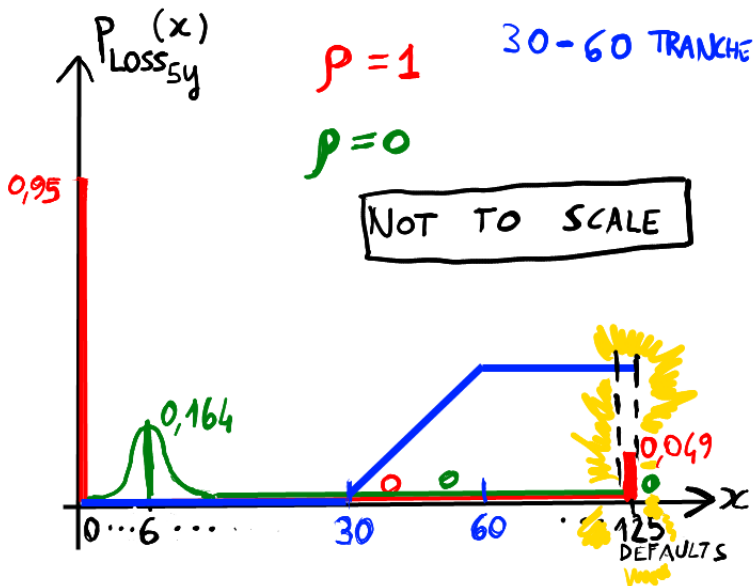
$$\rho = 0 \Rightarrow \mathbb{Q}(\text{Loss}_{5y} = 0.6 \frac{n}{125} \%) = \binom{125}{n} 0.049^n 0.951^{125-n},$$

loss distribution smooth and bell shaped, almost normal with mean number of defaults 6.125 and standard dev 2.41.

$$\rho = 0 \Rightarrow \mathbb{Q}(\text{Loss}_{5y} = 0\%) \approx 0.002, \quad \mathbb{Q}(\text{Loss}_{5y} = 100\%) \approx 2 \cdot 10^{-164}.$$







Tranches and Correlations I

With correlation zero, the 30-60 tranche has positive payoff in states where probability is always zero (green distribution).

The expected tranching loss is in this case is zero because probability is always zero.

With correlation one, the armageddon state of 125 defaults (total pool loss) has a positive probability of 0.049. The only other possible state is a loss of zero. The expected tranching loss:

$$60 * (1 - Rec) * 0.049 = 60 * 0.6 * 0.049 = 1.764$$

$\rho = 1$: every tranche with the same $B - A$ has the same price irrespective of A and B. Loss jumps directly at 125 defaults wiping off all tranches equally!

Tranches and Correlations II

$\rho \downarrow \Rightarrow$ equity tranche protection cost \uparrow , senior tranche protection cost \downarrow , mezzanine mixed.

Protection buyer in equity (senior) tranche is short (long) correlation.

HOWEVER this presupposes a flat correlation that is constant across the pool. The mini-crisis of 2005 showed that this notion of long and short correlation can be quite dangerous. The loss distribution is much more structured, as we'll see in a minute.

Even in our simple example however the cost of protection for 30-60 names is very different in the two cases: it is zero with $\rho = 0$ and 1.764 with $\rho = 1$ (interest rates 0).

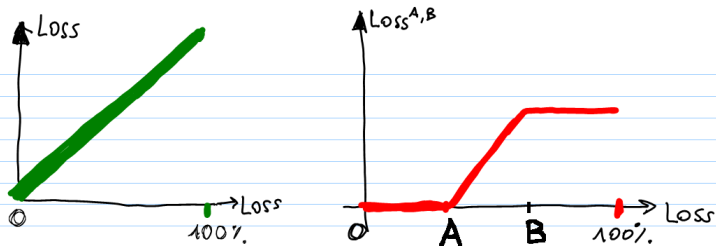
Tranches and Correlations

This model is applied in a very blunt way in the market.

The unique parameter ρ is reverse-engineered to reproduce the price of the liquid tranche under examination. "Implied correlation". Once obtained it is used to value related products.

Problem with this implied "compound correlation"

If at a given time the 3% – 6% tranche for a five year maturity has a given implied correlation, the 6% – 9% tranche for the same maturity will have a different one. The two tranches on the *same pool* are priced (and hedged!!!) with two inconsistent loss distributions



- Pricing (marking to market) a tranche: taking \mathbb{Q} expectation of the future tranche losses under the pricing measure.
- From nonlinearity, the tranche expectation will depend on the loss distribution: marginal distributions of the single names defaults **and** dependency among different names' defaults. Dependency is commonly called "correlation".
- Abuse of language: correlation is a complete description of dependence for jointly Gaussians, but more generally it is not.

Copulas

CDO Valuation: The culprit.

One-factor Gaussian copula

$$X_i = \sqrt{\rho_i} M + \sqrt{1 - \rho_i} Y_i$$

“MEA COPULA!” From Nobel award to universal scapegoat

Introduced in Credit Risk modeling by David X. Li. Commentators went from suggesting a Nobel award to blaming Li for the whole Crisis.

David Li, 2005, Wall Street Journal

[...] “The most dangerous part,” Mr. Li himself says of the model, “is when people believe everything coming out of it.” Investors who put too much trust in it or don’t understand all its subtleties may think they’ve eliminated their risks when they haven’t.

Indeed, these models are static. they ignore Credit Spread Volatilities, that in Credit can be 100%; this has paradoxical consequences

Tranches and Correlations

The dependence of the tranche on “correlation” is crucial. The market assumes a Gaussian Copula connecting the defaults of the 125 names, parametrized by a correlation matrix with $125 \times 124 / 2 = 7750$ entries. However, when looking at a tranche:

7750 parameters $\rho_{i,j}$ (or 125 ρ_i with 1-factor model) \longrightarrow 1 param $\bar{\rho}$

The unique parameter is reverse-engineered to reproduce the price of the liquid tranche under examination. “Implied correlation”. Once obtained it is used to value related products.

Problem with this implied “compound correlation”

If at a given time the 3% – 6% tranche for a five year maturity has a given implied correlation $\bar{\rho}_{3\%,6\%}$, the 6% – 9% tranche for the same maturity will have a different one $\bar{\rho}_{6\%,9\%}$. The two tranches on the *same pool* are priced (and hedged!!!) with two inconsistent loss distributions

09/06/2010

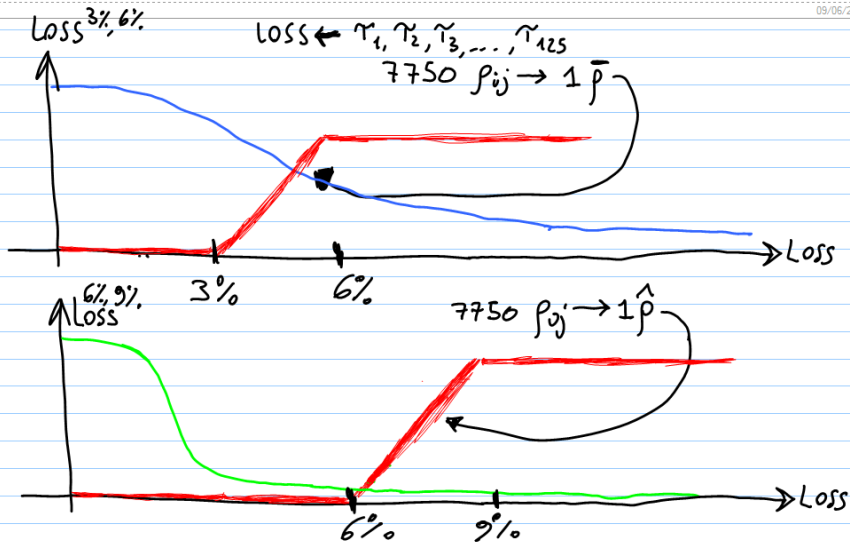


Figure: Compound correlation inconsistency

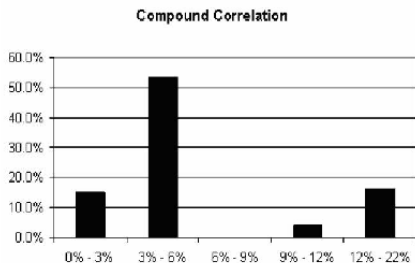


Figure: Plot $[A\%, B\%] \mapsto \bar{\rho}_{[A\%, B\%]}$ (After Edvard Munch's The Scream; Compound correlation DJ-iTraxx S5, 10y on 3 Aug 2005)

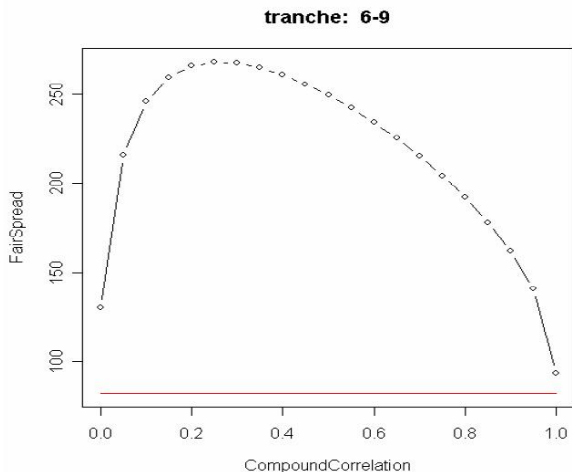


Figure: Non-invertibility compound correl DJ-iTraxx S5, 10y on 3 Aug 2005
 (red line is the market, curved line is the model for all possible $\bar{\rho}_{[6\%,9\%)}$)

Base correlation I

As a possible remedy for non-invertibility of compound correlation and other matters, the market introduced Base Correlation, $\tilde{\rho}_{B\%} = \bar{\rho}_{[0\%,B\%]}$ which is still prevailing in the market.

Problems with base correlation

Base correlation is easier to interpolate but is inconsistent even at single tranche level, in that it prices the 3% – 6% tranche by decomposing it into the 0% – 3% tranche and 0% – 6% tranche and using two different correlations (and hence distributions) for those. This inconsistency shows up occasionally in negative losses (i.e. in defaulted names resurrecting).

[in the graph we use put-call parity to simplify]

Base correlation II

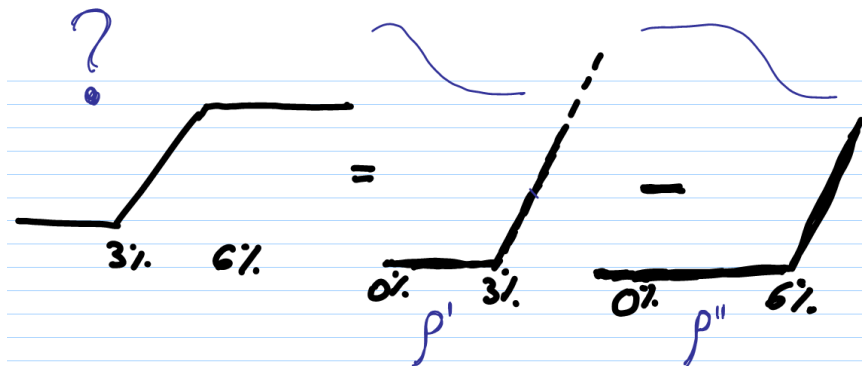


Figure: Base correlation inconsistency

Base correlation III

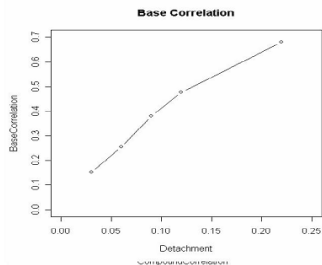
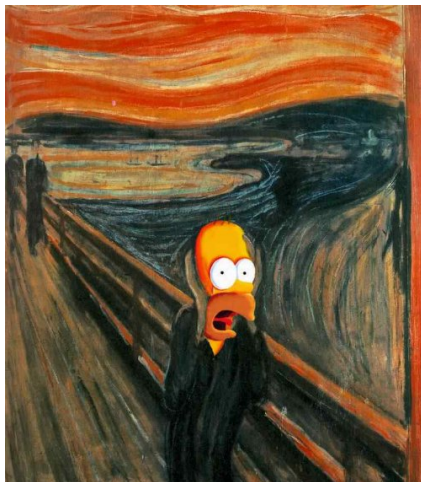


Figure: (Base correl $B\% \mapsto \tilde{\rho}_{B\%}$ for DJ-iTraxx S5, 10y on 3 Aug 2005)

Base correlation

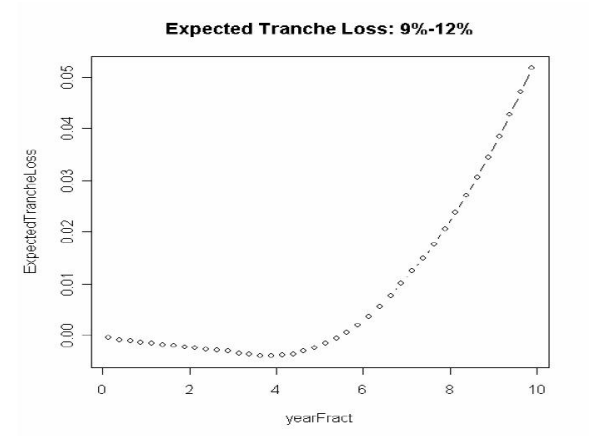


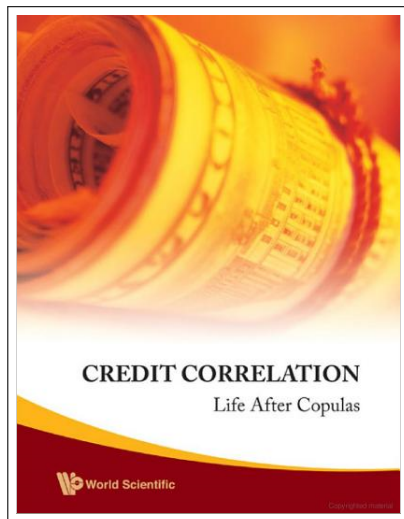
Figure: $T \mapsto \mathbb{E}[\text{Loss}_{[6\%,12\%]}^{tr}(T)]$: Expected tranche loss coming from Base correl calibration, 3/08/2005, 1st published 2006. Locally negative loss distrib \Rightarrow defaulted names RESURRECT with positive probability

Press: Nobody knew? Some facts

Proceedings of a Conference held in London in 2006 by Merrill Lynch.

A number of proposals to improve the static copula models used (and abused) for credit derivatives have been presented. I was there.

Quants and Academics were well aware (and had been for years) of the models limitations and were trying to overcome them.



A few journalist have very short memory...

12 Sept 2005. None of the commentators above was reading WSJ?
How a Formula [Base correlation + Gaussian Copula] Ignited Market
That Burned Some Big Investors (Wall St Journal).

Many other publications pre-2007 questioning the use of the Gaussian copula and the notion of implied and base correlation. For example:

B., Pallavicini & Torresetti. Implied Correlation: A paradigm to be handled with care, 2006, <http://ssrn.com/abstract=946755>

Now we turn to one of the solutions in 2006, beyond static and flat implied correlation models.

Top Down Dynamic Loss Model

Models that build the Loss from τ_1, \dots, τ_N are called bottom up. We now present a 2006 Top Down model that models the Loss directly as a stochastic process (under \mathbb{Q}) with sensible properties.

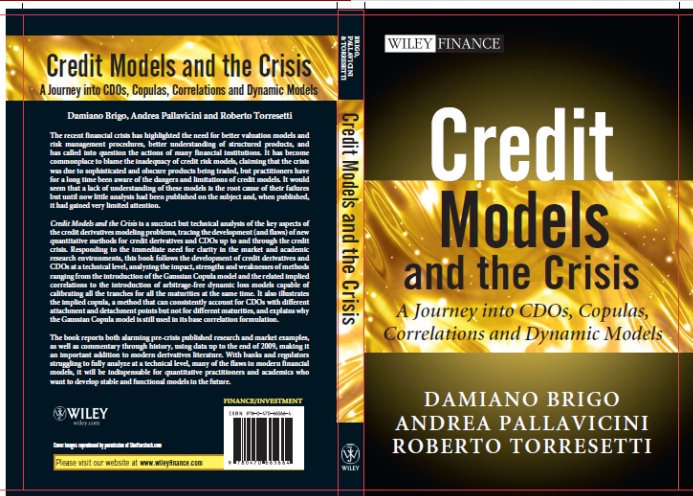


Figure: This book collects research published originally in 2006, warning against the flaws of the industry credit derivatives models. Related papers in the journals *Mathematical Finance*, *Risk Magazine*, *IJTA*

Beyond copulas: GPL and GPCL Models (2006-on)

We model the total number of defaults in the pool by t as

$$Z_t := \sum_{j=1}^n \delta_j Z_j(t)$$

(for integers δ_j) where Z_j are independent Poissons with intensity Λ^j . This is consistent with the Common Poisson Shock framework, where defaults are linked by a Marshall Olkin copula (Lindskog and McNeil).

Example : $n = 125$, $Z_t = 1 Z_1(t) + 2 Z_2(t) + \dots + 125 Z_{125}(t)$.

If Z_1 jumps there is just one default (idiosyncratic), if Z_{125} jumps there are 125 ones and the whole pool defaults one shot (total systemic risk), otherwise for other Z_i 's we have intermediate situations (sectors).

The GPL and GPCL Models: Default clusters?

- Thrifts in the early 90s at the height of the loan and deposit crisis.
- Airlines after 2001.
- Autos and financials more recently. From the September, 7 2008 to the October, 8 2008, we witnessed seven credit events: Fannie Mae, Freddie Mac, Lehman Brothers, Washington Mutual, Landsbanki, Glitnir, Kaupthing.

S&P ratings and default clusters

Moreover, S&P issued a request for comments related to changes in the rating criteria of corporate CDO. Tranches rated 'AAA' should be able to withstand the default of the largest single industry in the pool with zero recoveries. Stressed but plausible scenario that a cluster of defaults in the objective measure exists.

The GPL and GPCL Models

Problem: infinite defaults. Solution 1: **GPL**: Modify the aggregated pool default counting process so that this does not exceed the number of names, by simply capping Z_t to n , regardless of cluster structures:

$$C_t := \min(Z_t, n)$$

Solution 2: **GPCL**. Force clusters to jump only once and deduce single names defaults consistently.

The first choice is ok at top level but it does not really go down towards single names. The second choice is a real top down model, but combinatorially more complex.

Market data on March 6, 2006

Tranche data and DJi-TRAXX fixings, along with bid-ask spreads (I=index, T=Tranche, TI=Tranchelet). Focus on March 6. Base correlation would require 15 different inconsistent static models $\tilde{\rho}_{B\%}$ to explain this table. GPL will be a single, consistent, arbitrage-free dynamic model.

	Att-Det [A%,B%]	March, 1 2006		March, 6 2006		
		5y	7y	3y	5y	7y
I		35(1)	48(1)	20(1)	35(1)	48(1)
T	0-3	2600(50)	4788(50)	500(20)	2655(25)	4825(25)
	3-6	71.00(2.00)	210.00(5.00)	7.50(2.50)	67.50(1.00)	225.50(2.50)
	6-9	22.00(2.00)	49.00(2.00)	1.25(0.75)	22.00(1.00)	51.00(1.00)
	9-12	10.00(2.00)	29.00(2.00)	0.50(0.25)	10.50(1.00)	28.50(1.00)
	12-22	4.25(1.00)	11.00(1.00)	0.15(0.05)	4.50(0.50)	10.25(0.50)
TI	0-1	6100(200)	7400(300)			
	1-2	1085(70)	5025(300)			
	2-3	393(45)	850(60)			

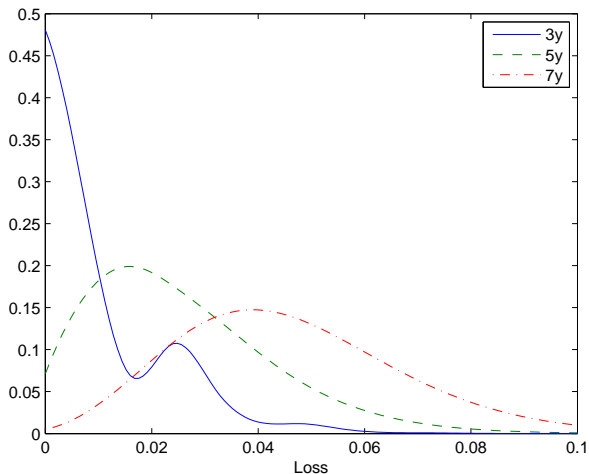
Calibration: All standard tranches up to seven years

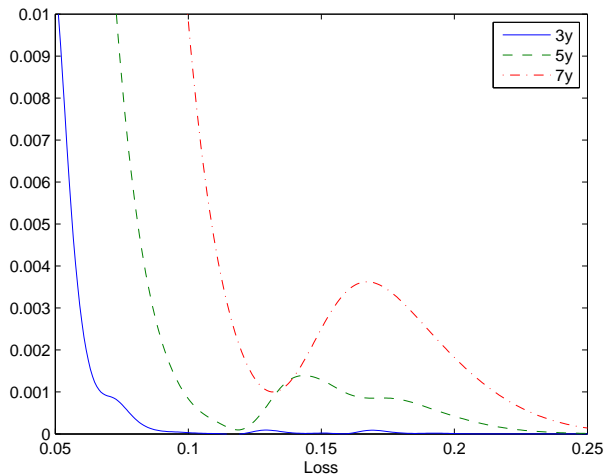
As a first calibration example we consider standard DJi-TRAXX tranches up to a maturity of 7y with constant recovery rate of 40%. The calibration procedure selects five Poisson processes. The 18 market quotes used by the calibration procedure are almost perfectly recovered. In particular all instruments are calibrated within the bid-ask spread (we show the ratio calibration error / bid ask spread).

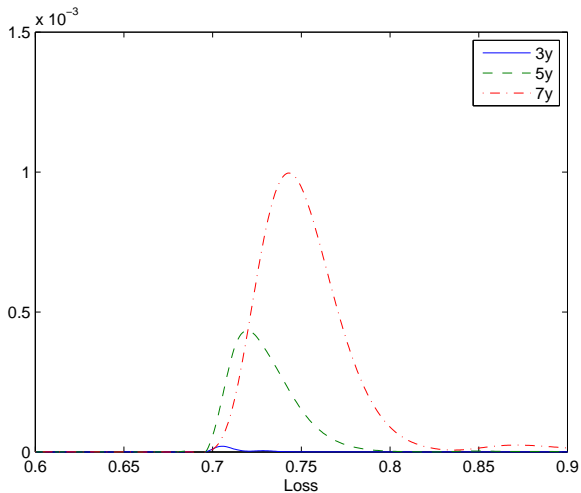
Calibr Error	Att-Det [A%,B%]	Maturities		
		3y	5y	7y
Index		-0.4	-0.2	-0.9
Tranche	0-3	0.1	0.0	-0.7
	3-6	0.0	0.0	0.7
	6-9	0.0	0.0	-0.2
	9-12	0.0	0.0	0.0
	12-22	0.0	0.0	0.2

δ	GPL $\Lambda(T)$		
	3y	5y	7y
1	0.535	2.366	4.930
3	0.197	0.266	0.267
16	0.000	0.007	0.024
21	0.000	0.003	0.003
88	0.000	0.002	0.007

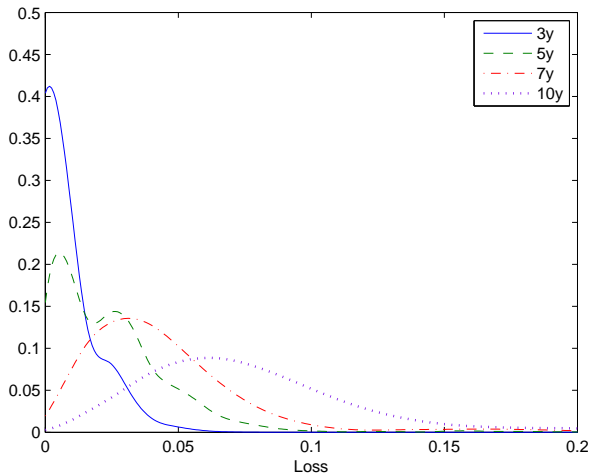
Loss distribution of the calibrated GPL model at different times



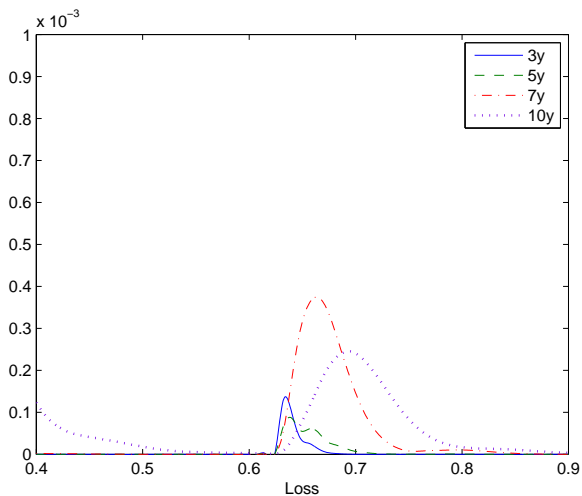




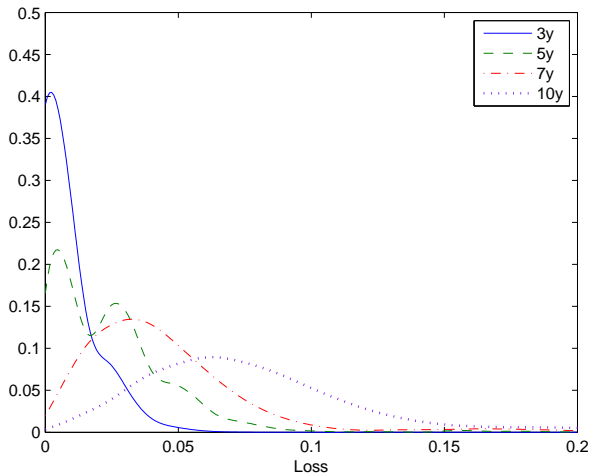
October 2 2006, GPL, Calibration up to 10y



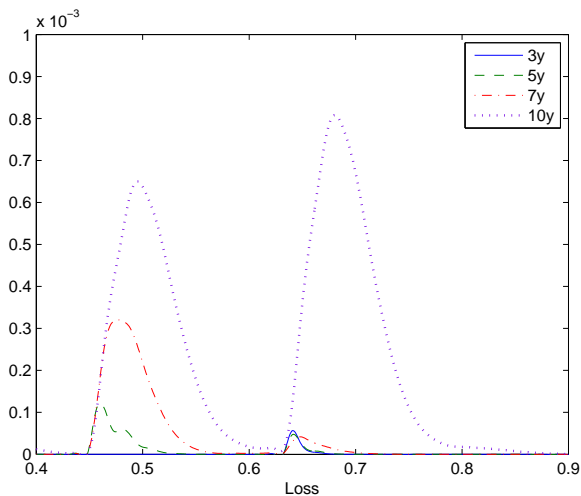
October 2 2006, GPL tail



October 2 2006, GPCL, Calibration up to 10y



October 2 2006, GPCL tail



Calibration comments I

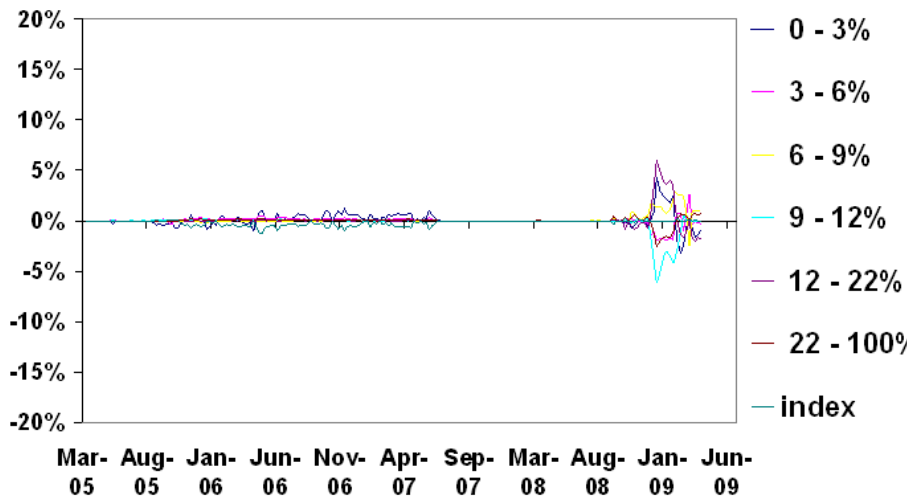
Sector / systemic calibration:

Notice the large portion of mass concentrated near the origin, the subsequent modes (default clusters) when moving along the loss distribution for increasing values, and the bumps in the far tail. Modes in the tail represent risk of default for large sectors. This is systemic risk as perceived by the dynamical model from the CDO quotes. With the crisis these probabilities have become larger, but they were already observable pre-crisis. Difficult to get this with parametric copula models.

History of calibration in-crisis with a different parametrization (α 's fixed a priori):

Calibration comments II

iTraxx 5 year - Relative Mispricing



Loss distribution in GPL through the crisis

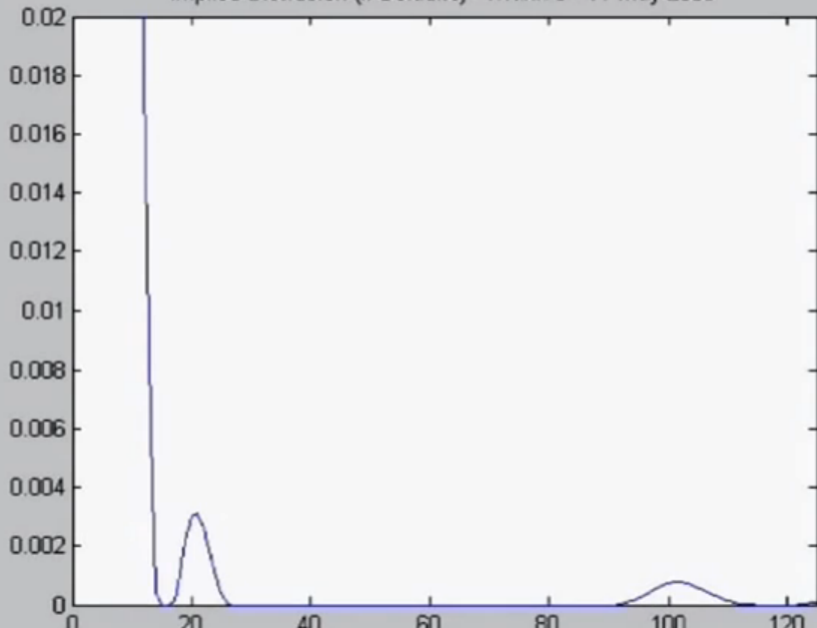
The loss distribution in the GPL model is arbitrage free, rich in structure and consistent with all market quotes, a feat impossible for implied correlation models.

The following movie shows how structured the loss dynamics can be, as highlighted by the GPL model.

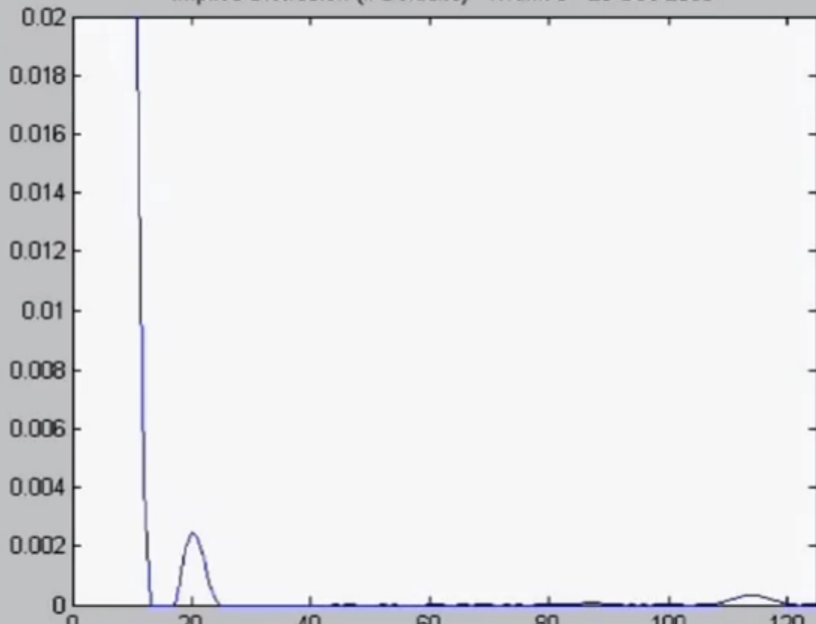
Animation showing how the loss distribution evolved in 2005+ is here

<http://www.youtube.com/watch?v=YZO-HeaGHkk&t=62m40s>

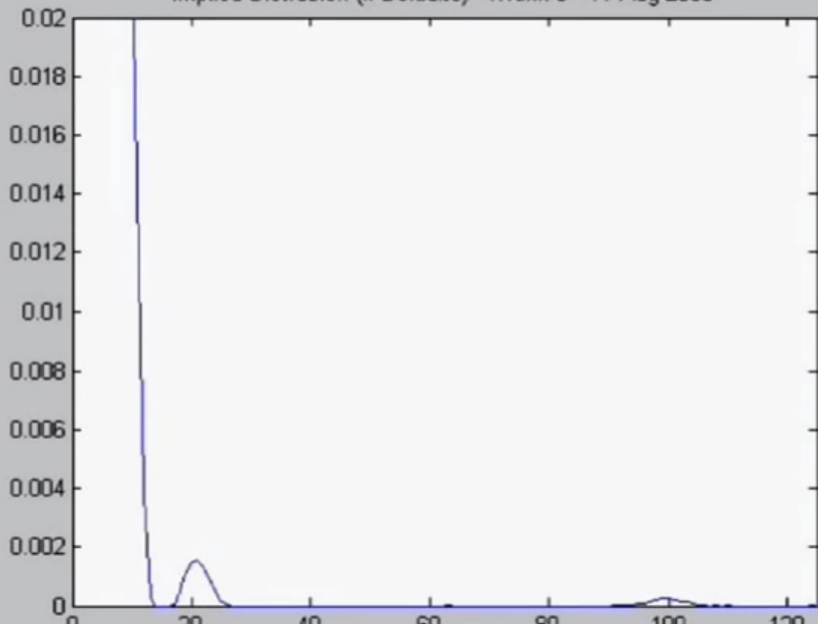
Implied Distribution (# Defaults) - iTraxx 5 - 11-May-2005



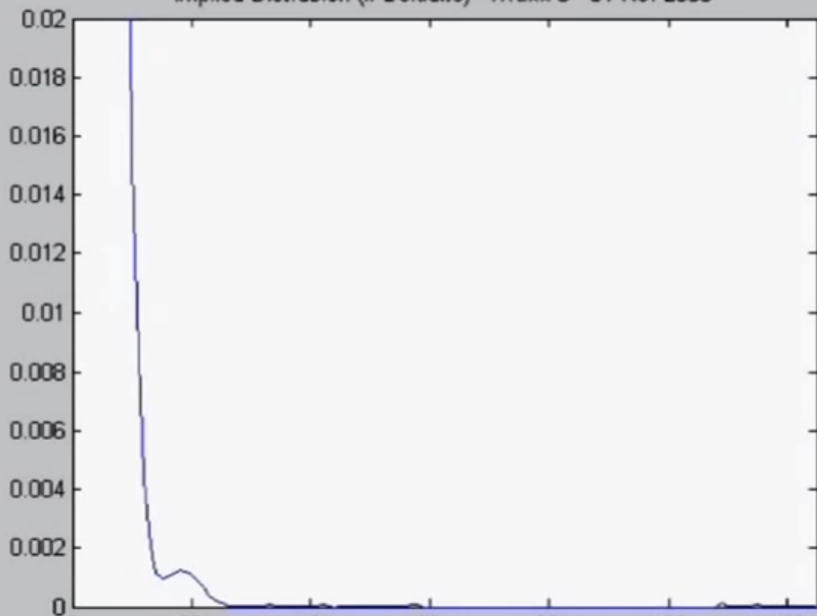
Implied Distrubion (# Defaults) - iTraxx 5 - 28-Dec-2005



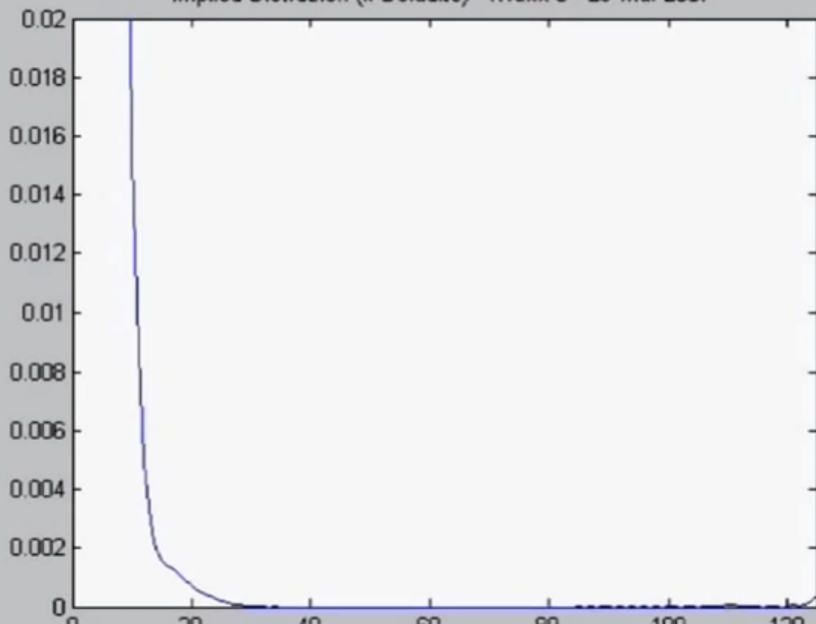
Implied Distrubion (# Defaults) - iTraxx 5 - 11-Aug-2006



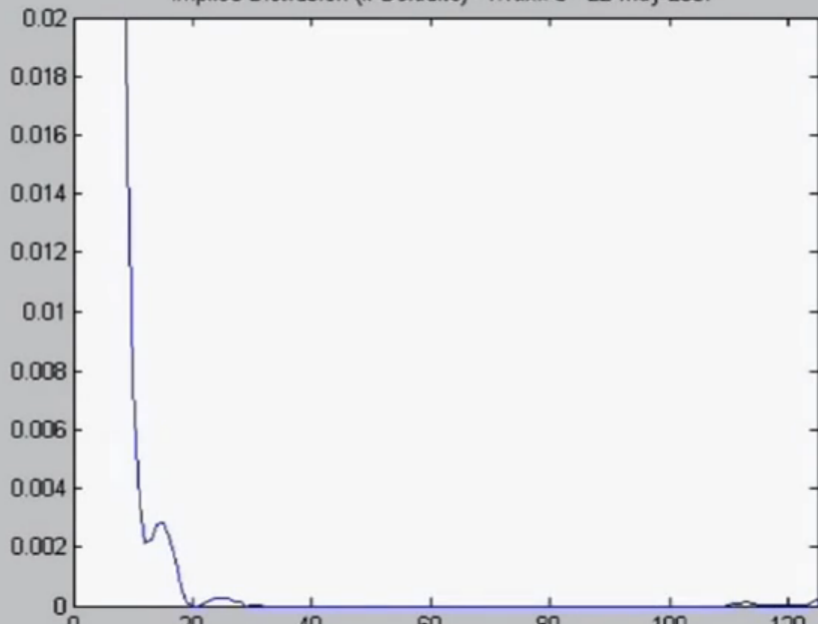
Implied Distrubion (# Defaults) - iTraxx 5 - 01-Nov-2006



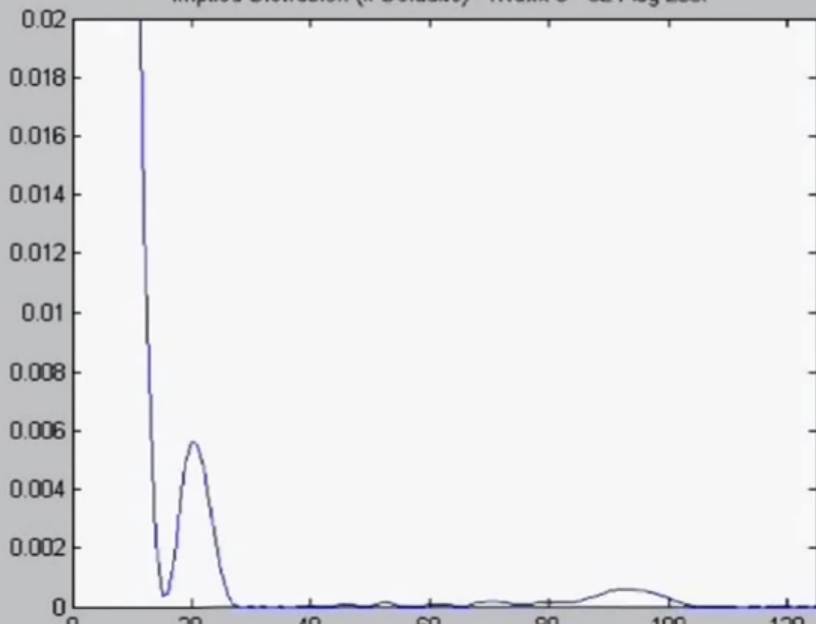
Implied Distrubion (# Defaults) - iTraxx 5 - 29-Mar-2007



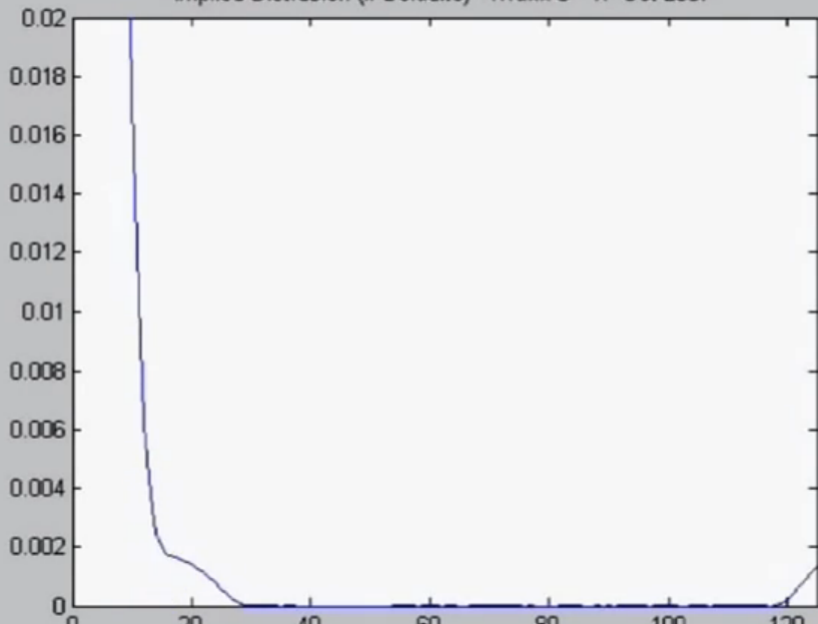
Implied Distrubion (# Defaults) - iTraxx 5 - 22-May-2007



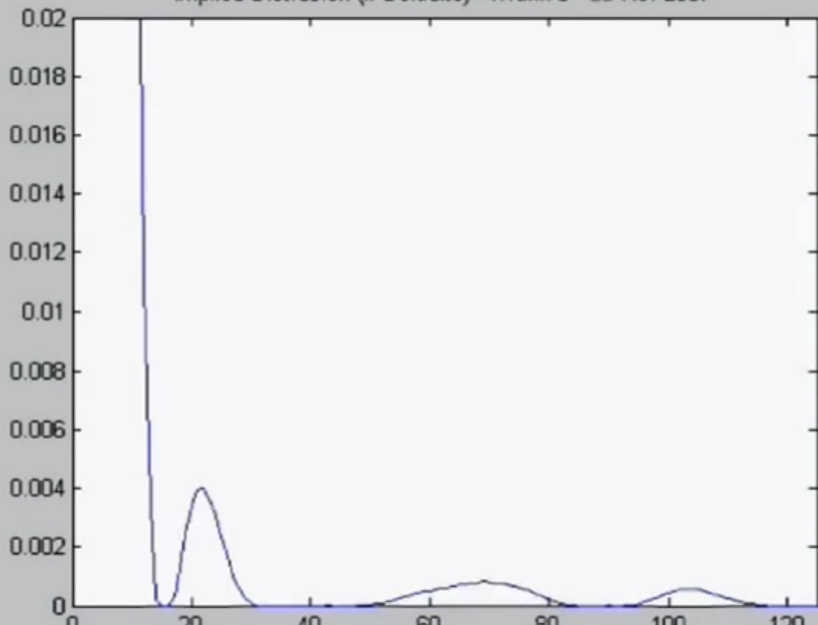
Implied Distrubion (# Defaults) - iTraxx 5 - 02-Aug-2007



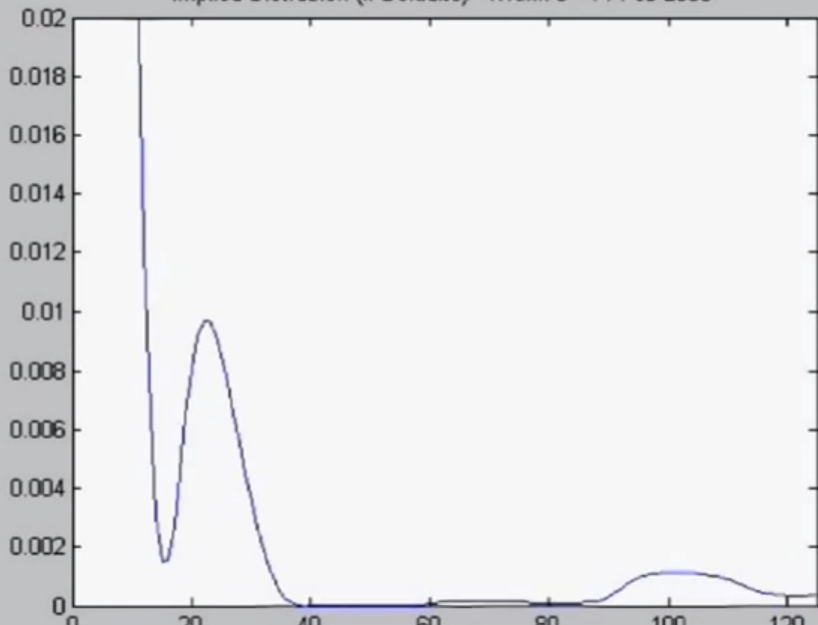
Implied Distrubion (# Defaults) - iTraxx 5 - 17-Oct-2007



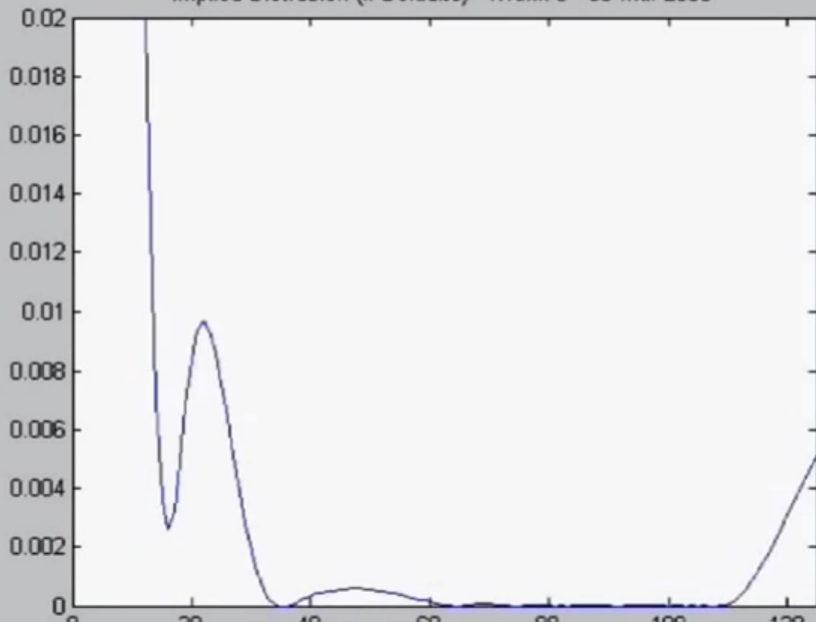
Implied Distrubion (# Defaults) - iTraxx 5 - 23-Nov-2007



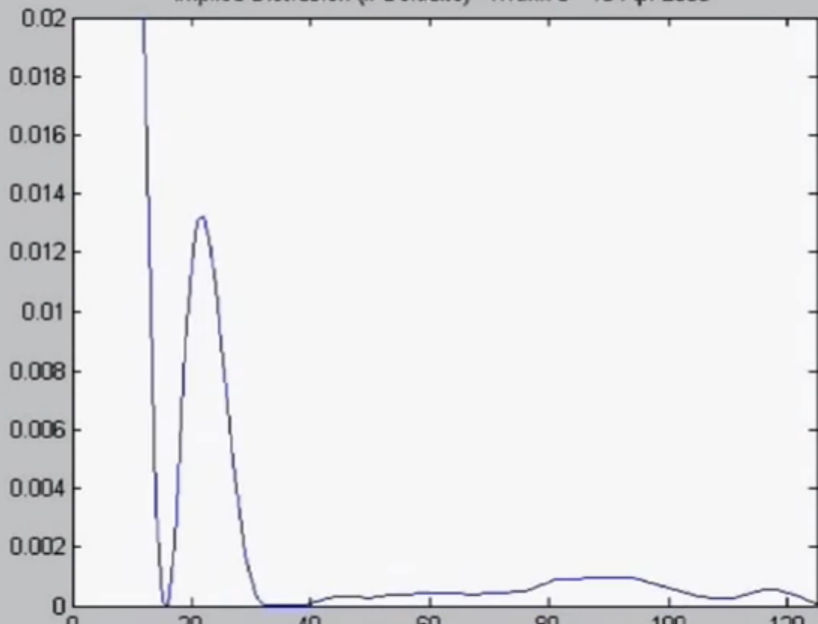
Implied Distribution (# Defaults) - iTraxx 5 - 14-Feb-2008



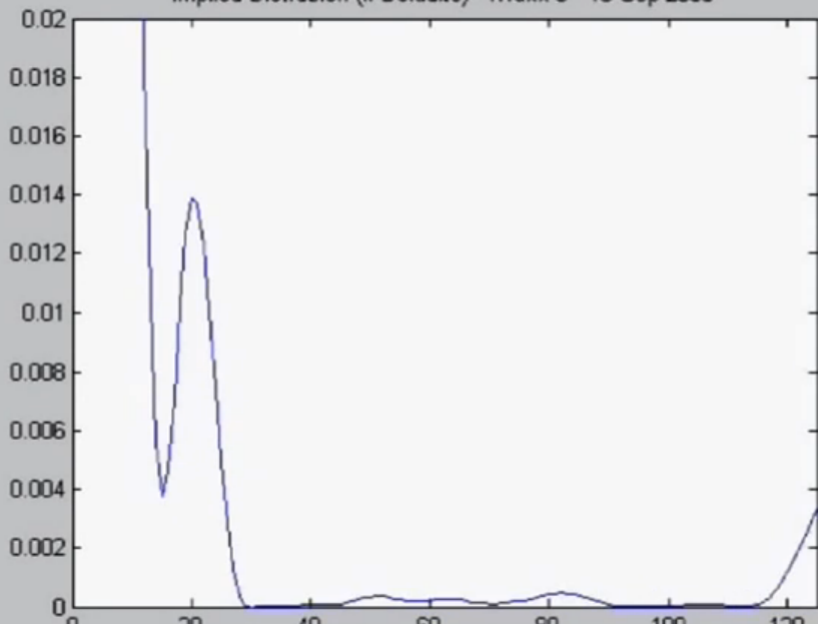
Implied Distribution (# Defaults) - iTraxx 5 - 06-Mar-2008



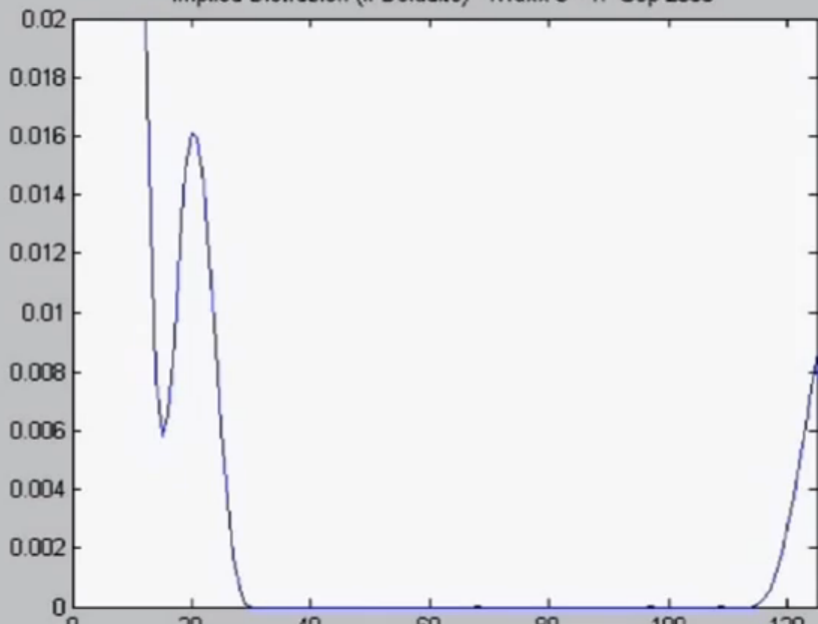
Implied Distribution (# Defaults) - iTraxx 5 - 15-Apr-2008



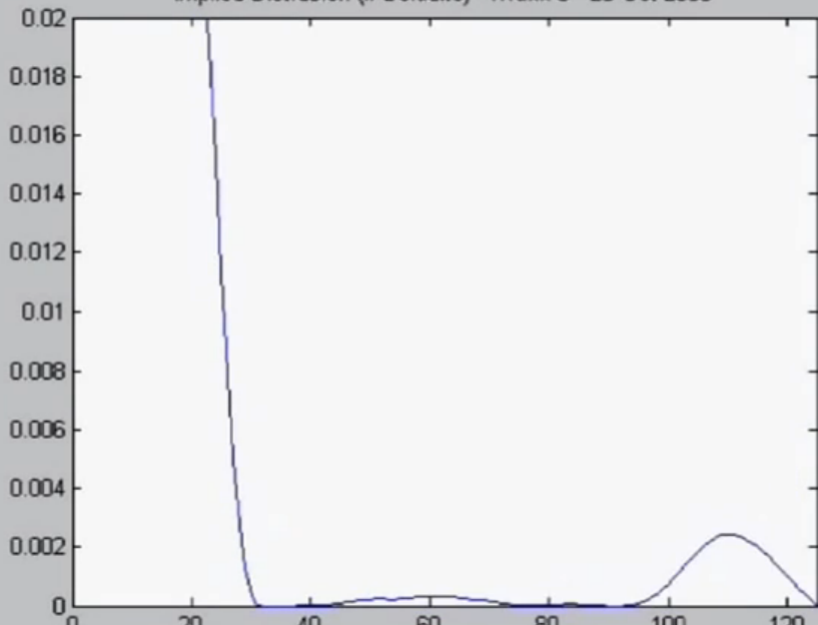
Implied Distribution (# Defaults) - iTraxx 5 - 10-Sep-2008



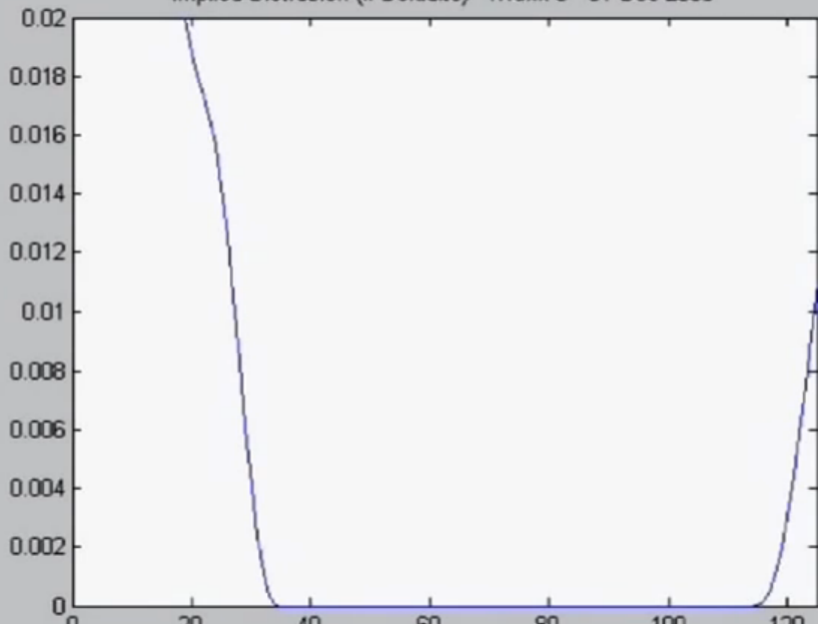
Implied Distrubion (# Defaults) - iTraxx 5 - 17-Sep-2008



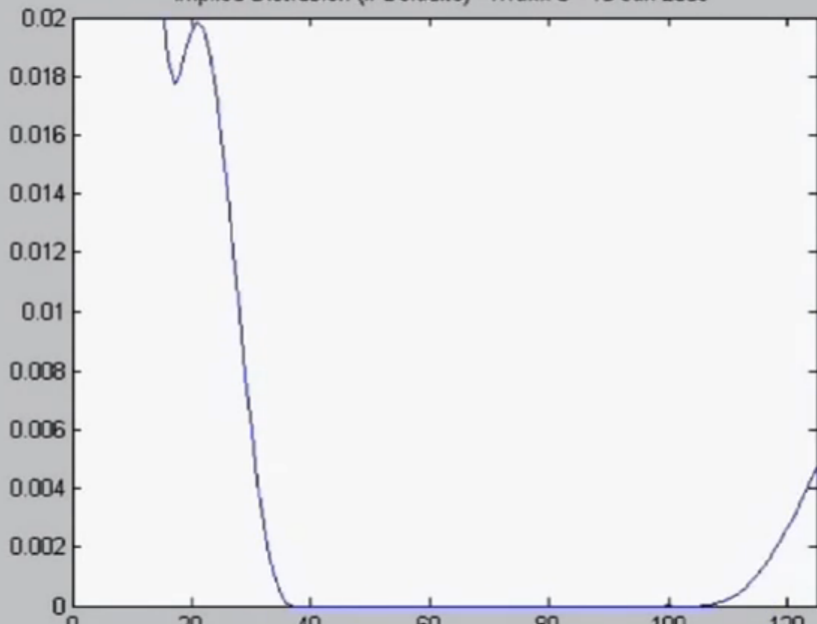
Implied Distrubion (# Defaults) - iTraxx 5 - 23-Oct-2008



Implied Distribution (# Defaults) - iTraxx 5 - 01-Dec-2008



Implied Distribution (# Defaults) - iTraxx 5 - 15-Jan-2009



The synthetic CDO case?

- We have illustrated how a complex situation in CDO markets has been trivialized by media and even regulators
- Models (such as base correl) were indeed inadequate, but...
- ... we have seen the example of the good GPL model 2006
- Industry did not adopt such models for a number of reasons: difficulty to go down to single names, IT/infrastructure inertia, slowing down business, almost death of credit correlation in 2007
- But why didn't the media pick up GPL, and especially missed the 2005 CDO crisis commentary in the Wall Street Journal?
- Losses came from Mortgage CDO's (RMBS), different products, base correl is not used there! RMBS have poor input data, below.
- Lack of rigour in a broad part of investigative journalism
- Cannot blame (even poor) modeling for policy, regulation, incentives, banking model, greedy culture, poor governance...

Is Mathematics Guilty and Wrong?

- Mathematical models are a simplification of reality, and as such, are always "wrong", even if they try to capture the salient features of the problem at hand.
- **"All models are wrong, but some models are useful"** (Prof. George E.P. Box)
- But mathematics is not wrong. The core mathematical theory behind derivatives valuation is correct...
- ... but the assumptions on which the theory is based may not reflect the real world when the market evolves over the years
- ... and input data may be inadequate for even simple models

Is Mathematics guilty?

- Although the models used in Credit Derivatives and counterparty risk have limits that have been highlighted before the crisis by several researchers, the ongoing crisis is due to factors that go well beyond any methodological inadequacy: the killer formula

$$\int_{-\infty}^{+\infty} \prod_{i=1}^{125} \Phi \left(\frac{\Phi^{-1}(1 - \exp(-\Lambda_i(T))) - \sqrt{\rho_i} m}{\sqrt{1 - \rho_i}} \right) \varphi(m) dm.$$

Versus

The Crisis:

US real estate policy, Originate to Distribute (to Hold?) system fragility, volatile monetary policies, myopic compensation and incentives system, lack of homogeneity in regulation, underestimation of liquidity risk, lack of data, fraud corrupted data...(Szegö 2009, The crash sonata in D major, JRMFI).

And what about the data?

Data and Inputs quality

For many financial products, and especially RMBS (Residential Mortgage Backed Securities), quite related to the asset class that triggered the crisis, the problem is in the data rather than in the models.

Risk of fraud

At times data for valuation in mortgages CDOs (RMBS and CDO of RMBS) can be distorted by fraud (see for example the FBI Mortgage fraud report, 2007,
www.fbi.gov/publications/fraud/mortgage_fraud07.htm).

Pricing a CDO on this underlying:



Figure: The above photos are from condos that were involved in a mortgage fraud. The appraisal described "recently renovated condominiums" to include Brazilian hardwood, granite countertops, and a value of 275,000 USD

And what about the data?

At times it is not even clear what is in the portfolio: From the offering circular of a huge RMBS (more than 300.000 mortgages)

Type of property	% of Total
Detached Bungalow	2.65%
Detached House	16.16%
Flat	13.25%
Maisonette	1.53%
Not Known	2.49%
New Property	0.02%
Other	0.21%
Semi Detached Bungalow	1.45%
Semi Detached House	27.46%
Terraced House	34.78%
Total	100.00%

Mathematics or Magic?

All this is before modeling. Models obey a simple rule that is popularly summarized by the acronym GIGO (Garbage In \rightarrow Garbage Out). As Charles Babbage (1791–1871) famously put it:



*On two occasions I have been asked,
“Pray, Mr. Babbage, if you put into the machine
wrong figures, will the right answers come out?”
I am not able rightly to apprehend
the kind of confusion of ideas
that could provoke such a question.*

Concluding: should we give up?

Models have been inadequate, but their fallacies were eclipsed by problems of policy, banking models, business culture, and data quality and availability. Should we give up modeling and mathematics in Finance, as some commentators implicitly suggested?

Interesting times. We need better models...

... as opposed to no models. We need models that account for the types of risks that had been neglected. Some such risks are nonlinear
⇒ We need to enhance consistency of models in different areas

We need to understand systemic risk, contagion, the dynamics of dependence, and how to deal with scarcity of data and data proxying...

Optimization is becoming more and more fundamental: Optimal trade execution, algorithmic trading, risk optimization...

All these areas, and many more, require quantitative input and good quantitative finance. Let's keep working and doing our best, and disregard blaming quantitative finance for failures that are more managerial, political and behavioural in nature.

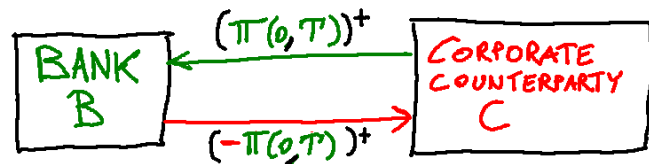
From credit derivatives to Valuation Adjustments...

This concludes our introduction to both single name and multi name credit derivatives and models.

We now turn to using such tools in one of the problems the industry is facing right now:

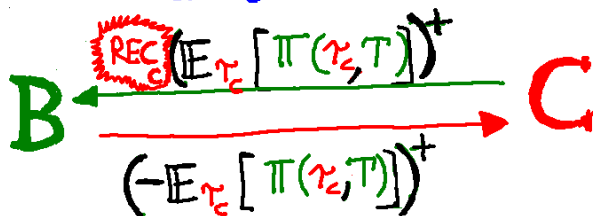
Pricing of counterparty credit risk, leading to the notion of Credit Valuation Adjustment (CVA)

Context



π :
PORTFOLIO
CASH FLOWS
TO B

τ_C
DEFAULT OF C
CLOSEOUT:



Q & A: What is Counterparty Credit Risk?

Q What is counterparty risk in general?

A *The risk taken on by an entity entering an OTC contract with a counterparty having a relevant default probability. As such, the counterparty might not respect its payment obligations.*

The counterparty credit risk is defined as the risk that the counterparty to a transaction could default before the final settlement of the transaction's cash flows. An economic loss would occur if the transactions or portfolio of transactions with the counterparty has a positive economic value at the time of default.

[Basel II, Annex IV, 2/A]

Q & A: Credit VaR and CVA

Q What is the difference between Credit VaR and CVA?

A *They are both related to credit risk.*

- *Credit VaR is a Value at Risk type measure, a Risk Measure. it measures a potential loss due to counterparty default.*
- *CVA is a price, it stands for Credit Valuation Adjustment and is a price adjustment. CVA is obtained by pricing the counterparty risk component of a deal, similarly to how one would price a credit derivative.*

Q & A: Credit VaR and CVA

Q What is the difference in practical use?

A *Credit VaR answers the question:*

- *"How much can I lose of this portfolio, within (say) one year, at a confidence level of 99%, due to default risk and exposure?"*
- *CVA instead answers the question:*
"How much discount do I get on the price of this deal due to the fact that you, my counterparty, can default? I would trade this product with a default free party. To trade it with you, who are default risky, I require a discount."

Clearly, a price needs to be more precise than a risk measure, so the techniques will be different.

Q & A: Credit VaR and CVA

Q Different? Are the methodologies for Credit VaR and CVA not similar?

A *There are analogies but CVA needs to be more precise in general. Also, Credit VaR should use statistics under the physical measure whereas CVA should use statistics under the pricing measure*

Q What are the regulatory bodies involved?

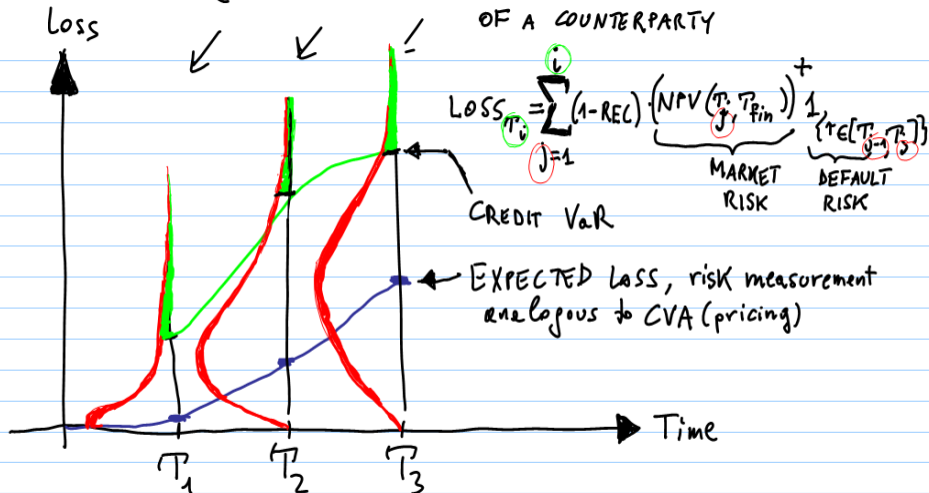
A *There are many, for Credit VaR type measures it is mostly Basel II and now III, whereas for CVA we have IAS, FASB and ISDA. But the picture is now blurring since Basel III is quite interested in CVA too and on measuring risk on future CVA losses in particular*

Q What is the focus of this presentation?

A *We will focus on CVA.*

Q & A: Credit VaR and CVA

LOSS DISTRIBUTIONS DUE TO DEFAULTS
OF A COUNTERPARTY



LOSSES ARE BUILT UNDER THE HISTORICAL MEASURE.

Q & A: CVA and Model Risk, WWR

Q What impacts counterparty risk CVA?

A *The OTC contract's underlying volatility, the statistical dependence ("correlation") between the underlying and default of the counterparty, and the counterparty credit spreads volatility.*

Q Is it model dependent?

A *It is highly model dependent even if the original portfolio without counterparty risk was not. There is a lot of model risk.*

Q What about wrong way risk?

A *The amplified risk when i) the reference underlying portfolio value in the future and ii) the counterparty default are strongly correlated in the wrong direction.*

Q & A: Collateral

Q What is collateral?

A *It is a guarantee (liquid and secure asset, cash) that is deposited in a collateral account in favour of the investor party facing the exposure. If the depositing counterparty defaults, thus not being able to fulfill payments associated to the above mentioned exposure, Collateral can be used by the investor to offset its loss.*

Q & A: Netting

Q What is netting?

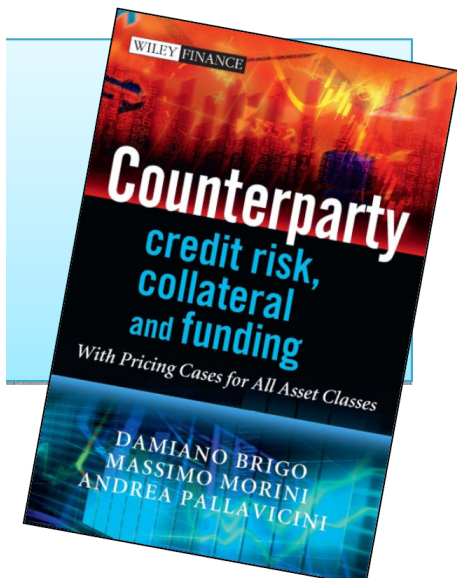
A *This is the agreement to net all positions towards a counterparty in the event of the counterparty default. This way, at counterparty default, positions with negative PV can be offset by positions with positive PV and the overall loss due to counterparty default is reduced. This because the option on a sum is smaller than the sum of the options: $(NPV1 + NPV2)^+ \leq NPV1^+ + NPV2^+$, $CVA_{1+2} \leq CVA_1 + CVA_2$. CVA is typically computed on large netting sets.*

For an introductory dialogue on Counterparty Risk see

CVA Q&A

D. Brigo (2012). Counterparty Risk Q&A: Credit VaR, CVA, DVA, Closeout, Netting, Collateral, Re-hypothecation, Wrong Way Risk, Basel, Funding, and Margin Lending. SSRN.com and arXiv.org.

Check also



General Notation

- We will call “Bank” or sometimes the “investor” the party interested in the counterparty adjustment. This is denoted by “B”
- We will call “counterparty” the party with whom the Bank is trading, and whose default may affect negatively the Bank. This is denoted by “C”.
- “1” will be occasionally used for the underlying name/risk factor(s) of the contract, or for the whole netting set underlying CVA
- The counterparty’s default time is denoted with τ_C and the recovery rate for unsecured claims with R_{EC_C} (we often use $LGD_C := 1 - R_{EC_C}$).
- $\Pi_B(t, T)$ is the netting set discounted payout without default risk seen by ‘B’ (sum of all future cash flows between t and T , discounted back at t). $\Pi_C(t, T) = -\Pi_B(t, T)$ is same but seen from ‘C’. When we omit the index B or C we mean ‘B’.

Examples of products Π

If "B" enters an interest rate swap where "B" pays fixed K and receives from "C" LIBOR L with tenor $T_\alpha, T_{\alpha+1}, \dots, T_\beta$, then the payout is written as

$$\Pi(0, T_\beta) = \sum_{i=\alpha+1}^{\beta} D(0, T_i)(T_i - T_{i-1})(L(T_{i-1}, T_i) - K).$$

where $L(S, T)$ is the LIBOR rate resetting at time S for maturity T . In this example the netting set consists of a single swap. The majority of the instruments that are subject to Counterparty risk is given by Interest Rate Swaps.

General Notation

- We define $NPV_B(t, T) = \mathbb{E}_t[\Pi(t, T)]$. When T is clear from the context we omit it and write $NPV(t)$.



$$\Pi(s, t) + D(s, t)\Pi(t, u) = \Pi(s, u)$$



$$\begin{aligned}\mathbb{E}_0[D(0, u)NPV(u, T)] &= \mathbb{E}_0[D(0, u)\mathbb{E}_u[\Pi(u, T)]] = \\ &= \mathbb{E}_0[D(0, u)\Pi(u, T)] = NPV(0, T) - \mathbb{E}_0[\Pi(0, u)] \\ &= NPV(0, T) - NPV(0, u)\end{aligned}$$

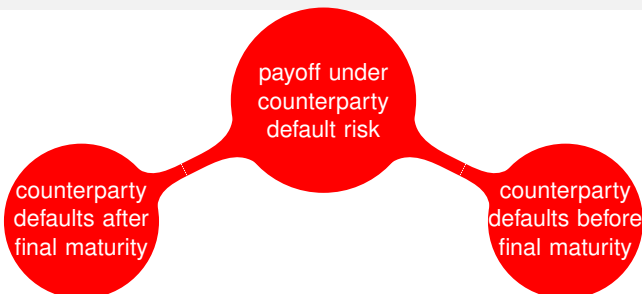
Unilateral counterparty risk

We now look into unilateral counterparty risk.

This is a situation where counterparty risk pricing is computed by assuming that only the counterparty can default, whereas the investor or bank doing the calculation is assumed to be default free.

Hence we will only consider here the default time τ_C of the counterparty. We will address the bilateral case later on.

The mechanics of Evaluating unilateral counterparty risk



original payoff of the instrument

all cash flows before default
⊕ recovery of the residual NPV at default if positive
⊖ Total residual NPV at default if negative

General Formulation under Asymmetry

$$\begin{aligned} \Pi_B^D(t, T) = & \mathbf{1}_{\tau_C > T} \Pi_B(t, T) + \mathbf{1}_{t < \tau_C \leq T} \left[\Pi_B(t, \tau_C) + \right. \\ & \left. + D(t, \tau_C) (\text{REC}_C (\text{NPV}_B(\tau_C))^+ - (-\text{NPV}_B(\tau_C))^+) \right] \quad (**) \end{aligned}$$

This last expression is the general payoff seen from the point of view of 'B' (Π_B , NPV_B) under unilateral counterparty default risk. Indeed,

- 1 if there is no early default, this expression reduces to first term on the right hand side, which is the payoff of a default-free claim.
- 2 In case of early default of the counterparty, the payments due before default occurs are received (second term)
- 3 and then if the residual net present value is positive only the recovery value of the counterparty REC_C is received (third term),
- 4 whereas if it is negative it is paid in full by the investor/ Bank (fourth term).

General Formulation under Asymmetry

If one simplifies the cash flows and takes the risk neutral expectation, one obtains the fundamental formula for the valuation of counterparty risk when the investor/ Bank B is default free:

$$\mathbb{E}_t \{ \Pi_B^D(t, T) \} = \mathbf{1}_{\{\tau_C > t\}} \mathbb{E}_t \{ \Pi_B(t, T) \} - \mathbb{E}_t \{ \text{LGD}_C \mathbf{1}_{\{t < \tau_C \leq T\}} D(t, \tau_C) [\text{NPV}_B(\tau_C)]^+ \} \quad (*)$$

- First term : Value without counterparty risk.
- Second term : Unilateral Counterparty Valuation Adjustment
- $\text{NPV}(\tau_C) = \mathbb{E}_{\tau_C} [\Pi(\tau_C, T)]$ is the value of the transaction on the counterparty default date. $\text{LGD} = 1 - \text{REC}_{\text{counterparty}}$.

$$\text{UCVA}_0 = \mathbb{E}_t \{ \text{LGD}_C \mathbf{1}_{\{t < \tau_C \leq T\}} D(t, \tau_C) [\text{NPV}_B(\tau_C)]^+ \}$$

Proof of the formula

In the proof we omit indices: $\tau = \tau_C$, $\text{REC} = \text{REC}_C$, $\text{LGD} = \text{LGD}_C$, $\text{NPV} = \text{NPV}_B$, $\Pi = \Pi_B$. The proof is obtained easily putting together the following steps to go from (*) to valuation of (**). Since

$$1_{\{\tau > t\}} \Pi(t, T) = 1_{\{\tau > T\}} \Pi(t, T) + 1_{\{t < \tau \leq T\}} \Pi(t, T)$$

we can rewrite the terms inside the expectation in the right hand side of the simplified formula (*) as

$$\begin{aligned} & 1_{\{\tau > t\}} \Pi(t, T) - \{ \text{LGD} 1_{\{t < \tau \leq T\}} D(t, \tau) [\text{NPV}(\tau)]^+ \} \\ &= 1_{\{\tau > T\}} \Pi(t, T) + 1_{\{t < \tau \leq T\}} \Pi(t, T) \\ &+ \{ (\text{REC} - 1) [1_{\{t < \tau \leq T\}} D(t, \tau) (\text{NPV}(\tau))^+] \} \\ &= 1_{\{\tau > T\}} \Pi(t, T) + 1_{\{t < \tau \leq T\}} \Pi(t, T) \\ &+ \text{REC} 1_{\{t < \tau \leq T\}} D(t, \tau) (\text{NPV}(\tau))^+ - 1_{\{t < \tau \leq T\}} D(t, \tau) (\text{NPV}(\tau))^+ \end{aligned}$$

Conditional on the information at τ the second and the fourth terms values (in red) are equal to

Proof (cont'd)

$$\begin{aligned}
 & E_{\tau}[1_{\{t < \tau \leq T\}} \Pi(t, T) - 1_{\{t < \tau \leq T\}} D(t, \tau) (\text{NPV}(\tau))^+] \\
 = & E_{\tau}[1_{\{t < \tau \leq T\}} [\Pi(t, \tau) + D(t, \tau) \Pi(\tau, T) - D(t, \tau) (E_{\tau}[\Pi(\tau, T)])^+]] \\
 = & 1_{\{t < \tau \leq T\}} [\Pi(t, \tau) + D(t, \tau) E_{\tau}[\Pi(\tau, T)] - D(t, \tau) (E_{\tau}[\Pi(\tau, T)])^+] \\
 = & 1_{\{t < \tau \leq T\}} [\Pi(t, \tau) - D(t, \tau) (-E_{\tau}[\Pi(\tau, T)])^+] \\
 = & 1_{\{t < \tau \leq T\}} [\Pi(t, \tau) - D(t, \tau) (-\text{NPV}(\tau))^+]
 \end{aligned}$$

since

$$1_{\{t < \tau \leq T\}} \Pi(t, T) = 1_{\{t < \tau \leq T\}} \{ \Pi(t, \tau) + D(t, \tau) \Pi(\tau, T) \}$$

and $f = f^+ - (-f)^+$.

Proof (cont'd)

Hence

$$\begin{aligned}
 & E_t \left[\mathbf{1}_{\{\tau > t\}} \Pi(t, T) - \{LGD \mathbf{1}_{\{t < \tau \leq T\}} D(t, \tau) [\text{NPV}(\tau)]^+\} \right] \\
 &= E_t \left[E_\tau \left[\mathbf{1}_{\{\tau > t\}} \Pi(t, T) - \{LGD \mathbf{1}_{\{t < \tau \leq T\}} D(t, \tau) [\text{NPV}(\tau)]^+\} \right] \right] \\
 &= E_t \left[E_\tau \left[\mathbf{1}_{\{\tau > T\}} \Pi(t, T) + \mathbf{1}_{\{t < \tau \leq T\}} \Pi(t, T) \right. \right. \\
 &+ \left. \left. R_{\text{EC}} \mathbf{1}_{\{t < \tau \leq T\}} D(t, \tau) (\text{NPV}(\tau))^+ - \mathbf{1}_{\{t < \tau \leq T\}} D(t, \tau) (\text{NPV}(\tau))^+ \right] \right] \\
 &= E_t \left[\mathbf{1}_{\{\tau > T\}} \Pi(t, T) + R_{\text{EC}} \mathbf{1}_{\{t < \tau \leq T\}} D(t, \tau) (\text{NPV}(\tau))^+ \right. \\
 &+ \left. \mathbf{1}_{\{t < \tau \leq T\}} [\Pi(t, \tau) - D(t, \tau) (-\text{NPV}(\tau))^+] \right]
 \end{aligned}$$

We have reached valuation of (**), starting from (*), QED

What we can observe

$$\mathbb{E}_t \{ \Pi_B^D(t, T) \} = \mathbf{1}_{\{\tau_C > t\}} \mathbb{E}_t \{ \Pi_B(t, T) \} - \mathbb{E}_t \{ \text{LGD}_C \mathbf{1}_{\{t < \tau_C \leq T\}} D(t, \tau_C) [\text{NPV}_B(\tau_C)]^+ \} \quad (*)$$

- Including counterparty risk in the valuation of an otherwise default-free derivative \implies credit (hybrid) derivative.
- Counterparty risk adds a level of optionality to the payoff. In particular, model independent products become model dependent also in the underlying market.
 \implies **Counterparty Risk analysis incorporates an opinion about the underlying market dynamics and volatility.**

The point of view of the counterparty “C”

The deal from the point of view of ‘C’, while staying in a world where only ‘C’ may default.

$$\Pi_C^D(t, T) = \mathbf{1}_{\tau_C > T} \Pi_C(t, T) + \mathbf{1}_{t < \tau_C \leq T} \left[\Pi_C(t, \tau_C) + D(t, \tau_C) \left((\text{NPV}_C(\tau_C))^+ - \text{REC}_C (-\text{NPV}_C(\tau_C))^+ \right) \right]$$

This last expression is the general payoff seen from the point of view of ‘C’ (Π_C , NPV_C) under unilateral counterparty default risk. Indeed,

- 1 if there is no early default, this expression reduces to first term on the right hand side, which is the payoff of a default-free claim.
- 2 In case of early default of the counterparty ‘C’, the payments due before default occurs go through (second term)
- 3 and then if the residual net present value is positive to the defaulted ‘C’, it is received in full from ‘B’ (third term),
- 4 whereas if negative, only REC_C fraction it is paid to ‘B’ (term 4).

The point of view of the counterparty “C”

The above formula simplifies to

$$\mathbb{E}_t \{ \Pi_C^D(t, T) \} = \mathbf{1}_{\tau_C > t} \mathbb{E}_t \{ \Pi_C(t, T) \} + \mathbb{E}_t \{ \text{LGD}_C \mathbf{1}_{t < \tau_C \leq T} D(t, \tau_C) [-\text{NPV}_C(\tau_C)]^+ \}$$

and the adjustment term with respect to the risk free price

$\mathbb{E}_t \{ \Pi_C(t, T) \}$ is called

UNILATERAL DEBIT VALUATION ADJUSTMENT

$$\text{UDVA}_C(t) = \mathbb{E}_t \{ \text{LGD}_C \mathbf{1}_{\{t < \tau_C \leq T\}} D(t, \tau_C) [-\text{NPV}_C(\tau_C)]^+ \}$$

The cash flow is triggered only when NPV is negative to the calculating agent, hence when the calculating agent is in “debt”. We note that

$$\text{UDVA}_C = \text{UCVA}_B.$$

Notice also that in this universe $\text{UDVA}_B = \text{UCVA}_C = 0$ as “B” is assumed default-free.

Including the investor/ Bank default or not?

Often the investor, when computing a counterparty risk adjustment, considers itself to be default-free. This can be either a unrealistic assumption or an approximation for the case when the counterparty has a much higher default probability than the investor.

If this assumption is made when no party is actually default-free, the unilateral valuation adjustment is asymmetric: if “C” were to consider itself as default free and “B” as counterparty, and if “C” computed the counterparty risk adjustment, this would not be the opposite of the one computed by “B” in the straight case.

Also, the total NPV including counterparty risk is similarly asymmetric, in that the total value of the position to “B” is not the opposite of the total value of the position to “C”. There is no *cash conservation*.

Including the investor/ Bank default or not?

We get back symmetry if we allow for default of the investor/ Bank in computing counterparty risk. This also results in an adjustment that is cheaper to the counterparty “C”.

The counterparty “C” may then be willing to ask the investor/ Bank “B” to include the investor default event into the model, when the Counterparty risk adjustment is computed by the investor

The case of symmetric counterparty risk

Suppose now that we allow for both parties to default. Counterparty risk adjustment allowing for default of “B”?

“B” : the investor; “C”: the counterparty;

τ_B, τ_C : default times of “B” and “C”. T : final maturity

We consider the following events, forming a partition

Four events ordering the default times

We assume simultaneous defaults are excluded, ie $\mathbb{Q}(\tau_C = \tau_B) = 0$. This assumption can be removed (eg Marshall Olkin).

$$\begin{aligned} \mathcal{A} &= \{\tau_B \leq \tau_C \leq T\} & \mathcal{E} &= \{T \leq \tau_B \leq \tau_C\} \\ \mathcal{B} &= \{\tau_B \leq T \leq \tau_C\} & \mathcal{F} &= \{T \leq \tau_C \leq \tau_B\} \\ \mathcal{C} &= \{\tau_C \leq \tau_B \leq T\} \\ \mathcal{D} &= \{\tau_C \leq T \leq \tau_B\} \end{aligned}$$

Define $\text{NPV}_{\{B,C\}}(t) := \mathbb{E}_t[\Pi_{\{B,C\}}(t, T)]$, and recall $\Pi_B = -\Pi_C$.

The case of symmetric counterparty risk

$$\Pi_B^D(t, T) = \mathbf{1}_{EUF} \Pi_B(t, T)$$

$$+ \mathbf{1}_{CUD} \left[\Pi_B(t, \tau_C) + D(t, \tau_C) \left(REC_C (\text{NPV}_B(\tau_C))^+ - (-\text{NPV}_B(\tau_C))^+ \right) \right]$$

$$+ \mathbf{1}_{AUB} \left[\Pi_B(t, \tau_B) + D(t, \tau_B) \left((\text{NPV}_B(\tau_B))^+ - REC_B (-\text{NPV}_B(\tau_B))^+ \right) \right]$$

- 1 If no early default \Rightarrow payoff of a default-free claim (1st term).
- 2 In case of early default of the counterparty, the payments due before default occurs are received (second term),
- 3 and then if the residual net present value is positive only the recovery value of the counterparty REC_C is received (third term),
- 4 whereas if negative, it is paid in full by the investor/ Bank (4th term).
- 5 In case of early default of the investor, the payments due before default occurs are received (fifth term),
- 6 and then if the residual net present value is positive it is paid in full by the counterparty to the investor/ Bank (sixth term),
- 7 whereas if it is negative only the recovery value of the investor/ Bank REC_B is paid to the counterparty (seventh term).

The case of symmetric counterparty risk

$$\mathbb{E}_t \left\{ \Pi_B^D(t, T) \right\} = \mathbf{1}_{\{\tau^{1st} > t\}} \mathbb{E}_t \left\{ \Pi_B(t, T) \right\} + \text{DVA}_B(t) - \text{CVA}_B(t)$$

$$\text{DVA}_B(t) = \mathbb{E}_t \left\{ \text{LGD}_B \cdot \mathbf{1}(t < \tau^{1st} = \tau_B < T) \cdot D(t, \tau_B) \cdot [-\text{NPV}_B(\tau_B)]^+ \right\}$$

$$\text{CVA}_B(t) = \mathbb{E}_t \left\{ \text{LGD}_C \cdot \mathbf{1}(t < \tau^{1st} = \tau_C < T) \cdot D(t, \tau_C) \cdot [\text{NPV}_B(\tau_C)]^+ \right\}$$

$$\mathbf{1}(A \cup B) = \mathbf{1}(t < \tau^{1st} = \tau_B < T), \quad \mathbf{1}(C \cup D) = \mathbf{1}(t < \tau^{1st} = \tau_C < T)$$

- Obtained simplifying the previous formula and taking expectation.
- 2nd term : adj due to scenarios $\tau_B < \tau_C$. This is positive to the investor/ Bank B and is called "Debit Valuation Adjustment" (DVA)
- 3d term : Counterparty risk adj due to scenarios $\tau_C < \tau_B$
- Bilateral Valuation Adjustment as seen from B :
 $\text{BVA}_B = \text{DVA}_B - \text{CVA}_B$. Note: $\tau^{1st} = \min(\tau_B, \tau_C)$, the 1st default.
- If computed from the opposite point of view of "C" having counterparty "B", $\text{BVA}_C = -\text{BVA}_B$. Symmetry. "Your CVA is my DVA & your DVA is my CVA".

CVA, DVA: Closeout

$$\bar{V}_t = \mathbb{E}_t \left\{ \Pi_B^D(t, T) \right\} = \overbrace{\mathbb{E}_t \left\{ \Pi_B(t, T) \right\}}^{V_t^0} + \text{DVA}_B(t) - \text{CVA}_B(t)$$

$$\text{DVA}_B(t) = \mathbb{E}_t \left\{ \text{LGD}_B \cdot \mathbf{1}_{\{t < \tau^{\text{1st}} = \tau_B < T\}} \cdot D(t, \tau_B) \cdot \left[\underbrace{-\text{NPV}_B(\tau_B)}_{v_{\tau_B}^0 \text{ or } \bar{V}_{\tau_B}^?} \right]^+ \right\}$$

$$\text{CVA}_B(t) = \mathbb{E}_t \left\{ \text{LGD}_C \cdot \mathbf{1}_{\{t < \tau^{\text{1st}} = \tau_C < T\}} \cdot D(t, \tau_C) \cdot \left[\underbrace{\text{NPV}_B(\tau_C)}_{v_{\tau_C}^0 \text{ or } \bar{V}_{\tau_C}^?} \right]^+ \right\}$$

V^0 risk free closeout (much easier but discontinuity), \bar{V} replacement closeout - recursive problem but more continuous. More in a minute.

The case of symmetric counterparty risk

Strange consequences of the formula new mid term, i.e. DVA

- credit quality of investor WORSENS \Rightarrow books POSITIVE MARK TO MKT
- credit quality of investor IMPROVES \Rightarrow books NEGATIVE MARK TO MKT
- Citigroup in its press release on the first quarter revenues of 2009 reported a *positive* mark to market due to its *worsened* credit quality: “Revenues also included [...] a net 2.5\$ billion positive CVA on derivative positions, excluding monolines, mainly due to the widening of Citi’s CDS spreads”

The case of symmetric counterparty risk: DVA?

October 18, 2011, 3:59 PM ET, WSJ. Goldman Sachs Hedges Its Way to Less Volatile Earnings.

Goldman's DVA gains in the third quarter totaled \$450 million [...] That amount is comparatively smaller than the \$1.9 billion in DVA gains that J.P. Morgan Chase and Citigroup each recorded for the third quarter. Bank of America reported \$1.7 billion of DVA gains in its investment bank. Analysts estimated that Morgan Stanley will record \$1.5 billion of net DVA gains when it reports earnings on Wednesday [...]

Is DVA real? **DVA Hedging**. Buying back bonds? Proxying?

DVA hedge? One should sell protection on oneself, impossible, unless one buys back bonds that he had issued earlier. Very Difficult.

Most times: proxying. Instead of selling protection on oneself, one sells protection on a number of names that one thinks are highly correlated to oneself.

The case of symmetric counterparty risk: DVA?

Again from the WSJ article above:

[...] Goldman Sachs CFO David Viniar said Tuesday that the company attempts to hedge [DVA] using a basket of different financials.

A Goldman spokesman confirmed that the company did this by selling CDS on a range of financial firms. [...] Goldman wouldn't say what specific financials were in the basket, but Viniar confirmed [...] that the basket contained 'a peer group.'

This can approximately hedge the spread risk of DVA, but not the jump to default risk. Merrill hedging DVA risk by selling protection on Lehman would not have been a good idea. Worsens systemic risk.

DVA or no DVA? Accounting VS Capital Requirements

NO DVA: Basel III, page 37, July 2011 release

This CVA loss is calculated without taking into account any offsetting debit valuation adjustments which have been deducted from capital under paragraph 75.

YES DVA: FAS 157

Because nonperformance risk (the risk that the obligation will not be fulfilled) includes the reporting entity's credit risk, the reporting entity should consider the effect of its credit risk (credit standing) on the fair value of the liability in all periods in which the liability is measured at fair value under other accounting pronouncements FAS 157 (see also IAS 39)

DVA or no DVA? Accounting VS Capital Requirements

Stefan Walter says:

”The potential for perverse incentives resulting from profit being linked to decreasing creditworthiness means capital requirements cannot recognise it, says Stefan Walter, *secretary-general of the Basel Committee*: The main reason for not recognising DVA as an offset is that it would be inconsistent with the overarching supervisory prudence principle under which we do not give credit for increases in regulatory capital arising from a deterioration in the firms own credit quality.”

Funding and DVA

We will look at this more carefully when dealing with funding costs. For now:

DVA a component of FVA?

DVA is related to funding costs when the payout is uni-directional, eg shorting/issuing a bond, borrowing in a loan, or going short a call option.

Indeed, if we are short simple products that are uni-directional, we are basically borrowing.

As we shorted a bond or a call option, for example, we received cash V_0 in the beginning, and we will have to pay the product payout in the end.

This cash can be used by us to fund other activities, and allows us to spare the costs of funding this cash V_0 from our treasury.

Funding and DVA

Our treasury usually funds in the market, and the market charges our treasury a cost of funding that is related to the borrowed amount V_0 , to the period T and to our own bank credit risk $\tau_B < T$.

In this sense the funding cost we are sparing when we avoid borrowing looks similar to DVA: it is related to the price of the object we are shorting and to our own credit risk.

However quite a number of assumptions is needed to identify DVA with a pure funding benefit, as we will see below.

The case of symmetric counterparty risk: DVA?

When allowing for the investor to default: symmetry

- DVA: One more term with respect to the unilateral case.
- depending on credit spreads and correlations, the total adjustment to be subtracted (CVA-DVA) can now be either positive or negative. In the unilateral case it can only be positive.
- Ignoring the symmetry is clearly more expensive for the counterparty and cheaper for the investor.
- Hedging DVA is difficult. Hedging “by peers” ignores jump to default risk
- We assume the unilateral case in most of the numerical presentations
- WE TAKE THE POINT OF VIEW OF ‘B’ from now on, so we omit the subscript ‘B’. We denote the counterparty as ‘C’.

CVA, DVA: A useful derivation in view of funding

- Immersion hypothesis for credit risk: work under default-free filtration \mathcal{F}_t as much as possible.
- Assume conditional independence of defaults: spreads λ 's may be correlated, but jump to defaults ξ 's will be independent.

Conditional independence of defaults I

Recall that we are assuming

$$\mathcal{G}_t = \mathcal{F}_t \vee (\bigvee_i \sigma(\{\tau_i \leq u\}, u \leq t))$$

with i indexing all the default times in the system. Working under \mathcal{F} -immersion usually means that the risks in the basic cash flows Π are assumed not to be credit sensitive but to depend only on the filtration \mathcal{F} of pre-default or default-free market information, eg default free interest rate swaps portfolio.

Conditional independence of defaults II

We are also assuming default times to be \mathcal{F} -conditionally independent:

$$\text{if } \tau_B = \Lambda_B^{-1}(\xi_B), \quad \tau_C = \Lambda_C^{-1}(\xi_C), \quad (\tau = \min(\tau_B, \tau_C))$$

then this means assuming that ξ_B and ξ_C are independent. Intensities $\lambda_B(t)$ and $\lambda_C(t)$ are taken \mathcal{F}_t adapted (& can be correlated) and

$$\mathbb{Q}(\tau > t) = \mathbb{Q}(\min(\tau_B, \tau_C) > t) = \mathbb{Q}(\tau_B > t \cap \tau_C > t) =$$

We use the tower property + independence of ξ 's on each other and \mathcal{F} :

$$\begin{aligned} &= \mathbb{E}[\mathbb{Q}(\tau_B > t \cap \tau_C > t | \mathcal{F}_t)] = \mathbb{E}[\mathbb{Q}(\tau_B > t | \mathcal{F}_t) \mathbb{Q}(\tau_C > t | \mathcal{F}_t)] = \\ &= \mathbb{E}[e^{-\Lambda_B(t)} e^{-\Lambda_C(t)}] = \mathbb{E}[e^{-\Lambda_B(t) - \Lambda_C(t)}] = \mathbb{E}[e^{-\int_0^t (\lambda_B(s) + \lambda_C(s)) ds}] \end{aligned}$$

Similarly, one can show the first to default time τ intensity λ

$$\text{is } \mathbb{Q}(\tau \in [t, t + dt] | \tau > t, \mathcal{F}_t) = \lambda_t dt = (\lambda_B(t) + \lambda_C(t)) dt.$$

Conditional independence of defaults III

Summing up:

Whenever we use the immersion hypothesis, meaning that we switch filtration from \mathcal{G} to \mathcal{F} , we assume the ξ to be conditionally independent and the basic cash flows $\Pi(s, t)$ to be \mathcal{F}_t adapted for all $s \leq t$.

Switching to the filtration \mathcal{F} typically transforms indicators such as $1_{\{\tau > t\}}$ into their \mathcal{F} expectations $e^{-\int_0^t (\lambda_B(s) + \lambda_C(s)) ds}$. This is often collected in the discount term $D(0, t; r)$ that becomes $D(0, t; r + \lambda)$.

$$D(0, t; r) 1_{\{\tau > t\}} = e^{-\int_0^t r_s ds} 1_{\{\tau > t\}} \text{ goes } e^{-\int_0^t r_s ds} e^{-\int_0^t \lambda_s ds} = D(0, t; r + \lambda)$$

The switching also transforms $1_{\{\tau \in dt\}}$ into $\lambda_t e^{-\int_0^t \lambda_s ds} dt$.

We now present the calculation of CVA and DVA under immersion. Here V will denote either V^0 or \bar{V} .

CVA, DVA: A useful derivation in view of funding

$$\begin{aligned}
 \text{CVA}_B(t) &= \mathbb{E}_t \left\{ \text{LGD}_C \cdot \mathbf{1}(t < \tau^{1st} = \tau_C < T) \cdot D(t, \tau_C) \cdot [V(\tau_C)]^+ \right\} \\
 &= \mathbb{E}_t \left\{ \text{LGD}_C \int_t^T \mathbf{1}_{\{\tau^{1st} \in du\}} \mathbf{1}_{\{\tau_B > u\}} D(t, u) [V(u)]^+ \right\} \\
 &= \text{LGD}_C \int_t^T \mathbb{E}_t \left\{ \mathbf{1}_{\{\tau_C \in du\}} \mathbf{1}_{\{\tau_B > u\}} D(t, u) (V(u))^+ \right\} \\
 &= \text{LGD}_C \int_t^T \mathbb{E}_t \left\{ \mathbb{E} \left[\mathbf{1}_{\{\tau_C \in du\}} \mathbf{1}_{\{\tau_B > u\}} D(t, u) (V(u))^+ \mid \mathcal{F}_T \right] \right\} \\
 &= \text{LGD}_C \int_t^T \mathbb{E}_t \left\{ D(t, u) (V(u))^+ \mathbb{E} \left[\mathbf{1}_{\{\tau_C \in du\}} \mathbf{1}_{\{\tau_B > u+du\}} \mid \mathcal{F}_T \right] \right\} = \dots \\
 &\left(\mathbb{E} \left[\mathbf{1}_{\{\tau_C \in du\}} \mathbf{1}_{\{\tau_B > u+du\}} \mid \mathcal{F}_T \right] = \mathbb{E} \left[\mathbf{1}_{\{\tau_C \in du\}} \mid \mathcal{F} \right] \mathbb{E} \left[\mathbf{1}_{\{\tau_B > u+du\}} \mid \mathcal{F} \right] = \right. \\
 &= \lambda_C(u) du e^{-\int_t^u \lambda_C(s) ds} e^{-\int_t^u \lambda_B(s) ds} = \lambda_C(u) du e^{-\int_t^u (\lambda_C(s) + \lambda_B(s)) ds} \\
 &\left. = \lambda_C(u) e^{-\int_t^u \lambda(s) ds} du \right) = \mathbb{E}_t \left\{ \text{LGD}_C \int_t^T D(t, u; r + \lambda) \lambda_C(u) (V(u))^+ du \right\}
 \end{aligned}$$

CVA, DVA: A useful derivation in view of funding

$$CVA_B(t) = \mathbb{E}_t \left\{ \int_t^T D(t, u; r + \lambda) L_{GD_C} \lambda_C(u) (V(u))^+ du \right\}$$

$$DVA_B(t) = \mathbb{E}_t \left\{ \int_t^T D(t, u; r + \lambda) L_{GD_B} \lambda_B(u) (-V(u))^+ du \right\}$$

and we will see later that (without collateral and under the Reduced Borrowing Benefit case) Funding Cost and Benefit Adjustments (FCA, FBA) are (notice the formal analogies, used in industry)

$$FCA_B(t) = \mathbb{E}_t \left\{ \int_t^T D(t, u; r + \lambda) L_{GD_B} \lambda_B(u) (V(u))^+ du \right\}$$

$$FBA_B(t) = \mathbb{E}_t \left\{ \int_t^T D(t, u; r + \lambda) L_{GD_B} \lambda_B(u) (-V(u))^+ du \right\} = DVA_B(t)$$

Closeout: Replacement (ISDA?) VS Risk Free

When we computed the bilateral adjustment formula from

$$\begin{aligned} \Pi_B^D(t, T) = & \mathbf{1}_{EUF} \Pi_B(t, T) \\ & + \mathbf{1}_{CUD} [\Pi_B(t, \tau_C) + D(t, \tau_C) (REC_C (NPV_B(\tau_C))^+ - (-NPV_B(\tau_C))^+)] \\ & + \mathbf{1}_{AUB} [\Pi_B(t, \tau_B) + D(t, \tau_B) ((-NPV_C(\tau_B))^+ - REC_B (NPV_C(\tau_B))^+)] \end{aligned}$$

(where we now substituted $NPV_B = -NPV_C$ in the last two terms) we used the risk free NPV upon the first default, to close the deal. But what if upon default of the first entity, the deal needs to be valued by taking into account the credit quality of the surviving party (we ignore the credit risk of the new entity replacing the defaulted one for now)? What if we make the Replacements

$$NPV_B(\tau_C) \rightarrow NPV_B(\tau_C) + UDVA_B(\tau_C)$$

$$NPV_C(\tau_B) \rightarrow NPV_C(\tau_B) + UDVA_C(\tau_B)?$$

Closeout: Replacement (ISDA?) VS Risk Free

ISDA (2009) Close-out Amount Protocol.

”In determining a Close-out Amount, the Determining Party may consider any relevant information, including, [...] quotations (either firm or indicative) for replacement transactions supplied by one or more third parties that **may take into account the creditworthiness of the Determining Party** at the time the quotation is provided”

This makes valuation more continuous: upon default we still price including the DVA, as we were doing before default.

Closeout: Replacement (ISDA?) VS Risk Free

The final formula with substitution closeout is quite complicated:

$$\begin{aligned}
 \Pi_B^D(t, T) &= \mathbf{1}_{E_{UF}} \Pi_B(t, T) \\
 &+ \mathbf{1}_{C_{UD}} \left[\Pi_B(t, \tau_C) + D(t, \tau_C) \right. \\
 &\cdot (REC_C (NPV_B(\tau_C) + UDVA_B(\tau_C))^+ - (-NPV_B(\tau_C) - UDVA_B(\tau_C))^+) \\
 &+ \mathbf{1}_{A_{UB}} \left[\Pi_B(t, \tau_B) + D(t, \tau_B) \right. \\
 &\cdot ((-NPV_C(\tau_B) - UDVA_C(\tau_B))^+ - REC_B (NPV_C(\tau_B) + UDVA_C(\tau_B))^+)
 \end{aligned}$$

Closeout: Replacement (ISDA?) VS Risk Free

B. and Morini (2010)

We analyze the Risk Free closeout formula in Comparison with the replacement Closeout formula for a Zero coupon bond when:

1. Default of 'B' and 'C' are independent
2. Default of 'B' and 'C' are co-monotonic

Suppose 'B' (the lender) holds the bond, and 'C' (the borrower) will pay the notional 1 at maturity T .

The risk free price of the bond at time 0 to 'B' is denoted by $P(0, T)$.

In this case $UDVA_B = 0$ at any time, since the NPV to B is positive at any point in time. Instead, $UDVA_C > 0$ at any time as NPV to C is always negative.

Closeout: Replacement (ISDA?) VS Risk Free

Suppose 'B' (the lender) holds the bond, and 'C' (the borrower) will pay the notional 1 at maturity T .

The risk free price of the bond at time 0 to 'B' is denoted by $P(0, T)$.

If we assume deterministic interest rates, the above formulas reduce to

$$P^{D,Repl}(0, T) = P(0, T)[\mathbb{Q}(\tau_C > T) + REC_C \mathbb{Q}(\tau_C \leq T)]$$

$$P^{D,Free}(0, T) = P(0, T)[\mathbb{Q}(\tau_C > T) + \mathbb{Q}(\tau_B < \tau_C < T) \\ + REC_C \mathbb{Q}(\tau_C \leq \min(\tau_B, T))]$$

$$= P(0, T)[\mathbb{Q}(\tau_C > T) + REC_C \mathbb{Q}(\tau_C \leq T) + LGD_C \mathbb{Q}(\tau_B < \tau_C < T)]$$

Risk Free Closeout and Credit Risk of the Lender

The adjusted price of the bond **DEPENDS ON THE CREDIT RISK OF THE LENDER 'B'** IF WE USE THE RISK FREE CLOSEOUT. This is counterintuitive and undesirable.

Closeout: Replacement (ISDA?) VS Risk Free

Co-Monotonic Case

If we assume the default of B and C to be co-monotonic, and the spread of the lender 'B' to be larger, we have that the lender 'B' defaults first in ALL SCENARIOS (e.g. 'C' is a subsidiary of 'B', or a company whose well being is completely driven by 'B': 'C' is a tyre factory whose only client is car producer 'B'). In this case

$$P^{D,Repl}(0, T) = P(0, T)[\mathbb{Q}(\tau_C > T) + REC_C \mathbb{Q}(\tau_C \leq T)]$$

$$P^{D,Free}(0, T) = P(0, T)[\mathbb{Q}(\tau_C > T) + \mathbb{Q}(\tau_C < T)] = P(0, T)$$

Risk free closeout is correct. Either 'B' does not default, and then 'C' does not default either, or if 'B' defaults, at that precise time C is solvent, and B recovers the whole payment. Credit risk of 'C' should not impact the deal.

Closeout: Replacement (ISDA?) VS Risk Free

Contagion. What happens at default of the Lender

$$P^{D,Subs}(t, T) = P(t, T)[\mathbb{Q}_t(\tau_C > T) + REC_C \mathbb{Q}_t(\tau_C \leq T)]$$

$$P^{D,Free}(t, T) = P^{D,Subs}(t, T) + P(t, T)LGD_C \mathbb{Q}_t(\tau_B < \tau_C < T)$$

We focus on two cases:

- τ_B and τ_C are independent. Take $t < T$.

$$\mathbb{Q}_{t-\Delta t}(\tau_B < \tau_C < T) \mapsto \{\tau_B = t\} \mapsto \mathbb{Q}_{t+\Delta t}(\tau_C < T)$$

and this effect can be quite sizeable.

- τ_B and τ_C are comonotonic. Take an example where $\tau_B = t < T$ implies $\tau_C = u < T$ with $u > t$. Then

$$\mathbb{Q}_{t-\Delta t}(\tau_C > T) \mapsto \{\tau_B = t, \tau_C = u\} \mapsto 0$$

$$\mathbb{Q}_{t-\Delta t}(\tau_C \leq T) \mapsto \{\tau_B = t, \tau_C = u\} \mapsto 1$$

$$\mathbb{Q}_{t-\Delta t}(\tau_B < \tau_C < T) \mapsto \{\tau_B = t, \tau_C = u\} \mapsto 1$$

Closeout: Replacement (ISDA?) VS Risk Free

Let us put the pieces together:

- τ_B and τ_C are independent. Take $t < T$.

$$P^{D,Subs}(t - \Delta t, T) \mapsto \{\tau_B = t\} \mapsto \text{no change}$$

$$P^{D,Free}(t - \Delta t, T) \mapsto \{\tau_B = t\} \mapsto \text{add } \mathbb{Q}_{t-\Delta t}(\tau_B > \tau_C, \tau_C < T)$$

and this effect can be quite sizeable.

- τ_B and τ_C are comonotonic. Take an example where $\tau_B = t < T$ implies $\tau_C = u < T$ with $u > t$. Then

$$P^{D,Subs}(t - \Delta t, T) \mapsto \{\tau_B = t\} \mapsto \text{subtract } X$$

$$X = LGD_C P(t, T) \mathbb{Q}_{t-\Delta t}(\tau_C > T)$$

$$P^{D,Free}(t - \Delta t, T) \mapsto \{\tau_B = t\} \mapsto \text{no change}$$

Closeout: Replacement (ISDA?) VS Risk Free

The independence case: Contagion with Risk Free closeout

The Risk Free closeout shows that *upon default of the lender*, the mark to market to the lender itself jumps up, or equivalently **the mark to market to the borrower jumps down**. The effect can be quite dramatic.

The replacement closeout instead shows no such contagion, as the mark to market does not change upon default of the lender.

The co-monotonic case: Contagion with replacement closeout

The Risk Free closeout behaves nicely in the co-monotonic case, and there is no change upon default of the lender.

Instead the replacement closeout shows that *upon default of the lender* the mark to market to the lender jumps down, or equivalently **the mark to market to the borrower jumps up**.

Closeout: Replacement (ISDA?) VS Risk Free

Impact of an early default of the Lender on the value of a loan (we would like to have no impact)

Dependence(τ_B, τ_C) : \rightarrow	independence	co-monotonicity
Closeout \downarrow		
Risk Free	Negatively affects Borrower (bad)	No impact (good)
Replacement	No impact (good)	Further Negatively affects Lender (bad)

For a numerical case study and more details see Brigo and Morini (2010, 2011).

A simplified formula without τ^{1st} for bilateral VA

$$DVA_B(t) = \mathbb{E}_t \left\{ LGD_B \cdot \mathbf{1}(t < \cancel{\tau^{1st}} = \tau_B < T) \cdot D(t, \tau_B) \cdot [-NPV_B(\tau_B)]^+ \right\}$$

$$CVA_B(t) = \mathbb{E}_t \left\{ LGD_C \cdot \mathbf{1}(t < \cancel{\tau^{1st}} = \tau_C < T) \cdot D(t, \tau_C) \cdot [+NPV_B(\tau_C)]^+ \right\}$$

- Simplified formula is only a simplified representation of bilateral risk & neglects that upon the first default closeout proceedings are started, thus involving double counting
- It is attractive because it allows for the construction of a bilateral counterparty risk pricing system based only on a unilateral one.
- The correct formula involves default dependence between the two parties through τ^{1st} and allows no such incremental construction
- A simplified bilateral formula is possible also in case of replacement closeout, but it turns out to be identical to the simplified formula of the risk free closeout case.
- We analyze the impact of default dependence between investor 'B' and counterparty 'C' on the difference between the two formulas by looking at an equity forward.

A simplified formula without τ^{1st} for bilateral VA

One can show easily that the difference between the full correct DVA-CVA formula and the simplified DVA-DVA formula is

$$\begin{aligned}
 & E_0[1_{\{\tau_B < \tau_C < T\}} LGD_C D(0, \tau_C) (E_{\tau_C}(\Pi(\tau_C, T)))^+] \\
 - & E_0[1_{\{\tau_C < \tau_B < T\}} LGD_B D(0, \tau_B) (-E_{\tau_B}(\Pi(\tau_B, T)))^+].
 \end{aligned} \tag{22}$$

A simplified formula without τ^{1st} : The case of a Zero Coupon Bond

We work under deterministic interest rates. We consider $P(t, T)$ held by 'B' (lender) who will receive the notional 1 from 'C' (borrower) at final maturity T if there has been no default of 'C'.

The difference between the correct bilateral formula and the simplified one is, under risk free closeout,

$$LGD_C P(0, T) \mathbb{Q}(\tau_B < \tau_C < T).$$

The case with replacement closeout is instead trivial and the difference is null. For a bond, the simplified formula coincides with the full substitution closeout formula.

Therefore the difference above is the same difference between risk free closeout and replacement closeout formulas, and has been examined earlier, also in terms of contagion.

A simplified formula without τ^{1st} : The case of an Equity forward

In this case the payoff at maturity time T is given by $S_T - K$

where S_T is the price of the underlying equity at time T and K the strike price of the forward contract (typically $K = S_0$, 'at the money', or $K = S_0/P(0, T)$, 'at the money forward').

We compute the difference D^{BC} between the correct bilateral risk free closeout formula for DVA-CVA and the simplified one.

A simplified formula without τ^{1st} : The case of an Equity forward

$D^{BC} := A_1 - A_2$, where

$$A_1 = E_0 \left\{ \mathbf{1}_{\{\tau_B < \tau_C < T\}} LGD_C D(0, \tau_C) (S_{\tau_C} - P(\tau_C, T)K)^+ \right\}$$

$$A_2 = E_0 \left\{ \mathbf{1}_{\{\tau_C < \tau_B < T\}} LGD_B D(0, \tau_B) (P(\tau_B, T)K - S_{\tau_B})^+ \right\}$$

The worst cases will be the ones where the terms A_1 and A_2 do not compensate. For example assume there is a high probability that $\tau_B < \tau_C$ and that the forward contract is deep in the money. In such case A_1 will be large and A_2 will be small.

Similarly, a case where $\tau_C < \tau_B$ is very likely and where the forward is deep out of the money will lead to a large A_2 and to a small A_1 .

However, we show with a numerical example that even when the forward is at the money the difference can be relevant. For more details see Brigo and Buescu (2011).

CVA difference as a function of Kendall's tau

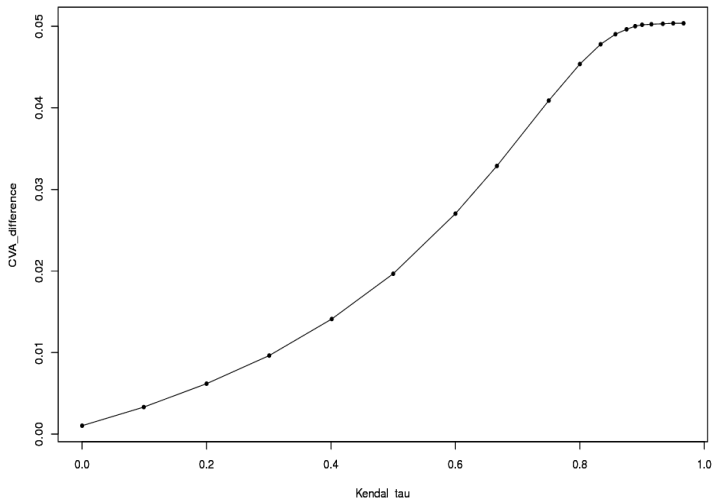


Figure: D^{BC} plotted against Kendall's tau between τ_B and τ_C , all other quantities being equal: $S_0 = 1$, $T = 5$, $\sigma = 0.4$, $K = 1$, $\lambda_B = 0.1$, $\lambda_C = 0.05$.

PAYOFF RISK

The exact payout corresponding with the Credit and Debit valuation adjustment is not clear.

- DVA or not?
- Which Closeout?
- First to default risk or not?
- How are collateral and funding accounted for exactly?

Worse than model risk: Payout risk. WHICH PAYOUT?

At a recent industry panel (WBS) on CVA it was stated that 5 banks might compute CVA in 15 different ways.

Methodology

- 1 Assumption: The *Bank/investor* enters a transaction with a *counterparty* and, when dealing with Unilateral Risk, the investor considers itself default free.
Note : All the payoffs seen from the point of view of the *investor*.
- 2 We model and calibrate the default time of the *counterparty* using a stochastic intensity default model, except in the equity case where we will use a firm value model.
- 3 We model the transaction underlying and estimate the deal NPV at default.
- 4 We allow for the counterparty default time and the contract underlying to be correlated.
- 5 We start however from the case when such correlation can be neglected.

Approximation: Default Bucketing

General Formulation

- 1 Model (underlying) to estimate the NPV of the transaction.
- 2 Simulations are run allowing for correlation between the credit and underlying models, to determine the counterparty default time and the underlying deal NPV respectively.

Approximated Formulation under default bucketing

$$\begin{aligned}
 \mathbb{E}_0 \Pi^D(0, T) &:= \mathbb{E}_0 \Pi(0, T) - \text{LGD} \mathbb{E}_0 [1_{\{\tau < T_b\}} D(0, \tau) (\mathbb{E}_\tau \Pi(\tau, T))^+] \\
 &= \mathbb{E}_0 \Pi(0, T) - \text{LGD} \mathbb{E}_0 \left[\left(\sum_{j=1}^b 1_{\{\tau \in (T_{j-1}, T_j]\}} \right) D(0, \tau) (\mathbb{E}_\tau \Pi(\tau, T))^+ \right] \\
 &= \mathbb{E}_0 \Pi(0, T) - \text{LGD} \sum_{j=1}^b \mathbb{E}_0 [1_{\{\tau \in (T_{j-1}, T_j]\}} D(0, \tau) (\mathbb{E}_\tau \Pi(\tau, T))^+] \\
 &\approx \mathbb{E}_0 \Pi(0, T) - \text{LGD} \sum_{j=1}^b \mathbb{E}_0 [1_{\{\tau \in (T_{j-1}, T_j]\}} D(0, T_j) (\mathbb{E}_{T_j} \Pi(T_j, T))^+]
 \end{aligned}$$

Approximation: Default Bucketing and Independence

- 1 In this formulation defaults are bucketed but we still need a joint model for τ and the underlying Π including their correlation.
- 2 Option model for Π is implicitly needed in τ scenarios.

Approximated Formulation under independence (and 0 correlation)

$$\mathbb{E}_0 \Pi^D(0, T) := \mathbb{E}_0 \Pi(0, T) - \text{LGD} \sum_{j=1}^b \mathbb{Q}\{\tau \in (T_{j-1}, T_j]\} \mathbb{E}_0 [D(0, T_j) (\mathbb{E}_{T_j} \Pi(T_j, T))^+]$$

- 1 In this formulation defaults are bucketed and only survival probabilities are needed (no default model).
- 2 Option model is STILL needed for the underlying of Π .

Ctrparty default model: CIR++ stochastic intensity

If we cannot assume independence, we need a default model.

Counterparty instantaneous credit spread: $\lambda(t) = y(t) + \psi(t; \beta)$

- 1 $y(t)$ is a CIR process with possible jumps

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dW_t^y + dJ_t, \quad \tau_C = \Lambda^{-1}(\xi), \quad \Lambda(T) = \int_0^T \lambda(s)ds$$

- 2 $\psi(t; \beta)$ is the shift that matches a given CDS curve
- 3 ξ is standard exponential independent of all brownian driven stochastic processes
- 4 In CDS calibration we assume deterministic interest rates.
- 5 Calibration : Closed form Fitting of model survival probabilities to counterparty CDS quotes
- 6 B and El Bachir (2010) (Mathematical Finance) show that this model with jumps has closed form solutions for CDS options.

Literature on CVA across asset classes

Impact of dynamics, volatilities, correlations, wrong way risk

- **Interest Rate Swaps and Derivatives Portfolios** (B. Masetti (2005), B. Pallavicini 2007, 2008, B. Capponi P. Papatheodorou 2011, B. C. P. P. 2012 with collateral and gap risk)
- **Commodities swaps (Oil)** (B. and Bakkar 2009)
- **Credit: CDS on a reference credit** (B. and Chourdakis 2009, B. C. Pallavicini 2012 Mathematical Finance)
- **Equity Return Swaps** (B. and Tarenghi 2004, B. T. Morini 2011)
- Equity uses AT1P firm value model of B. and T. (2004) (barrier options with time-inhomogeneous GBM) and extensions (random barriers for risk of fraud).

Further asset classes are studied in the literature. For example see Biffis et al (2011) for CVA on **longevity swaps**.

Interest Rate and Commodities swaps and derivatives

We now examine UCVA with WWR for:

- Interest Rate Swaps and Derivatives Portfolios
- Commodities swaps
- CDS
- Equity Returns Swaps

Interest rate swaps are the vast majority of market contracts on which CVA is computed.

Interest Rates Swap Case

Formulation for IRS under independence (no correlation)

$$\text{IRS}^D(t, K) = \text{IRS}(t, K) - \text{LGD} \sum_{i=a+1}^{b-1} \mathbb{Q}\{\tau \in (T_{i-1}, T_i]\} \text{SWAPTION}_{i,b}(t; K, S_{i,b}(t), \sigma_{i,b})$$

Modeling Approach with corr.

Gaussian 2-factor G2++ short-rate $r(t)$ model:

$$r(t) = x(t) + z(t) + \varphi(t; \alpha), r(0) = r_0$$

$$dx(t) = -ax(t)dt + \sigma dW_x$$

$$dz(t) = -bz(t)dt + \eta dW_z$$

$$dW_x dW_z = \rho_{x,z} dt$$

$$\alpha = [r_0, a, b, \sigma, \eta, \rho_{1,2}]$$

$$dW_x dW_y = \rho_{x,y} dt, dW_z dW_y = \rho_{z,y} dt$$

Calibration

- The function $\varphi(\cdot; \alpha)$ is deterministic and is used to calibrate the initial curve observed in the market.
- We use swaptions and zero curve data to calibrate the model.
- The r factors x and z and the intensity are taken to be correlated.

Interest Rates Swap Case

Total Correlation Counterparty default / rates

$$\bar{\rho} = \text{Corr}(dr_t, d\lambda_t) = \frac{\sigma\rho_{x,y} + \eta\rho_{z,y}}{\sqrt{\sigma^2 + \eta^2 + 2\sigma\eta\rho_{x,z}} \sqrt{1 + \frac{2\beta\gamma^2}{\nu^2 y_t}}}$$

where β is the intensity of arrival of λ jumps and γ is the mean of the exponentially distributed jump sizes.

Without jumps ($\beta = 0$)

$$\bar{\rho} = \text{Corr}(dr_t, d\lambda_t) = \frac{\sigma\rho_{x,y} + \eta\rho_{z,y}}{\sqrt{\sigma^2 + \eta^2 + 2\sigma\eta\rho_{x,z}}}$$

IRS: Case Study

1) Single Interest Rate Swaps (IRS)

At-the-money fix-receiver forward interest-rate-swap (IRS) paying on the EUR market.

The IRS's fixed legs pay annually a 30E/360 strike rate, while the floating legs pay LIBOR twice per year.

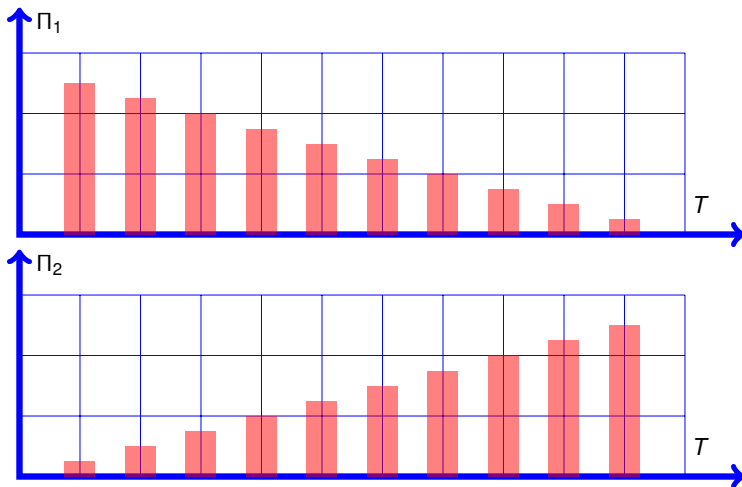
2) Netted portfolios of IRS.

- Portfolios of at-the-money IRS either with different starting dates or with different maturities.

- 1 (P1) annually spaced dates $\{T_i : i = 0 \dots N\}$, T_0 two business days from trade date; portfolio of swaps maturing at each T_i , with $i > 0$, all starting at T_0 .
- 2 (P2) portfolio of swaps starting at each T_i all maturing at T_N .

Can also do exotics (Ratchets, CMS spreads, Bermudan)

IRS Case Study: Payment schedules



IRS Results

Counterparty risk price for netted receiver IRS portfolios $\Pi 1$ and $\Pi 2$ and simple IRS (maturity 10Y). Every IRS, constituting the portfolios, has unit notional and is at equilibrium. Prices are in bps.

λ	correlation $\bar{\rho}$	$\Pi 1$	$\Pi 2$	IRS
3%	-1	-140	-294	-36
	0	-84	-190	-22
	1	-47	-115	-13
5%	-1	-181	-377	-46
	0	-132	-290	-34
	1	-99	-227	-26
7%	-1	-218	-447	-54
	0	-173	-369	-44
	1	-143	-316	-37

Compare with "Basel 2" deduced adjustments

Basel 2, under the "Internal Model Method", models wrong way risk by means of a 1.4 multiplying factor to be applied to the zero correlation case, even if banks have the option to compute their own estimate of the multiplier, which can never go below 1.2 anyway.

Is this confirmed by our model?

$$(140 - 84)/84 \approx 66\% > 40\%$$

$$(54 - 44)/44 \approx 23\% < 40\%$$

So this really depends on the portfolio and on the situation.

A bilateral example and correlation risk

Finally, in the bilateral case for Receiver IRS, 10y maturity, high risk counterparty and mid risk investor, we notice that depending on the correlations

$$\bar{\rho}_0 = \text{Corr}(dr_t, d\lambda_t^0), \quad \bar{\rho}_2 = \text{Corr}(dr_t, d\lambda_t^2), \quad \rho_{0,2}^{\text{Copula}} = 0$$

the DVA - CVA or Bilateral CVA does change sign, and in particular for portfolios $\Pi 1$ and IRS the sign of the adjustment follows the sign of the correlations.

$\bar{\rho}_2$	$\bar{\rho}_0$	$\Pi 1$	$\Pi 2$	10×IRS
-60%	0%	-117(7)	-382(12)	-237(16)
-40%	0%	-74(6)	-297(11)	-138(15)
-20%	0%	-32(6)	-210(10)	-40(14)
0%	0%	-1(5)	-148(9)	31(13)
20%	0%	24(5)	-96(9)	87(12)
40%	0%	44(4)	-50(8)	131(11)
60%	0%	57(4)	-22(7)	159(11)

Payer vs Receiver

- Counterparty Risk (CR) has a relevant impact on interest-rate payoffs prices and, in turn, correlation between interest-rates and default (intensity) has a relevant impact on the CR adjustment.
- The (positive) CR adjustment to be subtracted from the default free price **decreases with correlation for receiver payoffs**.
Natural: If default intensities increase, with high positive correlation their correlated interest rates will increase more than with low correlation, and thus a receiver swaption embedded in the adjustment decreases more, reducing the adjustment.
- The adjustment for payer payoffs increases with correlation.

Further Stylized Facts

- As the default probability implied by the counterparty CDS increases, the size of the adjustment increases as well, but the impact of correlation on it decreases.
- Financially reasonable: Given large default probabilities for the counterparty, fine details on the dynamics such as the correlation with interest rates become less relevant
- **The conclusion is that we should take into account interest-rate/ default correlation in valuing CR interest-rate payoffs.**
- In the bilateral case correlation risk can cause the adjustment to change sign

Exotics

For examples on exotics, including Bermudan Swaptions and CMS spread Options, see

Papers with Exotics and Bilateral Risk

- Brigo, D., and Pallavicini, A. (2007). Counterparty Risk under Correlation between Default and Interest Rates. In: Miller, J., Edelman, D., and Appleby, J. (Editors), Numerical Methods for Finance, Chapman Hall.
- Brigo, D., Pallavicini, A., and Papatheodorou, V. (2009). Bilateral counterparty risk valuation for interest-rate products: impact of volatilities and correlations. Available at Defaultrisk.com, SSRN and arXiv

Commodities and WWR

The correlation between interest rates dr_t (LIBOR, OIS) and credit intensities $d\lambda_t$, if measured historically, is often quite small in absolute value. Hence interest rates are a case where including correlation is good for stress tests and conservative hedging of CVA, but a number of market participants think that CVA can be computed by assuming zero correlations.

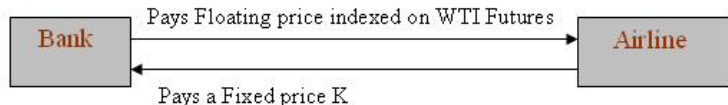
Whether one agrees or not, there are other asset classes on which CVA can be computed and where there is agreement on the necessity of including correlation in CVA pricing. We provide an example: Oil swaps traded with an airline.

It's natural to think that the future credit quality of the airline will be correlated with prices of oil.

Commodities: Futures, Forwards and Swaps

- **Forward:** OTC contract to buy a commodity to be delivered at a maturity date T at a price specified today. The cash/commodity exchange happens at time T .
- **Future:** Listed Contract to buy a commodity to be delivered at a maturity date T . Each day between today and T margins are called and there are payments to adjust the position.
- **Commodity Swap: Oil Example:**

FIXED-FLOATING (for hedge purposes)



Commodities: Modeling Approach

Schwartz-Smith Model

$$\begin{aligned} \ln(S_t) &= x_t + l_t + \varphi(t) \\ dx_t &= -kx_t dt + \sigma_x dW_x \\ dl_t &= \mu dt + \sigma_l dW_l \\ dW_x dW_l &= \rho_{x,l} dt \end{aligned}$$

Correlation with credit

$$\begin{aligned} dW_x dW_y &= \rho_{x,y} dt, \\ dW_l dW_y &= \rho_{l,y} dt \end{aligned}$$

Variables

S_t : Spot oil price;
 x_t, l_t : short and long term components of S_t ;
 This can be re-cast in a classic convenience yield model

Calibration

φ : defined to exactly fit the oil forward curve.
 Dynamic parameters k, μ, σ, ρ are calibrated to At the money implied volatilities on Futures options.

Commodities

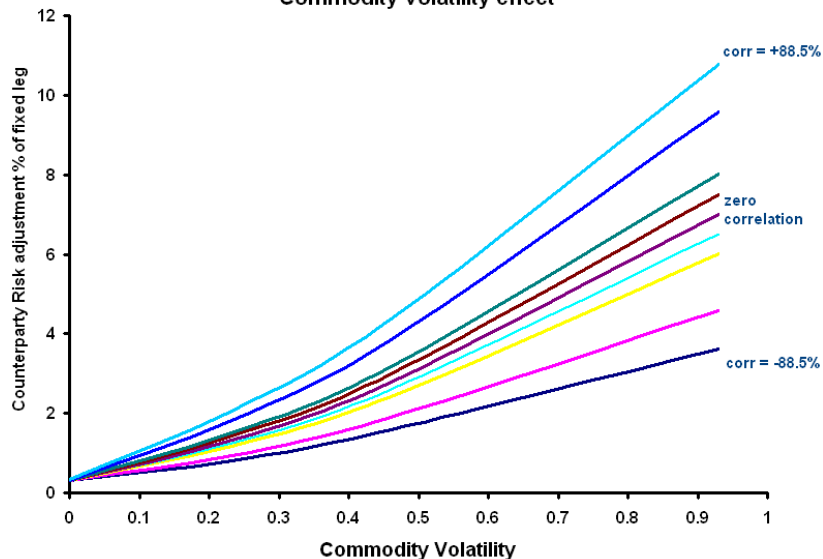
Total correlation Commodities - Counterparty default

$$\bar{\rho} = \text{corr}(d\lambda_t, dS_t) = \frac{\sigma_x \rho_{x,y} + \sigma_L \rho_{L,y}}{\sqrt{\sigma_x^2 + \sigma_L^2 + 2\rho_{x,L} \sigma_x \sigma_L}}$$

We assumed no jumps in the intensity

We show the counterparty risk CVA computed by the AIRLINE on the BANK. This is because after 2008 a number of bank's credit quality deteriorated and an airline might have checked CVA on the bank with whom the swap was negotiated.

Commodities: Commodity Volatility Effect

Counterparty Risk adjustment for 7Y Payer WTI Swap
Commodity volatility effect

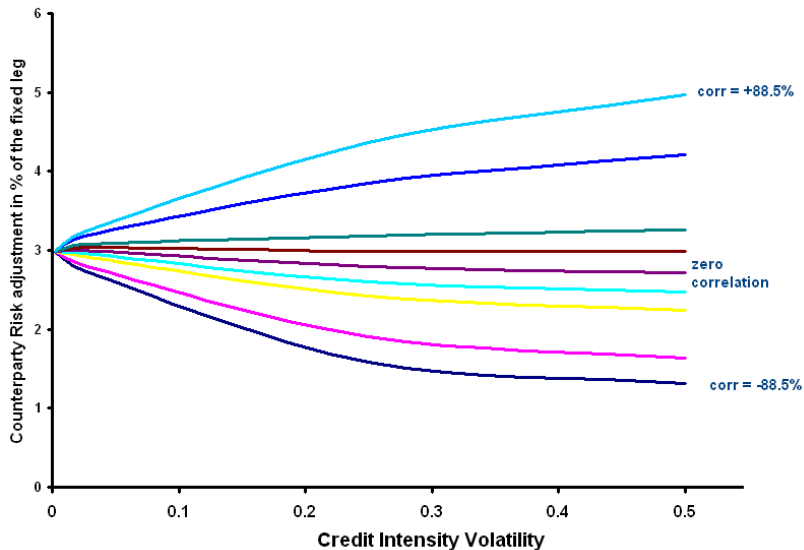
Commodities: Commodity Volatility Effect

Notice: In this example where CVA is calculated by the AIRLINE, positive correlation implies larger CVA.

This is natural: if the Bank credit spread widens, and the bank default becomes more likely, with positive correlation also OIL goes up.

Now CVA computed by the airline is an option, with maturity the default of the bank=counterparty, on the residual value of a Payer swap. As the price of OIL will go up at default due to the positive correlation above, the *payer* oil-swap will move in-the-money and the OIL option embedded in CVA will become more in-the-money, so that CVA will increase.

Counterparty Risk adjustment for 7Y Payer WTI Swap Credit volatility effect



Commodities¹ : Credit volatility effect

$\bar{\rho}$	intensity volatility ν_R	0.025	0.25	0.50
-88.5	Payer adj	2.742	1.584	1.307
	Receiver adj	1.878	2.546	3.066
-63.2	Payer adj	2.813	1.902	1.63
	Receiver adj	1.858	2.282	2.632
-25.3	Payer adj	2.92	2.419	2.238
	Receiver adj	1.813	1.911	2.0242
-12.6	Payer adj	2.96	2.602	2.471
	Receiver adj	1.802	1.792	1.863
0	Payer adj	2.999	2.79	2.719
	Receiver adj	1.79	1.676	1.691
+12.6	Payer adj	3.036	2.985	2.981
	Receiver adj	1.775	1.562	1.527
+25.3	Payer adj	3.071	3.184	3.258
	Receiver adj	1.758	1.45	1.371
+63.2	Payer adj	3.184	3.852	4.205
	Receiver adj	1.717	1.154	0.977
+88.5	Payer adj	3.229	4.368	4.973
	Receiver adj	1.664	0.988	0.798

Fixed Leg Price maturity 7Y: 7345.39 USD for a notional of 1 Barrel per Month

¹ adjustment expressed as % of the fixed leg price

Commodities² : Commodity volatility effect

$\bar{\rho}$	Commodity spot volatility σ_S	0.0005	0.232	0.46	0.93
-88.5	Payer adj	0.322	0.795	1.584	3.607
	Receiver adj	0	1.268	2.546	4.495
-63.2	Payer adj	0.322	0.94	1.902	4.577
	Receiver adj	0	1.165	2.282	4.137
-25.3	Payer adj	0.323	1.164	2.419	6.015
	Receiver adj	0	0.977	1.911	3.527
-12.6	Payer adj	0.323	1.246	2.602	6.508
	Receiver adj	0	0.917	1.792	3.325
0	Payer adj	0.324	1.332	2.79	6.999
	Receiver adj	0	0.857	1.676	3.115
+12.6	Payer adj	0.324	1.422	2.985	7.501
	Receiver adj	0	0.799	1.562	2.907
+25.3	Payer adj	0.324	1.516	3.184	8.011
	Receiver adj	0	0.742	1.45	2.702
+63.2	Payer adj	0.325	1.818	3.8525	9.581
	Receiver adj	0	0.573	1.154	2.107
+88.5	Payer adj	0.326	2.05	4.368	10.771
	Receiver adj	0	0.457	0.988	1.715

Fixed Leg Price maturity 7Y: 7345.39 USD for a notional of 1 Barrel per Month

²adjustment expressed as % of the fixed leg price

Wrong Way Risk?

Basel 2, under the "Internal Model Method", models wrong way risk by means of a 1.4 multiplying factor to be applied to the zero correlation case, even if banks have the option to compute their own estimate of the multiplier, which can never go below 1.2 anyway.

What did we get in our cases? Two examples:

$$(4.973 - 2.719)/2.719 = 82\% \gg 40\%$$

$$(1.878 - 1.79)/1.79 \approx 5\% \ll 20\%$$

CVA for Credit Default Swaps (CDS) I

- Model equations: ("1" = CDS underlying, "2" = "C" = counterparty)

$$d\lambda_j(t) = k_j(\mu_j - \lambda_j(t))dt + \nu_j\sqrt{\lambda_j(t)}dZ_j(t), \quad j = 1, 2$$

- CIR models. The Brownian motions Z_1 and Z_2 driving λ_1 and λ_2 are assumed to be independent.
- Cumulative intensities are defined as : $\Lambda(t) = \int_0^t \lambda(s)ds$.
- Default times are $\tau_j = \Lambda_j^{-1}(\xi_j)$. Exponential triggers ξ_1 and ξ_2 are connected through a gaussian copula with correlation parameter ρ :

$$\xi_1 = -\ln(1 - \Phi(\sqrt{\rho}M + \sqrt{1 - \rho}Y_1)), \quad \xi_2 = -\ln(1 - \Phi(\sqrt{\rho}M + \sqrt{1 - \rho}Y_2)),$$

with M, Y_1, Y_2 i.i.d. standard normals, Φ the standard normal CDF. M : systemic factor.

- Copula functions have undergone a lot of criticism, see again the 2007 credit crisis and the related discussion, especially in relation with Collateralized Debt Obligations (CDOs).

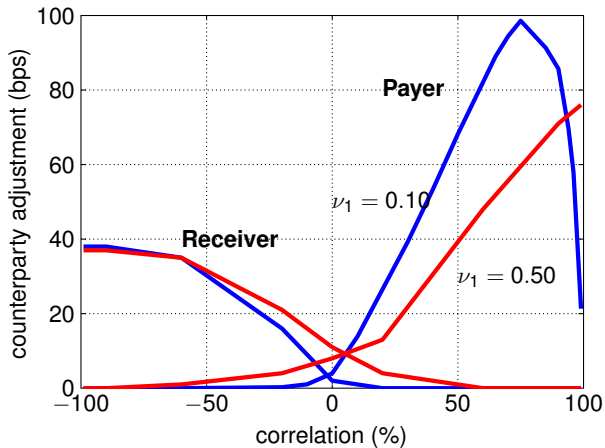
CVA for Credit Default Swaps (CDS) II

- Copulas however are necessary here: simply correlating τ 's via a non-zero covariation in the Brownians Z_1 and Z_2 in the λ 's, while keeping ξ_1 and ξ_2 independent, would not create a strong enough link between τ_1 and τ_2 . We need to put a copula between ξ_1 and ξ_2 and give up independence on ξ_1 ξ_2 to achieve a strong link. The strong link is fundamental in the case of CDS as wrong way risk is a key risk for CVA on CDS.
- In our approach, we take into account default correlation between default times τ_1 and τ_C & credit spreads volatility $\nu_j, j = 1, 2$.
- Important: volatility can amplify default time uncertainty, while high correlation reduces conditional default time uncertainty. Taking into account ρ and $\nu \implies$ better representation of market information and behavior, especially for wrong way risk.

Numerical example

We now show a CVA calculation on a CDS (both payer and receiver) in two cases: one with low credit spread vol ν_1 (blue) and one with high ν_1 (red). We plot the CVA for the CDS when the correlation parameter ρ in the copula increases from -1 to 1 . The interesting case is the payer case, where we compute the CVA on a CDS where we are buying protection from another bank “C”.

The plot shows the wrong way risk profile in the payer case when the correlation increases. However, something strange happens to the blue line of wrong way risk on the right hand side: for very high correlations, CVA suddenly drops dramatically and wrong way risk seems to disappear. What is going on?



Credit Spread Volatility as a Smoothing Parameter

The dropping blue correlation pattern is due to a feature inherent in the copula notion (any copula).

Take for example the case with constant deterministic (zero volatility) intensities for simplicity. Push dependence to co-monotonicity ($\rho = 1$ in the Gaussian case and $\xi_1 = \xi_2 =: \xi$), so that

$$\tau_1 = \frac{\xi}{\lambda_1} \quad \tau_C = \frac{\xi}{\lambda_C} \quad (*)$$

Usually $\lambda_1 > \lambda_C$ because one does not buy default protection for name 1 from an entity C that is riskier than 1.

Then $\tau_1 < \tau_C$ in all scenarios.

Then whenever τ_C hits, the CDS has already defaulted and there is no loss faced by B. This is why CVA drops to zero when $\rho \rightarrow 1$.

Credit Spread Volatility as a Smoothing Parameter

$$\tau_1 = \frac{\xi}{\lambda_1} \quad \tau_C = \frac{\xi}{\lambda_C} \quad (*)$$

However, if we increase Credit Volatility ν to values that are realistic (Brigo 2005 on CDS options) the uncertainty in (*) comes back in the "denominator" and the pattern goes back to be increasing.

The fundamental role of Credit Volatility

Credit Vol is a fundamental risk factor and should be taken into account. Current models for multiname credit derivatives (CDO, Default Baskets) ignore credit volatility assuming it is zero. This can lead to very funny results when the correlation becomes very high (unrealistic representation of systemic risk)

Equity: Intensity vs Firm value models

If we have equity S_t of a name '1' as contract underlying and we have the default of the counterparty

$$\tau_C = \Lambda_C^{-1}(\xi_C)$$

it's hard to correlate τ_C and S_1 enough, given that the exponential random variable ξ_C and any Brownian motion W_1 driving S_1 will necessarily be independent.

Underlying Equity/ Counterparty Default correlation

The only hope to create correlation is to put a stochastic λ_C and correlate it with W_1 driving S_1 . However, since most of the randomness of τ_C comes from ξ_C , this does not create enough correlation.

With equity we change family of credit models, and resort to Firm Value (or structural) models for the default of the counterparty.

Equity: Intensity vs Firm Value models

Intensity VS Firm Value models

$$\tau_C = \Lambda_C^{-1}(\xi_C) \quad \text{vs} \quad \tau_C = \inf\{t : V(t) \leq H(t)\}$$

Default of the counterparty is the first time when the counterparty firm value V hits a default barrier H .

Equity/Credit Correlation with Firm Value Models

Now if the underlying equity S_1 is driven by a brownian motion W_1 ,

$$dS_1(t) = (r - y_1)S_1(t)dt + \sigma_1(t)S_1(t)dW_1(t)$$

and the counterparty $V = V_C$ is also driven by a brownian motion W_C ,

$$dV(t) = (r - q)V(t)dt + \sigma(t)V(t)dW_C(t)$$

then an effective way to create correlation is $dW_1 dW_C = \rho_{1C} dt$

Equity: Firm Value models for the counterparty default

AT1P model

Let the risk neutral firm value V dynamics and the default barrier $\hat{H}(t)$ of the counterparty 'C' be

$$dV(t) = V(t)(r(t) - q(t))dt + V(t)\sigma(t)dW_C(t)$$

$$H(t) = \frac{H}{V_0} \mathbb{E} [V_t] e^{(-B \int_0^t \sigma_s^2 ds)}$$

and let the default time τ be **the 1st time V_C hits $H(t)$ from above**, starting from $V_0 > H$. Here $H > 0$ and B are free parameters we may use to shape the barrier.

Then the survival probability is given analytically in close form by a barrier option type formula (see Brigo and Tarengi (2005) and Brigo, Morini and Tarengi (2011)).

Firm Value model Calibration to CDS data

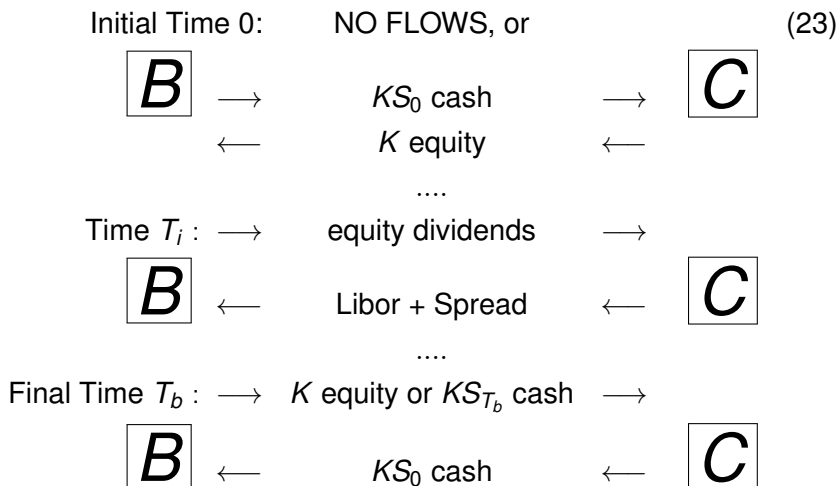
It is possible to fit exactly the CDS spreads for the counterparty through the firm value volatility $\sigma(t)$ using a bootstrapping procedure.

$$\left. \begin{array}{l} S_{0,1y}^{\text{MktCDS}} \\ S_{0,2y}^{\text{MktCDS}} \\ \vdots \\ S_{0,10y}^{\text{MktCDS}} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} dV(t) = (r - q)V(t)dt + \sigma_V(t)V(t)dW(t) \\ H(t) \\ \text{model parameters: } \sigma_V(t) \end{array} \right.$$

This ensures that the firm value model is consistent with liquid credit data of the counterparty.

In the papers we give examples based on Lehman and Parmalat.

Counterparty risk in equity return swap (ERS)



Counterparty risk in equity return swap (ERS)

- We are a default-free company (bank) “B” entering a contract with counterparty “C” (corporate). The reference underlying equity is “1”.
- “B” and “C” agree on an amount K of stocks of “1” (with price S) to be taken as nominal ($N = K S_0$). The contract starts in $T_a = 0$ and has final maturity $T_b = T$.
- At $t = 0$ there is no exchange of cash (alternatively, we can think that “C” delivers to “B” an amount K of “1” stock and receives a cash amount equal to KS_0).
- At intermediate times “B” pays to “C” the dividend flows of the stocks (if any) in exchange for a periodic risk free rate plus a spread X .
- At final maturity $T = T_b$, “B” pays KS_T to “C” (or gives back the amount K of stocks) and receives a payment KS_0 .

The (fair) spread X is chosen in order to obtain a contract whose value at inception is zero

Counterparty risk in equity return swap (ERS)

$S_0 = 20$, volatility $\sigma = 20\%$ and constant dividend yield $y = 0.80\%$. The simulation date is September 16th, 2009. The contract has maturity $T = 5y$ and the settlement of the risk free rate has a semi-annual frequency. Finally, we included a recovery rate $R_{EC} = 40\%$ for the counterparty default.

T_i	$S_i^{BID,CDS}$ (bps)	$S_i^{ASK,CDS}$ (bps)
1y	25	31
3y	34	39
5y	42	47
7y	46	51
10y	50	55

Table: CDS spreads used for the counterparty “B” credit quality in the valuation of the equity return swap.

Counterparty risk in equity return swap (ERS)

Fair spread X is driven by CVA

We compute the unilateral CVA adjustment by simulation in the model above. We search for the spread X such that the total value of the ERS INCLUDING THE CVA ADJUSTMENT is zero. In fact, it can be proven that without counterparty credit risk the theoretical fair spread X would be 0. We see that the spread X is due entirely to counterparty risk.

ρ	X (AT1P)
-1	0.0
-0.2	3.0
0	5.5
0.5	14.7
1	24.9

Table: Fair spread X (in basis points) of the Equity Return Swap in five different correlation cases for AT1P.

Compare with "Basel 2" deduced adjustments

Basel 2, under the "Internal Model Method", models wrong way risk by means of a 1.4 multiplying factor to be applied to the zero correlation case, even if banks have the option to compute their own estimate of the multiplier, which can never go below 1.2 anyway.

Is this confirmed by our model?

$$(24.9 - 5.5)/5.5 \approx 353\% \gg 40\%$$

Model Risk?

We have seen earlier that CVA/DVA may be subject to Payout Risk, in that we are not sure about the payout (closeout? First to default?)

However, as we have seen comparing the equity with the rates or credit examples, models can be different too, leading to model risk (eg firm value vs intensity models for credit)

Precise assessment of model risk is very difficult. Model validation departments should be looking into this.

A possibility is calibrating different models to the same data and see how the pricing of CVA/DVA changes.

Collateral Management and Gap Risk I

Collateral (CSA) is considered to be the solution to counterparty risk.

Periodically, the position is re-valued ("marked to market") and a quantity related to the change in value is posted on the collateral account from the party who is penalized by the change in value.

This way, the collateral account, at the periodic dates, contains an amount that is close to the actual value of the portfolio and if one counterparty were to default, the amount would be used by the surviving party as a guarantee (and viceversa).

Gap Risk is the residual risk that is left due to the fact that the realignment is only periodical. If the market were to move a lot between two realigning ("margining") dates, a significant loss would still be faced.

Folklore: Collateral completely kills CVA and gap risk is negligible.

Collateral Management and Gap Risk I

Folklore: Collateral completely kills CVA and gap risk is negligible.

We are going to show that there are cases where this is not the case at all (B. Capponi and Pallavicini 2012, Mathematical Finance)

- Risk-neutral evaluation of counterparty risk in presence of collateral management can be a difficult task, due to the complexity of clauses.
- Only few papers in the literature deal with it. Among them we cite Cherubini (2005), Alavian *et al.* (2008), Yi (2009), Assefa *et al.* (2009), Brigo et al (2011) and citations therein.
- Example: Collateralized bilateral CVA for a netted portfolio of IRS with 10y maturity and 1y coupon tenor for different default-time correlations with (and without) collateral re-hypothecation. See B, Capponi, Pallavicini and Papatheodorou (2011)

Collateral Management and Gap Risk II

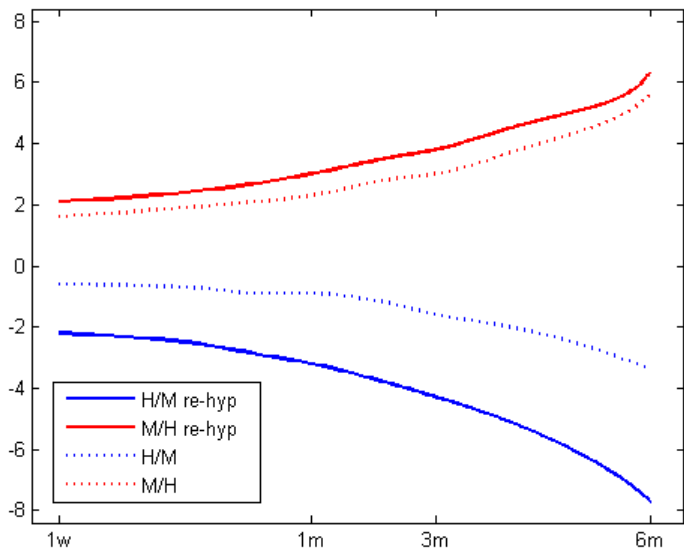


Figure explanation

Bilateral valuation adjustment, margining and rehypothecation

The figure shows the BVA(DVA-CVA) for a ten-year IRS under collateralization through margining as a function of the update frequency δ with zero correlation between rates and counterparty spread, zero correlation between rates and investor spread, and zero correlation between the counterparty and the investor defaults. The model allows for nonzero correlations as well.

Continuous lines represent the re-hypothecation case, while **dotted lines** represent the opposite case. The *red line* represents an investor riskier than the counterparty, while the *blue line* represents an investor less risky than the counterparty. All values are in basis points.

See the full paper by Brigo, Capponi, Pallavicini and Papatheodorou ‘Collateral Margining in Arbitrage-Free Counterparty Valuation Adjustment including Re-Hypothecation and Netting’

available at <http://arxiv.org/abs/1101.3926>



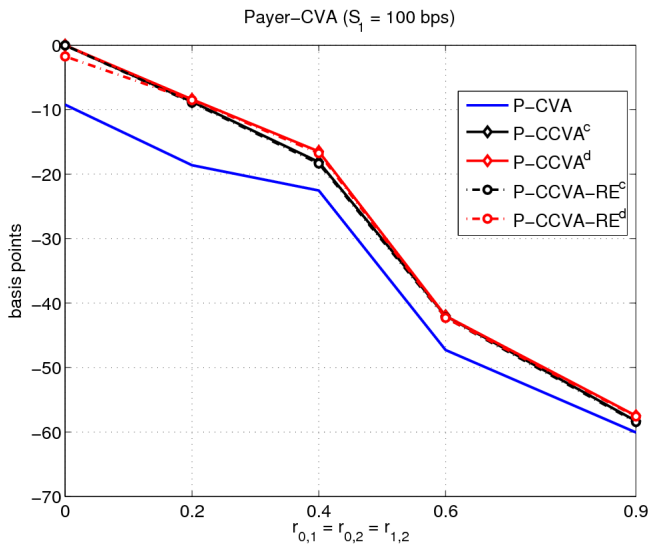
Figure explanation

From the fig, we see that the case of an investor riskier than the counterparty (M/H) leads to positive value for DVA-CVA, while the case of an investor less risky than the counterparty has the opposite behaviour. If we inspect the DVA and CVA terms as in the paper we see that when the investor is riskier the DVA part of the correction dominates, while when the investor is less risky the counterparty has the opposite behaviour.

Re-hypothecation enhances the absolute size of the correction, a reasonable behaviour, since, in such case, each party has a greater risk because of being unsecured on the collateral amount posted to the other party in case of default.

Let us now look at a case with more contagion: a CDS.

Collate



Collateral Management and Gap Risk II

The figure refers to a payer CDS contract as underlying. See the full paper by Brigo, Capponi and Pallavicini (2011) for more cases.

If the investor holds a payer CDS, he is buying protection from the counterparty, i.e. he is a protection buyer.

We assume that the spread in the fixed leg of the CDS is 100 while the initial equilibrium spread is about 250.

Given that the payer CDS will be positive in most scenarios, when the investor defaults it is quite unlikely that the net present value be in favor of the counterparty.

We then expect the CVA term to be relevant, given that the related option will be mostly in the money. This is confirmed by our outputs.

Collateral Management and Gap Risk III

We see in the figure a relevant CVA component (part of the bilateral DVA - CVA) starting at 10 and ending up at 60 bps when under high correlation.

We also see that, for zero correlation, collateralization succeeds in completely removing CVA, which goes from 10 to 0 basis points.

However, collateralization seems to become less effective as default dependence grows, in that collateralized and uncollateralized CVA become closer and closer, and for high correlations we still get 60 basis points of CVA, even under collateralization.

The reason for this is the instantaneous default contagion that, under positive dependency, pushes up the intensity of the survived entities, as soon as there is a default of the counterparty.

Collateral Management and Gap Risk IV

Indeed, the term structure of the on-default survival probabilities (see paper) lies significantly below the one of the pre-default survival probabilities conditioned on $\mathcal{G}_{\tau-}$, especially for large default correlation.

The result is that the default leg of the CDS will increase in value due to contagion, and instantaneously the Payer CDS will be worth more. This will instantly increase the loss to the investor, and most of the CVA value will come from this jump.

Given the instantaneous nature of the jump, the value at default will be quite different from the value at the last date of collateral posting, before the jump, and this explains the limited effectiveness of collateral under significantly positive default dependence.

Collateral Management and Gap Risk V

The precise payout of residual CVA and DVA adjustment cash flows after collateralization will be introduced in the Funding Costs modeling part below, and will be called $\Pi_{CVAcoll}$ and $\Pi_{DVAcoll}$. These are the terms that have been priced in the above examples.

Inclusion of Funding Cost

When we work on CVA and DVA we are focusing on cash flows contingent on the first default, and on their valuation.

There are however other cash flows that are not related directly to the default event, but to the funding costs. We work on this now.

Inclusion of Funding Cost

When managing a trading position, one needs to obtain cash in order to do a number of operations:

- borrowing / lending cash to implement the replication strategy,
- possibly repo-lending or stock-lending the replication risky asset,
- borrowing cash to post collateral
- receiving interest on posted collateral
- paying interest on received collateral
- using received collateral to reduce borrowing from treasury
- borrowing to pay a closeout cash flow upon default

and so on. Where are such funds obtained from?

- Obtain cash from her Treasury department or in the market.
- receive cash as a consequence of being in the position.

All such flows need to be remunerated:

- if one is "borrowing", this will have a cost,
- and if one is "lending", this will provide revenues.

Introduction to Quant. Analysis of Funding Costs I

We now present an introduction to funding costs modeling. Motivation?

Funding Value Adjustment Proves Costly to J.P. Morgan's 4Q Results
(Michael Rapoport, Wall St Journal, Jan 14, 2014)

"[...] So what is a funding valuation adjustment, and why did it cost J.P. Morgan Chase \$1.5 billion?"

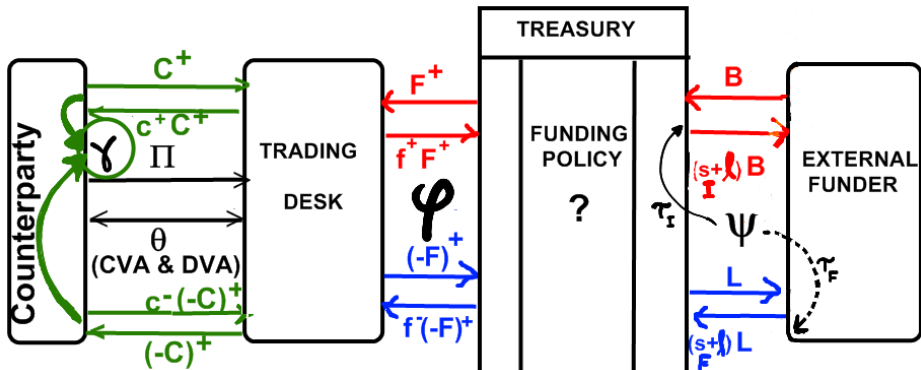
We now approach funding costs modeling by incorporating funding costs into valuation.

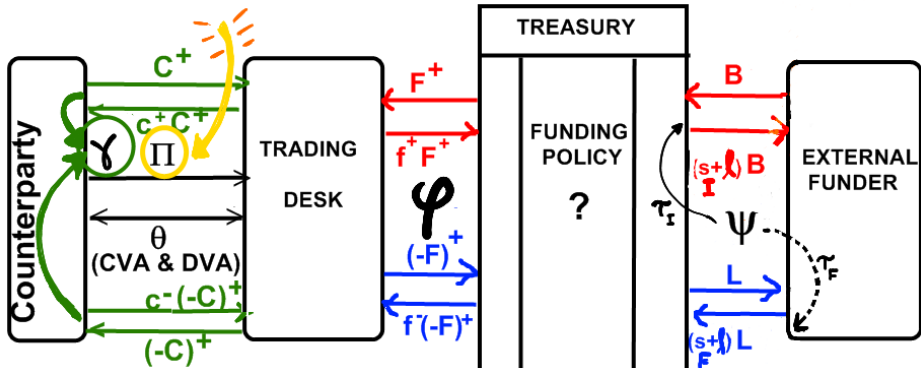
We start from scratch from the product cash flows and add collateralization, cost of collateral, CVA and DVA after collateral, and funding costs for collateral and for the replication of the product.

In the following τ_I denotes the default time of the investor / bank doing the calculation of the price (previously τ_B). "C": counterparty as before.

Adjustments Cash Flows

- $\Pi(t, T)$ will be the credit-collateral-funding free cash flows of the trade from t to T , discounted back at t with the risk free rate, as in our CVA notation earlier;
- γ will be the cost-of-collateralization cash flows, representing flows of interest remuneration or cost due to collateral posting or receiving;
- θ will be the closeout cash flows at the first default $\tau = \tau^1 = \min(\tau_I, \tau_C)$, inclusive of the trading CVA and trading DVA cash flows after collateralization;
- φ will be cost-of-funding-the-hedge cash flows for the replication strategy of the trade, representing flows of interest remuneration or cost due to the implementation of the hedging strategy;
- ψ will be the closeout cash flows for the external borrowing and lending activity the treasury of our bank is doing to fund our trading activities;





Basic Payout plus Credit and Collateral: Cash Flows I

- We calculate prices by discounting cash-flows under the pricing measure. Collateral and funding are modeled as additional cashflows (as for CVA and DVA)
- We start from derivative's basic cash flows without credit, collateral of funding risks

$$\bar{V}_t := \mathbb{E}_t[\Pi(t, T \wedge \tau) + \dots]$$

where

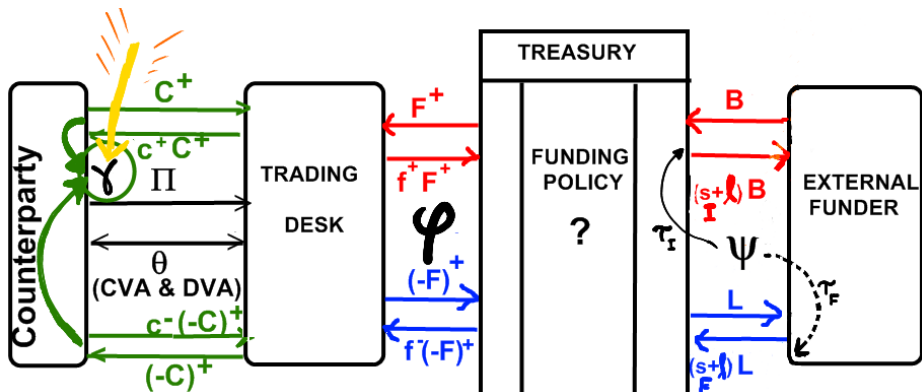
- $\tau := \tau_C \wedge \tau_I$ is the first default time, and
- $\Pi(t, u)$ is the sum of all payoff terms from t to u , discounted at t

Cash flows are stopped either at the first default or at portfolio's expiry if defaults happen later.

Basic Payout plus Credit and Collateral: Cash Flows II

- Note that we can write the credit-collateral-funding free price as

$$V_t^0 := \mathbb{E}_t[\Pi(t, T)] = \mathbb{E}_t\left[\Pi(t, T \wedge \tau) + \mathbf{1}_{\{\tau \leq T\}} D(t, \tau) V_\tau^0\right]$$



Basic Payout plus Credit and Collateral: Cash Flows

- As second contribution we consider the collateralization procedure and we add its cash flows.

$$\bar{V}_t := \mathbb{E}_t[\Pi(t, T \wedge \tau)] + \mathbb{E}_t[\gamma(t, T \wedge \tau; C) + \dots]$$

where

- C_t is the collateral account defined by the CSA,
- $\gamma(t, u; C)$ are the collateral margining costs up to time u .
- The second expected value originates what is occasionally called Liquidity Valuation Adjustment (LVA) in simplified versions of this analysis. We will show this in detail later.
- If $C > 0$ collateral has been overall posted by the counterparty to protect us, and we have to pay interest c^+ .
- If $C < 0$ we posted collateral for the counterparty (and we are remunerated at interest c^-).

Basic Payout plus Credit and Collateral: Cash Flows

- The cash flows due to the margining procedure are

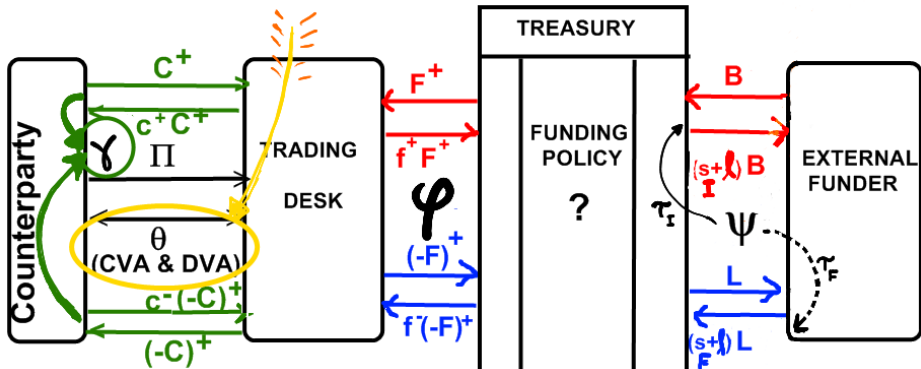
$$\gamma(t, u; C) = \int_t^u D(t, s) C_s (r_s - \tilde{c}_s) ds$$

where the collateral accrual rates are given by

$$\tilde{c}_t := c_t^+ \mathbf{1}_{\{C_t > 0\}} + c_t^- \mathbf{1}_{\{C_t < 0\}}$$

Note that if the collateral rates in \tilde{c} are both equal to the risk free rate, then this term is zero.

We should also include all the collateral updates and how the collateral nets with the mark to market at each margin call daily, including thresholds, minimum transfer amounts, etc. This can be done. Most terms cancel via telescopic sums and what is left is the γ term here. For full details see the B. Morini Pallavicini book.



Close-Out: Trading-CVA/DVA under Collateral – I

- As third contribution we consider the cash flow happening at 1st default, and we have

$$\begin{aligned}\bar{V}_t &:= \mathbb{E}_t[\Pi(t, T \wedge \tau)] \\ &+ \mathbb{E}_t[\gamma(t, T \wedge \tau; \mathbf{C})] \\ &+ \mathbb{E}_t[\mathbf{1}_{\{\tau < T\}} D(t, \tau) \theta_\tau(\mathbf{C}, \varepsilon) + \dots]\end{aligned}$$

where

→ ε_τ is the close-out amount, or residual value of the deal at default, which we called NPV earlier,

- $\varepsilon_\tau = V_\tau^0 = NPV(\tau, T) = \mathbb{E}_\tau[\Pi(\tau, T)]$ with risk free closeout, whereas $\varepsilon_\tau = \bar{V}_\tau$ with replacement closeout;

→ $\theta_\tau(\mathbf{C}, \varepsilon)$ is the on-default cash flow.

- θ_τ will contain collateral adjusted CVA and DVA payouts for the instrument cash flows

Close-Out: Trading-CVA/DVA under Collateral – II

- We define θ_τ including the pre-default value of the collateral account since it is used by the close-out netting rule to reduce exposure.
- The final expressions for θ we give below result from accounting formulas similar to the ones leading to the bilateral CVA/ DVA but inclusive of collateral netting. Full calculations and proofs are in the B. Morini Pallavicini book, here we only present the final formulas;
- An important issue here is collateral re-hypothecation. Often the collateral agreement (CSA) grants the collateral taker relatively unrestricted use of the collateral for his liquidity and trading needs until it is returned to the collateral provider.
- Under rehypothecation, Collateral can be re-invested and this lowers its remuneration cost.

Close-Out: Trading-CVA/DVA under Collateral – III

- However, while without rehypothecation the collateral provider can expect to get any excess collateral returned after honoring the amount payable on the deal, if rehypothecation is allowed the collateral provider runs the risk of losing a fraction or all of the excess collateral in case of default on the collateral taker's part.

Suppose we have a mark to market today of 100 against us. We post 100 cash collateral in the account. There is re-hypothecation. The other party can re-invest the collateral. One day later the mark to market swings heavily in our favor. We now have a mark to market of 80 in our favor. What should happen now is that the the counterparty gives us back the 100 collateral we posted yesterday and further posts 80 cash collateral in our favor as a guarantee for the trade. However before any of this can happen, the counterparty defaults.

Close-Out: Trading-CVA/DVA under Collateral – IV

- In this scenario we face two losses: we lose the collateral we posted yesterday, receiving only a recovery of it from the defaulted counterparty that reinvested it, and we face a full uncollateralized loss on the 80 we were expecting now from the mark to market.

We denote the recovery fraction on the rehypothecated collateral by REC'_I (and set $LGD'_I = 1 - REC'_I$) when the investor is the collateral taker and by REC'_C (and set $LGD'_C = 1 - REC'_C$) when the counterparty is the collateral taker. The collateral provider typically has precedence over other creditors of the defaulting party in getting back any excess capital, which means $REC_I \leq REC'_I \leq 1$ and similarly for REC'_C . If no rehypothecation is allowed and the collateral is kept safe in a segregated account, we have that $REC'_I = REC'_C = 1$.

Close-Out: Trading-CVA/DVA under Collateral – V

- The on-default cash flow $\theta_\tau(\mathbf{C}, \varepsilon)$ can be calculated by following ISDA documentation. We obtain

$$\begin{aligned}\theta_\tau(\mathbf{C}, \varepsilon) &:= \varepsilon_\tau - \mathbf{1}_{\{\tau=\tau_C < \tau_I\}} \Pi_{\text{CVAcoll}} + \mathbf{1}_{\{\tau=\tau_I < \tau_C\}} \Pi_{\text{DVAcoll}} \\ \Pi_{\text{CVAcoll}} &= \text{LGD}_C(\varepsilon_\tau^+ - \mathbf{C}_{\tau-}^+)^+ + \text{LGD}'_C(-(-\varepsilon_\tau)^+ + (-\mathbf{C}_{\tau-})^+)^+ \\ \Pi_{\text{DVAcoll}} &= \text{LGD}_I((- \varepsilon_\tau)^+ - (-\mathbf{C}_{\tau-})^+)^+ + \text{LGD}'_I(\mathbf{C}_{\tau-}^+ - \varepsilon_\tau^+)^+\end{aligned}$$

- In case of re-hypothecation, when $\text{LGD}_C = \text{LGD}'_C$ and $\text{LGD}_I = \text{LGD}'_I$, we obtain a simpler relationship with

$$\Pi_{\text{DVAcoll}} = \text{LGD}_I(-(\varepsilon_\tau - \mathbf{C}_{\tau-}))^+, \quad \Pi_{\text{CVAcoll}} = \text{LGD}_C(\varepsilon_\tau - \mathbf{C}_{\tau-})^+.$$

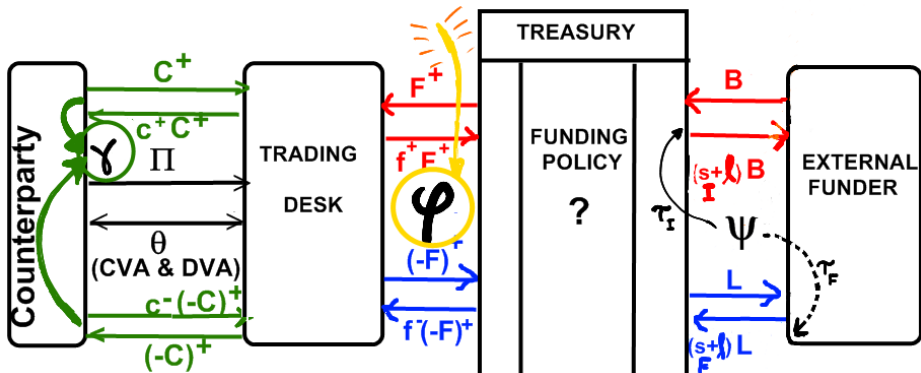
Close-Out: Trading-CVA/DVA under Collateral – VI

- In case of no collateral re-hypothecation (see full paper for all cases)

$$\Pi_{\text{CVAcoll}} = L_{\text{GD}_C}(\varepsilon_{\tau}^+ - C_{\tau^-}^+)^+$$

$$\Pi_{\text{DVAcoll}} = L_{\text{GD}_I}((-\varepsilon_{\tau})^+ - (-C_{\tau^-})^+)^+$$

- Recall once again that in all the above formulas under replacement closeout, $\varepsilon_{\tau} = \bar{V}_{\tau}$ (nonlinearity/recursion!). Under risk-free closeout, $\varepsilon_{\tau} = V_{\tau}^0$ (easier)



Funding Costs of the Replication Strategy – I

- As fourth contribution we consider the cost of funding for the hedging procedures and we add the relevant cash flows.

$$\begin{aligned}\bar{V}_t &:= \mathbb{E}_t[\Pi(t, T \wedge \tau)] + \mathbb{E}_t[\gamma(t, T \wedge \tau; \mathbf{C}) + \mathbf{1}_{\{\tau < T\}} D(t, \tau) \theta_\tau(\mathbf{C}, \varepsilon)] \\ &+ \mathbb{E}_t[\varphi(t, T \wedge \tau; F, H)]\end{aligned}$$

The last term, especially in simplified versions, is related to what is called FVA in the industry. We will point this out once we get rid of the rate r .

- F_t is the cash account for the replication of the trade,
- H_t is the risky-asset account in the replication,
- $\varphi(t, u; F, H)$ are the cash F and hedging H funding costs up to u .

- In classical Black Scholes on Equity, for a call option (no credit risk, no collateral, no funding costs),

$$\bar{V}_t^{\text{Call}} = \Delta_t S_t + \eta_t B_t =: H_t + F_t, \quad \tau = +\infty, \quad \mathbf{C} = \gamma = \varphi = 0.$$

Funding Costs of the Replication Strategy – II

- Now whenever we borrow amounts H and F from the treasury we need to pay interest f^+ , whereas when we lend amounts H and F in a short hedge position we receive interest f^- . We write such cash flows now.
- Continuously compounding format:

$$\begin{aligned} \varphi(t, u) &= \int_t^u D(t, s)(F_s + H_s) (r_s - \tilde{f}_s) ds \\ &\quad - \int_t^u D(t, s)H_s (r_s - \tilde{h}_s) ds \\ \tilde{f}_t &:= f_t^+ \mathbf{1}_{\{F_t + H_t > 0\}} + f_t^- \mathbf{1}_{\{F_t + H_t < 0\}} \quad \tilde{h}_t := h_t^+ \mathbf{1}_{\{H_t > 0\}} + h_t^- \mathbf{1}_{\{H_t < 0\}} \end{aligned}$$

Funding Costs of the Replication Strategy – III

- The expected value of φ is related to the so called FVA. If the treasury funding rates \tilde{f} are same as asset lending/borrowing \tilde{h}

$$\varphi(t, u) = \int_t^u D(t, s) F_s (r_s - \tilde{f}_s) ds$$

We will use this assumption when deriving the generalized Black Scholes example. In that case we assume

$$\tilde{f}_t := f_t^+ \mathbf{1}_{\{F_t \geq 0\}} + f_t^- \mathbf{1}_{\{F_t < 0\}}$$

- If further treasury borrows/lends at risk free $\tilde{f} = r \Rightarrow \varphi = FVA = 0$.

Funding Costs of the Replication Strategy – IV

Replica: F cash & H risky asset. Cash is borrowed $F > 0$ from the treasury at an interest f^+ (cost) or is lent $F < 0$ at a rate f^- (revenue)

Risky asset position in replica is worth H . Cash needed to buy $H > 0$ is borrowed at interest f from treasury; in this case H can be used for asset lending (Repo for example) at a rate h^+ (revenue); etc ($H < 0...$)

Include default risk of funder and funded ψ , leading to CVA_F & DVA_F .
 f^+ & f^- policy driven and related to λ_I, λ_C , more in a minute.

IMPORTANT: FVA coming from f^+ & f^- is largely offset by ψ terms valuation as we will see.

A Trader's explanation of the funding cash flows I

We now show how the adjusted cash flows originate assuming we buy a call option on an equity asset S_T with strike K .

We analyze the operations a trader would enact with the treasury and the repo market in order to fund the trade, and we map these operations to the related cash flows.

We go through the following steps in each small interval $[t, t + dt]$, seen from the point of view of the trader/investor buying the option.

A Trader's explanation of the funding cash flows II

Time t :

- 1 I wish to buy a call option with maturity T whose current price is $V_t = V(t, S_t)$. I need V_t cash to do that. So I borrow V_t cash from my bank treasury and buy the call.
- 2 I receive the collateral C_t for the call, that I give to the treasury.
- 3 Now I wish to hedge the call option I bought. To do this, I plan to repo-borrow $\Delta_t = \partial_S V_t$ stock on the repo-market.
- 4 To do this, I borrow $H_t = \Delta_t S_t$ cash at time t from the treasury.
- 5 I repo-borrow an amount Δ_t of stock, posting cash H_t guarantee.
- 6 I sell the stock I just obtained from the repo to the market, getting back the price H_t in cash.
- 7 I give H_t back to treasury.
- 8 Outstanding: I hold the Call; My debt to the treasury is $V_t - C_t$; I am Repo borrowing Δ_t stock.

A Trader's explanation of the funding cash flows III

Time $t + dt$:

- ⑨ I need to close the repo. To do that I need to give back Δ_t stock. I need to buy this stock from the market. To do that I need $\Delta_t S_{t+dt}$ cash.
- ⑩ I thus borrow $\Delta_t S_{t+dt}$ cash from the bank treasury.
- ⑪ I buy Δ_t stock and I give it back to close the repo and I get back the cash H_t deposited at time t plus interest $h_t H_t$.
- ⑫ I give back to the treasury the cash H_t I just obtained, so that the net value of the repo operation has been

$$H_t(1 + h_t dt) - \Delta_t S_{t+dt} = -\Delta_t dS_t + h_t H_t dt$$

Notice that this $-\Delta_t dS_t$ is the right amount I needed to hedge V in a classic delta hedging setting.

A Trader's explanation of the funding cash flows IV

- 13 I close the derivative position, the call option, and get V_{t+dt} cash.
- 14 I have to pay back the collateral plus interest, so I ask the treasury the amount $C_t(1 + c_t dt)$ that I give back to the counterparty.
- 15 My outstanding debt plus interest (at rate f) to the treasury is $V_t - C_t + C_t(1 + c_t dt) + (V_t - C_t)f_t dt = V_t(1 + f_t dt) + C_t(c_t - f_t dt)$. I then give to the treasury the cash V_{t+dt} I just obtained, the net effect being

$$V_{t+dt} - V_t(1 + f_t dt) - C_t(c_t - f_t) dt = dV_t - f_t V_t dt - C_t(c_t - f_t) dt$$

- 16 I now have that the total amount of flows is :

$$-\Delta_t dS_t + h_t H_t dt + dV_t - f_t V_t dt - C_t(c_t - f_t) dt$$

A Trader's explanation of the funding cash flows V

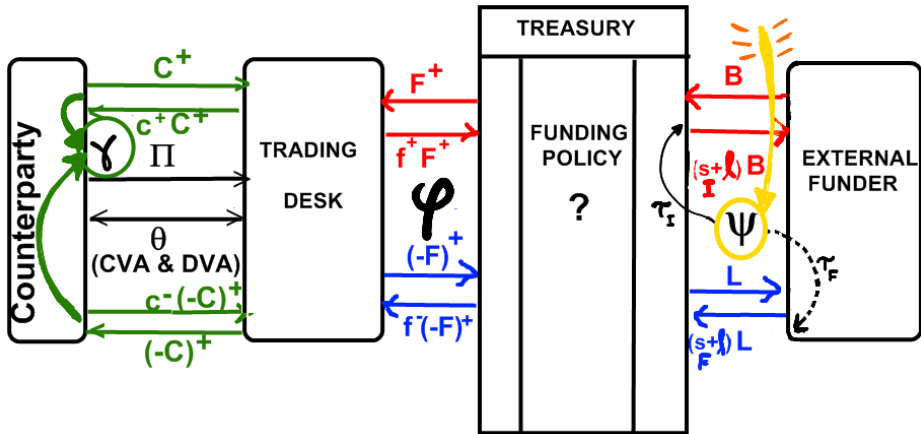
17 Now I present–value the above flows in t in a risk neutral setting.

$$\begin{aligned}
 & \mathbb{E}_t[-\Delta_t dS_t + h_t H_t dt + dV_t - f_t V_t dt - C_t(c_t - f_t) dt] = \\
 & = -\Delta_t(r_t - h_t)S_t dt + (r_t - f_t)V_t dt - C_t(c_t - f_t) dt - d\varphi(t) \\
 & = -H_t(r_t - h_t) dt + (r_t - f_t)(H_t + F_t + C_t) dt - C_t(c_t - f_t) dt - d\varphi(t) \\
 & = (h_t - f_t)H_t dt + (r_t - f_t)F_t dt + (r_t - c_t)C_t dt - d\varphi(t)
 \end{aligned}$$

This derivation holds assuming that $\mathbb{E}_t[dS_t] = r_t S_t dt$ and $\mathbb{E}_t[dV_t] = r_t V_t dt - d\varphi(t)$, where $d\varphi$ is a dividend of V in $[t, t + dt)$ expressing the funding costs. Setting the above expression to zero we obtain

$$d\varphi(t) = (h_t - f_t)H_t dt + (r_t - f_t)F_t dt + (r_t - c_t)C_t dt$$

which coincides with the definition given earlier.



Default flows ψ for the Funding part I

$$\begin{aligned} \bar{V}_t &:= \mathbb{E}_t[\Pi(t, T \wedge \tau)] + \mathbb{E}_t[\gamma(t, T \wedge \tau; \mathbf{C}) + \mathbf{1}_{\{\tau < T\}} D(t, \tau) \theta_\tau(\mathbf{C}, \varepsilon)] \\ &+ \mathbb{E}_t[\varphi(t, T \wedge \tau; F, H)] + \mathbb{E}_t[\psi(t, \tau_F, \tau, T)] \end{aligned}$$

When our bank treasury is borrowing in the market from bank F, F charges our bank a CVA due to our credit risk. Seen from our bank, this charge is a DVA_F (“treasury DVA”, as opposed to our earlier “trading DVA”) that makes the loan more expensive.

This means that if we fix the final notional, we will be able to borrow less than if we were default free. If we fix the amount borrowed now, we will have to repay more at the end. Overall the loan will be more expensive because of our bank credit risk. This is a cost.

Similarly, when our bank treasury lends externally, it measures a CVA_F (treasury CVA, as opposed to the trading CVA) on the loan due to the

Default flows ψ for the Funding part II

possibility that the borrower defaults. Loan is more remunerative due to upfront CVA_F charged to external Borrower (External Funder Benefit).

IMPORTANT

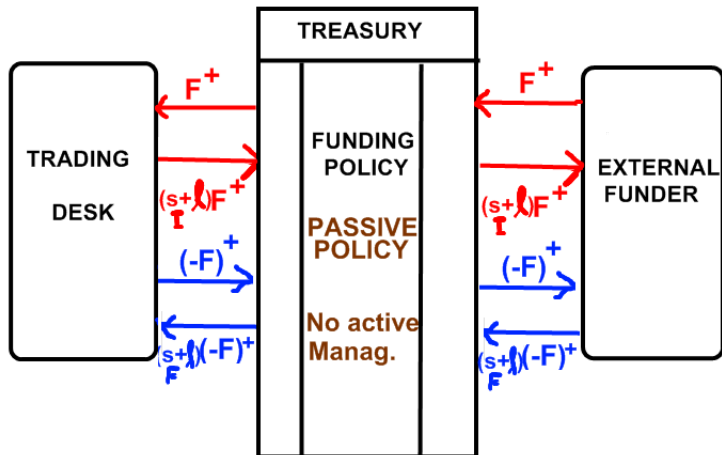
We are adding the ψ treasury DVA-CVA term to our Equation but the Eq terms would ideally sit in different parts of the bank.

- The value of the ψ part is with the treasury,
- while the other parts are with the trading desk.
- We will shortly see the different ways the treasury may pass the cost/benefits in ψ to the desk
- This is controlled with the rates f^+ and f^- in the funding cost-benefit term φ through suitable credit spreads

Default flows ψ for the Funding part III

In the following couple of slides we assume $\tilde{f} = \tilde{h}$ for simplicity, or $H = 0$ (perfectly collateralized risky hedge with collateral included in rehypothecation), since in this case external borrowing/lending reduces to F .

External Funder Benefit (EFB, blue arrow)



Default flows ψ for the Funding part I

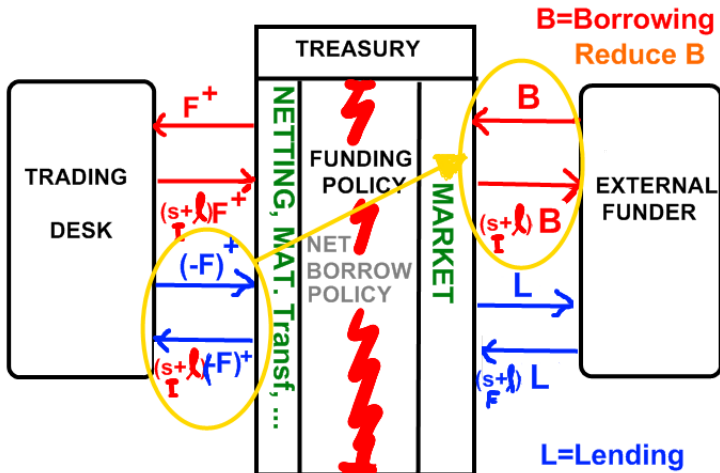
Total value of claim includes cash flows from debit and credit risk in the funding strategy that are seen by the treasury:

$$\begin{aligned} \psi_{EFB}(t, \tau_F, \tau, T) = & D(t, \tau) 1_{\{\tau = \tau_I < T\}} L_{GD_I}(F_\tau)^+ \\ & - D(t, \tau_F) 1_{\{\tau \wedge \tau_F = \tau_F < T\}} L_{GD_F}(-F_{\tau_F})^+ \end{aligned}$$

The first term on the right hand side is the funding DVA cash flow (leading to what is called occasionally DVA_2 or FDA, “Funding Debit Adjustment”). We will call the value of this cash flow DVA_F . This is triggered when our treasury is borrowing and defaults first, causing a loss to the external lender.

The second term on the right hand side is the funding credit valuation adjustment cash flow, that is triggered when our treasury is lending externally and the borrower defaults first. The value of this cash flow is called $-CVA_F$. *There is a possibly different definition for ψ :*

Reduced Borrowing Benefit (RBB).



Default flows ψ for the Funding part I

If the treasury considers the desk as net borrowing, the lending of $(-F)^+$ will be considered not as a loan but as a reduction in borrowing.

In this sense there will be no CVA_F term now, since no lending is considered by the treasury.

In this case the cash flows of the credit adjustment for the funding part consist only of the debit adjustment part and are called Reduced Borrowing Benefit:

$$\psi_{RBB}(t, \tau_F, \tau, T) = D(t, \tau) 1_{\{\tau = \tau_I < T\}} L_{GDJ}(F_\tau)^+$$

The two cases of External Funder Benefit (EFB) and Reduced Borrowing Benefit (RBB) will be discussed shortly also in connection with interest rates \tilde{f} .

$$(*) \quad \bar{V}_t = \mathbb{E}_t \left[\Pi(t, T \wedge \tau) + \gamma(t) + \mathbf{1}_{\{\tau < T\}} D(t, \tau) \theta_\tau + \varphi(t) + \psi(t, \tau_F, \tau) \right]$$

Can we interpret:

$$\mathbb{E}_t \left[\Pi(t, T \wedge \tau) + \mathbf{1}_{\{\tau < T\}} D(t, \tau) \theta_\tau(C, \varepsilon) \right] : \text{RiskFree Price} + \text{DVA} - \text{CVA?}$$

$$\mathbb{E}_t \left[\gamma(t, T \wedge \tau) + \varphi(t, T \wedge \tau; F, H) \right] : \text{Funding adjustment LVA+FVA?}$$

$$\mathbb{E}_t \left[\psi(t, \tau_F, \tau, T) \right] : \text{Treasury CVA}_F \text{ and DVA}_F$$

Not really. This is not a decomposition. It is an equation. In fact since

$$\bar{V}_t = F_t + H_t + C_t \quad (\text{re-hypo})$$

we see that the φ present value term depends (via \tilde{f}) on future $F_t + H_t = \bar{V}_t - C_t$ and generally the closeouts $\theta \psi$, via ε , F and C , depend on future \bar{V} too. All terms feed each other and there is no neat separation of risks. *Recursive pricing: Nonlinear PDE's / BSDEs for \bar{V}*
”FinalV = RiskFreeV (+ DVA?) - CVA + FVA” not possible...
 ... in theory. Approx and linearization in practice.

Funding inclusive valuation equations I

- We now substitute in (*) for \bar{V} all the earlier expressions for the terms Π , γ , θ , φ and ψ .
- We use the notation (in the sense of distributions, so π_t may contain dirac deltas in general for example)

$$\pi_t dt = \Pi(t, t + dt).$$

- Substituting we obtain

$$\begin{aligned} \bar{V}_t = \int_t^T \mathbb{E} \left\{ D(t, u) \left[\mathbf{1}_{\tau > u} (\pi_u + (r_u - \tilde{c}_u) C_u) du + \theta_u \mathbf{1}_{\tau \in du} + \right. \right. & \text{EQFund0} \\ & + \mathbf{1}_{\tau > u} ((r_u - \tilde{f}_u)(F_u + H_u) + (\tilde{h}_u - r_u) H_u) du \\ & \left. \left. + \mathbf{1}_{\{\tau > u, \tau_I \in du\}} \text{LGD}_I(F_u)^+ - \mathbf{1}_{\{\tau > u, \tau_F \in du\}} \text{LGD}_F(-F_u)^+ \right] \middle| \mathcal{G}_t \right\} \end{aligned}$$

Funding inclusive valuation equations II

- Switch to filtration \mathcal{F} , assume conditional independence of τ 's (independent ξ 's) and $\Pi(t, u)$ to be \mathcal{F}_u measurable (immersion)

$$\bar{V}_t = \int_t^T \mathbb{E}\{D(t, u; r + \lambda)[\pi_u + (r_u - \tilde{c}_u)C_u + \lambda_u\theta_u + \quad \text{EQFund1}$$

$$+(r_u - \tilde{f}_u)(F_u + H_u) + (\tilde{h}_u - r_u)H_u + \lambda_u^I \text{LGD}_I(F_u)^+ - \lambda_u^F \text{LGD}_F(-F_u)^+ | \mathcal{F}_t\} du$$

- Set $Z_u = \lambda_u^I \text{LGD}_I(F_u)^+ - \lambda_u^F \text{LGD}_F(-F_u)^+$, the Treasury DVA-CVA term, and subtract $\epsilon = \bar{V}$, assuming replacement closeout, from θ , so as to isolate the Trading CVA and DVA terms. Use $V = F + H + C$

$$\bar{V}_t = \int_t^T \mathbb{E}\{D(t, u; r + \lambda)[\pi_u + \lambda_u(\theta_u - \bar{V}_u) + (\tilde{f}_u - \tilde{c}_u)C_u + \quad \text{EQFund2}$$

$$+(r_u - \tilde{f}_u + \lambda_u)\bar{V}_u + (\tilde{h}_u - r_u)H_u + Z_u | \mathcal{F}_t\} du$$

Funding inclusive valuation equations III

- Use Feynman Kac: we know that

$$\bar{V}_t = \mathbb{E}_t \left[\int_t^T D(t, u; \mu) [\alpha_u + \beta_u \bar{V}_u] du \right] = \mathbb{E}_t \left[\int_t^T D(t, u; \mu - \beta) \alpha_u du \right]$$

- Then from EQFund2 we have, absorbing λV in the discount:

$$\begin{aligned} \bar{V}_t = \int_t^T \mathbb{E} \{ & D(t, u; r) [\pi_u + \lambda_u (\theta_u - \bar{V}_u) + (\tilde{f}_u - \tilde{c}_u) C_u + \quad \text{EQFund3} \\ & + (r_u - \tilde{f}_u) \bar{V}_u + (\tilde{h}_u - r_u) H_u + Z_u] | \mathcal{F}_t \} du \end{aligned}$$

Funding inclusive valuation equations IV

- or alternatively, absorbing the whole $(r - f + \lambda)V$

$$\bar{V}_t = \int_t^T \mathbb{E}\{D(t, u; \tilde{f})[\pi_u + \lambda_u(\theta_u - \bar{V}_u) + (\tilde{f}_u - \tilde{c}_u)C_u + (\tilde{h}_u - r_u)H_u + Z_u] | \mathcal{F}_t\} du \quad \text{EQFund4}$$

- Assuming $H = 0$ (rolled par swaps or, better, perfectly collateralized hedge with collateral included)

$$\bar{V}_t = \int_t^T \mathbb{E}\{D(t, u; \tilde{f})[\pi_u + \lambda_u(\theta_u - \bar{V}_u) + (\tilde{f}_u - \tilde{c}_u)C_u + Z_u] | \mathcal{F}_t\} du \quad \text{EQFund4'}$$

Funding inclusive valuation equations V

- If $H \neq 0$, assume now generalized delta hedging (in vector sense)

$$H_u = S_u \frac{\partial \bar{V}(u, S)}{\partial S}$$

and use Feynman Kac again:

$$\bar{V}_t = \mathbb{E}^r \int_t^T D(t, u; \mu) [\alpha_u + m(u, S_u) \frac{\partial \bar{V}}{\partial S}] du = \mathbb{E}_t^{r+m} \left[\int_t^T D(t, u; \mu) \alpha_u du \right]$$

where in general E^m is a probability measure where S grows at rate m , ie with drift mS .

Note: these are formal steps that are not fully justified mathematically, but they can be by using the theory of FBSDEs and semilinear PDEs. See B., Francischello and Pallavicini (2015) arXiv:1506.00686 or

<http://ssrn.com/abstract=2613010> for a fully rigorous treatment with proofs of existence and uniqueness of solutions.

Funding inclusive valuation equations VI

- EqFund4 with delta hedging becomes $((h - r)H = (h - r)\partial_S \bar{V})$

$$\bar{V}_t = \int_t^T \mathbb{E}^h \{ D(t, u; \tilde{f}) [\pi_u + \lambda_u (\theta_u - \bar{V}_u) + (\tilde{f}_u - \tilde{c}_u) C_u + Z_u] | \mathcal{F}_t \} du \quad \text{EQFund5}$$

- This last equation depends only on market rates. There is no theoretical risk free rate or risk neutral measure in this Eq.**
Invariance Theorem: The pricing equation is invariant wrt the specification of the short rate r_t .
- Recall: h are repo/stock lending rates for underlying risky assets,
- $(\theta_u - \bar{V}_u)$ are trading CVA and DVA after collateralization
- $(\tilde{f}_u - \tilde{c}_u) C_u$ is the cost of funding collateral with the treasury
- Z_u is the treasury CVA_F and DVA_F on the funding process
- NO Explicit funding term for the replica as this has been absorbed in the discount curve and in the collateral cost**

Funding inclusive valuation equations VII

- The last equation can be written as a semi-linear PDE or a BSDE
- As we explained, $\mathbb{E}^{\tilde{h}}$ is the expected value under a probability measure where the underlying assets evolve with a drift rate (return) of \tilde{h} . Remember that \tilde{h} depends on H , and hence on V .
- Therefore the PRICING MEASURE DEPENDS ON THE FUTURE VALUES OF THE VERY PRICE V WE ARE COMPUTING. NONLINEAR EXPECTATION. THE PRICING MEASURE BECOMES DEAL DEPENDENT.
- Under the assumption $H = 0$ we can avoid the last Feynman Kac step and the deal dependent measure: we still price under the risk neutral measure (\approx OIS) but the terms in EQFund4' bear the same description as EQFund5 we just commented.
- Notice that in EQFund5 or the simpler EQFund4' we DISCOUNT AT FUNDING directly. Some industry parties use this version and a funding discount curve.

Funding inclusive valuation equations VIII

Let's take a step back. Write EqFund1-2 more in detail.

$$\begin{aligned} \bar{V}_t = \int_t^T \mathbb{E}\{D(t, u; r + \lambda)[\pi_u + \lambda_u \theta_u + (r_u - \tilde{c}_u)C_u + \quad \text{EQFund1}' \\ + (r_u - \tilde{f}_u)(\bar{V}_u - C_u) + (\tilde{h}_u - r_u)H_u + Z_u] | \mathcal{F}_t\} du \end{aligned}$$

We can see easily that (again Feynman Kac)

$$\int_t^T \mathbb{E}\{D(t, u; r + \lambda)[\pi_u + \lambda_u V_u^0]\} du = \int_t^T \mathbb{E}\{D(t, u; r)\pi_u\} du = V_t^0$$

and, given $\theta_u = \varepsilon_u - \mathbf{1}_{\{u=\tau_C < \tau_I\}} \Pi_{\text{CVAcoll}}(u) + \mathbf{1}_{\{u=\tau_I < \tau_C\}} \Pi_{\text{DVAcoll}}(u)$,
under rehypotecation and under \mathcal{F} it is tempting to write EQFund1' as

Funding inclusive valuation equations IX

$$\bar{V} = \text{RiskFreePrice} - \text{CVA} + \text{DVA} + \text{LVA} + \text{FVA} - \text{CVA}_F + \text{DVA}_F$$

$$\text{RiskFreePrice} = V_t^0, \quad \text{LVA} = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda)(r_u - \tilde{c}_u) C_u | \mathcal{F}_t \right\} du$$

$$-\text{CVA} = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[-\text{LGD}_C \lambda_C(u) (\bar{V}_u - C_{u-})^+ \right] | \mathcal{F}_t \right\} du$$

$$\text{DVA} = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[\text{LGD}_I \lambda_I(u) (-(\bar{V}_u - C_{u-}))^+ \right] | \mathcal{F}_t \right\} du$$

$$\text{FVA} = - \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[(\tilde{f}_u - r_u)(\bar{V}_u - C_u) - (\tilde{h}_u - r_u) H_u \right] | \mathcal{F}_t \right\} du$$

$$-\text{CVA}_F = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[\text{LGD}_F \lambda_F(u) (-(\bar{V}_u - C_u - H_u))^+ \right] | \mathcal{F}_t \right\} du$$

$$\text{DVA}_F = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[\text{LGD}_I \lambda_I(u) (\bar{V}_u - C_u - H_u)^+ \right] | \mathcal{F}_t \right\} du$$

Funding inclusive valuation equations X

If we insist in applying these equations, rather than the r -independent EQFund5, then we need to find a proxy for r . This can be taken as the overnight rate (OIS discounting).

Further, if we assume that H_u is zero as it is perfectly collateralized and includes its collateral, then

$$FVA = - \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[(\tilde{f}_u - r_u)(\bar{V}_u - C_u) \right] | \mathcal{F}_t \right\} du$$

Notice that when we are borrowing cash $F = V - C$, since usually $f > r$, FVA is negative and is a cost. Also LVA can be negative. Occasionally LVA and FVA are added together in a sort of total $FVA_{tot} = LVA + FVA$.

$$FVA_{tot} = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[-(\tilde{f}_u - r_u)\bar{V}_u + (\tilde{f}_u - \tilde{c}_u)C_u \right] | \mathcal{F}_t \right\} du$$

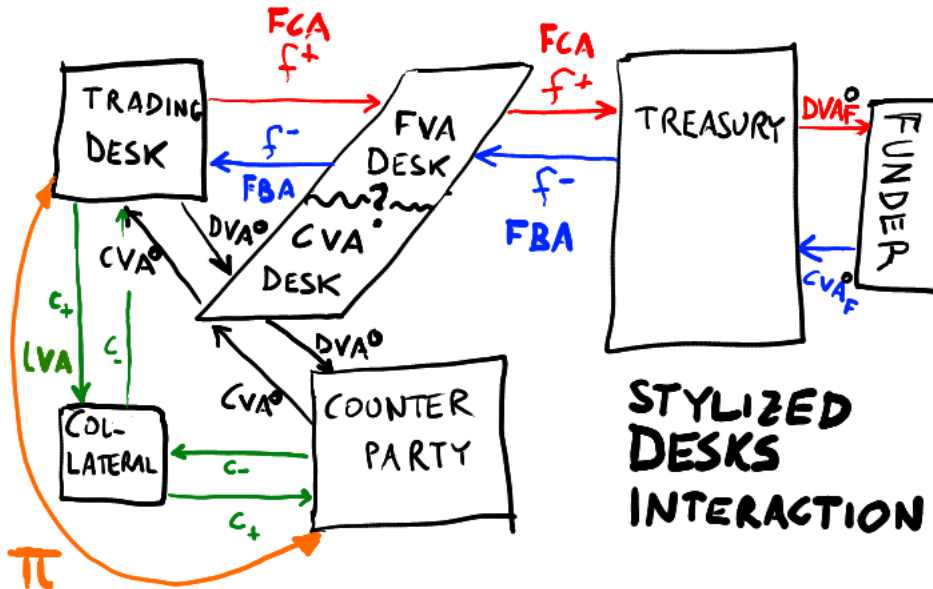
Funding inclusive valuation equations XI

Define $FVA = -FCA + FBA$ where $-FCA$ will be a Cost, and hence negative, while FBA will be a Benefit, hence positive.

$$FCA = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[(f_u^+ - r_u)(\bar{V}_u - C_u)^+ \right] \middle| \mathcal{F}_t \right\} du$$

$$FBA = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[(f_u^- - r_u)(-(\bar{V}_u - C_u))^+ \right] \middle| \mathcal{F}_t \right\} du$$

Notice the structural analogies with the expressions for CVA and DVA respectively.

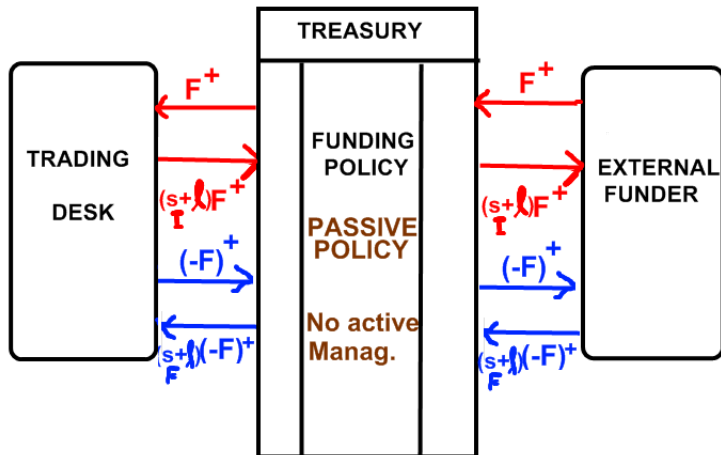


Funding inclusive valuation equations: EFB vs RBB

To further specify the equations we need to distinguish the assumptions on external lending by the treasury, and we will deal now separately with the two cases:

- External Funder Benefit (EFB)
- Reduced Borrower Benefit (RBB)

Funding inclusive valuation equations: EFB case



Funding inclusive valuation equations: EFB case

Assume that we use the EFB funding rates \tilde{f} inclusive of credit risk, so

that (set $s_{I,C,F} = \lambda_{I,C,F} L_{GD I,C,F}$)

$$f^+ = r + s_I + \ell^+, \quad f^- = r + s_F + \ell^-$$

$$-FCA = - \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[(s_I + \ell^+) (\bar{V}_u - C_u)^+ \right] \middle| \mathcal{F}_t \right\} du$$

$$FBA = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[(s_F + \ell^-) (-(\bar{V}_u - C_u))^+ \right] \middle| \mathcal{F}_t \right\} du$$

where FCA_ℓ is the part in ℓ^+ , and FBA_ℓ is the part in ℓ^- . We see $-FCA =: -DVA_F - FCA_\ell$, $FBA =: CVA_F + FBA_\ell$ where CVA_F and DVA_F are implicitly defined and coincide with the corresponding ψ valuation terms for treasury C/DVA's.

The presence of Credit Spreads in \tilde{f} leads to components in FBA and FCA that offset the Treasury DVA_F and CVA_F . Summing up:
 $V = V_0 - CVA + LVA + DVA - FCA + FBA + DVA_F - CVA_F$ where

$$V_0 = \int_t^T \mathbb{E} \left\{ D(t, u; r) \pi_u | \mathcal{F}_t \right\} du, \quad LVA = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) (r_u - \tilde{c}_u) C_u | \mathcal{F}_t \right\} du$$

$$-FCA (= -DVA_F - FCA_\ell) = - \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) (s_I(u) + \ell^+(t)) (\bar{V}_u - C_u)^+ | \mathcal{F}_t \right\} du$$

$$FBA (= CVA_F + FBA_\ell) = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[(s_F + \ell^-) (-(\bar{V}_u - C_u))^+ \right] | \mathcal{F}_t \right\} du$$

$$-CVA = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[-s_C (\bar{V}_u - C_u)^+ \right] | \mathcal{F}_t \right\} du$$

$$DVA = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[s_I (-(\bar{V}_u - C_u)^+) \right] | \mathcal{F}_t \right\} du$$

$$DVA_F = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) s_I(u) (\bar{V}_u - C_u)^+ | \mathcal{F}_t \right\} du$$

$$-CVA_F = - \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[s_F (-(\bar{V}_u - C_u)^+) \right] | \mathcal{F}_t \right\} du$$

Double Counting: EFB Case

Summing up: $V = V_0(\text{risk free}) +$

$$\underbrace{-CVA + DVA}_{\text{Trading CVA DVA}} \quad \underbrace{+LVA}_{\text{Coll cost \& benefit}} \quad \underbrace{-FCA + FBA}_{\text{Replica funding cost \& benefit}} \quad + \underbrace{DVA_F - CVA_F}_{\text{Funding CVA DVA}}$$

Remember also what we just found for FCA and FBA:

$$V = V_0 - CVA + DVA + LVA \quad \underbrace{-FCA}_{-DVA_F - FCA_\ell} \quad + \quad \underbrace{FBA}_{CVA_F + FBA_\ell} \quad + DVA_F - CVA_F$$

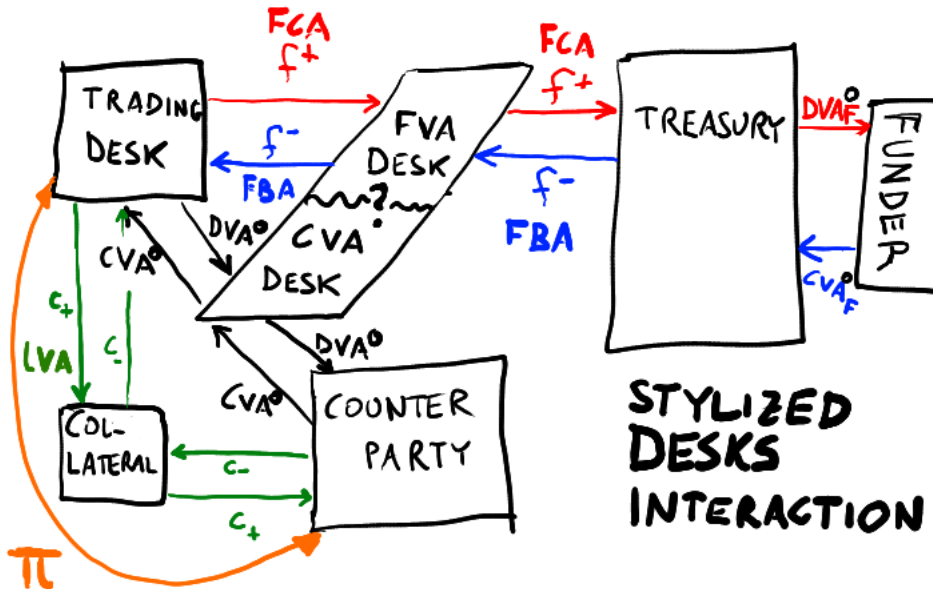
The blue and red terms are passed by the treasury to the desk so the total net value for the whole bank cancels

Double Counting: EFB Case

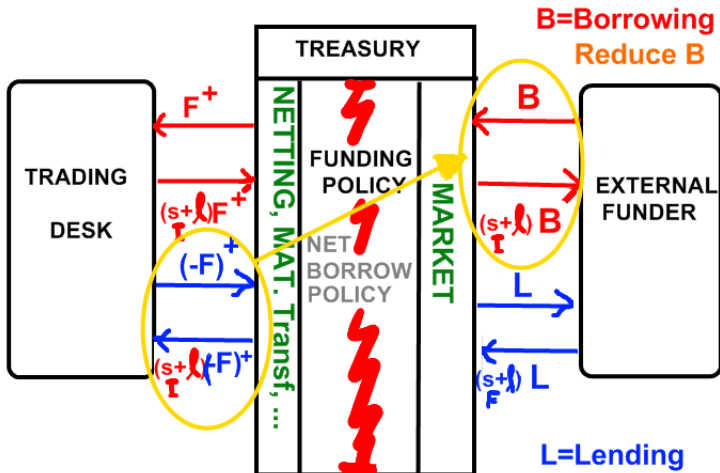
Keeping the full formula without simplifying

$$V = V_0 - CVA + DVA + LVA \underbrace{-FCA}_{-DVA_F - FCA_\ell} + \underbrace{FBA}_{CVA_F + FBA_\ell} + DVA_F - CVA_F$$

- If bases $\ell = 0$ then Funding costs are offset by the treasury CVA_F and DVA_F and "there are no funding costs" overall.
- However, for the trading desk (TDesk) there is still a cost $FCA = DVA_F + FCA_\ell$ to be paid to Treasury. This happens via the FVA desk if that exists, or via the CVA desk otherwise.
- TDesk also sees a benefit $FBA = CVA_F + FBA_\ell$ received from treasury via the FVA desk if existing, or CVA desk otherwise.
- Treasury pays DVA_F at time 0 to Funder, charging that as a cost FCA to Tdesk, and receives CVA_F at time 0 from funder, and passes that to the TDesk as benefit. All this via F/CVA desk
- CVA desk still deals with trading CVA and DVA



Funding inclusive valuation equations: RBB case



Funding inclusive valuation equations: RBB case

We are now going to specialize the funding equations

$$FCA = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[(f_u^+ - r_u)(\bar{V}_u - C_u)^+ \right] \middle| \mathcal{F}_t \right\} du$$

$$FBA = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[(f_u^- - r_u)(-(\bar{V}_u - C_u))^+ \right] \middle| \mathcal{F}_t \right\} du$$

to the RBB case where

$$f^+ - r = s_l + \ell^+, \quad f^- - r = s_l + \ell^-.$$

We also take $\psi = \psi_{RBB}$ (no CVA_F part).

Funding inclusive valuation equations: RBB case

The FCA term remains as in the EFB case.

However, notice what happens to FBA now, in the RBB case.

$$FBA = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[(s_l + \ell^-) (-(\bar{V}_u - C_u))^+ \right] | \mathcal{F}_t \right\} du = \text{DVA} + \text{FBA}_\ell$$

We have that FBA includes a copy of the *trading* DVA

$$V_0 = \int_t^T \mathbb{E} \left\{ D(t, u; r) \pi_u | \mathcal{F}_t \right\} du, \quad LVA = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) (r_u - \tilde{c}_u) C_u | \mathcal{F}_t \right\} du$$

$$-FCA (= -DVA_F - FCA_\ell) = - \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) (s_I(u) + \ell^+(t)) (\bar{V}_u - C_u)^+ | \mathcal{F}_t \right\} du$$

$$FBA (= DVA + FBA_\ell) = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[(s_I + \ell^-) (-\bar{V}_u - C_u)^+ \right] | \mathcal{F}_t \right\} du$$

$$-CVA = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[-s_C (\bar{V}_u - C_{u-})^+ \right] | \mathcal{F}_t \right\} du$$

$$DVA = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[s_I (-\bar{V}_u - C_{u-})^+ \right] | \mathcal{F}_t \right\} du$$

$$DVA_F = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) s_I(u) (\bar{V}_u - C_u)^+ | \mathcal{F}_t \right\} du$$

$$-CVA_F = 0;$$

One of the two DVA must go.

Funding inclusive valuation equations: RBB case I

$$V = V_0 - CVA + DVA + LVA \underbrace{-FCA}_{-DVA_F - FCA_\ell} + \underbrace{FBA}_{DVA + FBA_\ell} + DVA_F$$

Now we no longer have exact offsetting terms. The DVA inside FBA will not be offset by a CVA_F . The problem is that the formula contains two identical DVA's.

Compare with the EFB case:

$$V = V_0 - CVA + DVA + LVA \underbrace{-FCA}_{-DVA_F - FCA_\ell} + \underbrace{FBA}_{CVA_F + FBA_\ell} + DVA_F - CVA_F$$

Funding inclusive valuation equations: RBB case II

When we compute the funding rate f^- we use our own $s_I = \lambda_I L_{GD_I}$ as a gain spread, based on the “reduced borrowing” argument.

But receiving back interest s_I as a benefit of reduced borrowing means we are in fact computing a rolling-DVA for F as $[t, t + dt)$ spans the whole trading interval. Since $F = V - C$, we are basically computing again the trading DVA by means of the funding rate f^- .

We are thus counting our own default risk twice on the *same* exposure scenario $-(V - C)^+$. This is why, save for the basis term ℓ_- , we should take one of the two DVA's out to avoid double counting.

Funding inclusive valuation equations: RBB case III

We thus have two possible choices:

1: Privilege Credit Adjustments over Funding ones

$$V = V_0 - CVA + DVA + LVA \underbrace{-FCA}_{-DVA_F - FCA_\ell} + \underbrace{FBA}_{-DVA + FBA_\ell} + DVA_F$$

Treasury is charged initially DVA_F , and charges this back to TDesk as part of FCA via FVADesk if \exists , else CVADesk.

For the reduced borrowing TDesk sees a benefit FBA_ℓ , obtained from treasury via FVADesk as a payment reduction, and TDesk is still charged DVA at time 0 and receives CVA at time 0 from counterparty via CVADesk. Overall (notice that if $\ell = 0$ there's no funding adjustment)

$$V = V_0 - CVA + DVA + LVA - FCA_\ell + FBA_\ell.$$

Funding inclusive valuation equations: RBB case IV

2: Privilege Funding adjustments over the Credit ones

$$V = V_0 - CVA + \cancel{DVA} + LVA - \underbrace{FCA}_{-DVA_F - FCA_\ell} + \underbrace{FBA}_{DVA + FBA_\ell} + DVA_F$$

resulting in

$$V = V_0 - CVA + LVA - FCA_\ell + \underbrace{FBA}_{DVA + FBA_\ell}$$

Now DVA is managed by the FVA Desk. Notice that if liquidity basis $\ell = 0$ then $V = V_0 - CVA + LVA + \underbrace{FBA}_{DVA}$ and the only funding term is the benefit term given by trading DVA

Current best practice?

As of January 2015, the RBB formulas above with

V^0 , LVA, CVA, FBA (inclusive of trading DVA), FCA and DVA_F

are as close as possible to what one of the top global banks is doing.

From several conversations I believe that also other global banks are doing something very similar.

Adjustments go in different parts of the bank

The ψ term: treasury component of \bar{V}

As we said earlier, we added into the \bar{V} cash flows ψ , the treasury DVA-CVA term, leading to CVA_F and DVA_F in EFB or to 2 DVA in RBB. However, in reality the Eq terms sit in different parts of the bank.

- The value of the ψ part is with the treasury,
- while the other parts are with the trading desk.
- We have seen two different ways the treasury may pass the cost/benefits in ψ to the desk (EBF and RBB) but the desk is still charged, so the term ψ is often omitted when doing valuation from the desk
- and the ψ terms stay with the treasury
- Hence the trading desk applies the above formulas but without adding the ψ terms valuation

Funding valuation: Black Scholes (only funding) I

To get a feel for how all this translates into the familiar option pricing framework of Black Scholes we start with pricing a CALL option in the standard Black Scholes model under funding costs.

- Equity Call $(S_T - K)^+$, underlying S , maturity T , strike K .
- $dS_t = rS_t dt + \sigma S_t dW_t$ under \mathbb{Q} .
- $r = 0.01 = 100$ bps, $\sigma = 0.25 = 25\%$, $S_0 = 100$, $K = 80$, $T = 3y$
- Classic price (no credit / collateral / funding) $V_0^0 = 28.9$

$$V_t^0 = S_t \Phi(d_1(t)) - Ke^{-r(T-t)} \Phi(d_2(t)), \quad d_{1,2} = \frac{\ln \frac{S_t}{K} + (r \pm \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}$$

We first look at the case without default and collateralization (so only funding). We compare full Monte Carlo to four simplified approaches. $f^+ = f^- = \hat{f}$ and $s^f = \hat{f} - r$ funding spread. We assume $h = f$ (so the H term in φ goes to zero) and there is no repo rate effect on funding.

Funding valuation: Black Scholes (only funding) II

- (i) f discounting of the risk-free Black-Scholes (hat means $f^+ = f^-$)

$$V_t^{(i)} = e^{-\hat{f}T} \left(S_t \Phi(d_1(t)) - Ke^{-r(T-t)} \Phi(d_2(t)) \right)$$

- (ii) Black-Scholes with drift $h = f$ and discount f :

$$V_t^{(ii)} = S_t \Phi(g_1(t)) - Ke^{-\hat{f}(T-t)} \Phi(g_2(t)), \quad g_{1,2} = \frac{\ln(S_t/K) + (\hat{f} \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

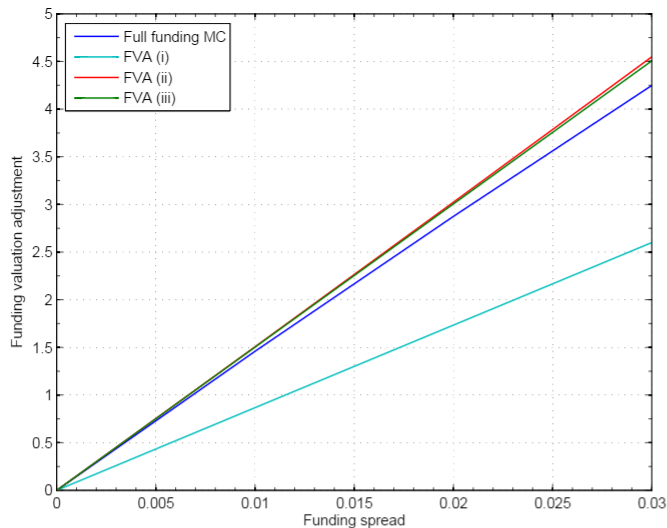
- (iii) FVA approximated formula keeping in mind

$$F_t = -Ke^{-r(T-t)} \Phi(d_2(t)):$$

$$\text{FVA}_0^{(iii)} = (r - \hat{f}) \int_0^T \mathbb{E}_0 \left\{ e^{-rs} [F_s] \right\} ds = (\hat{f} - r) Ke^{-rT} \int_0^T \mathbb{E}_0 \left\{ \Phi(d_2(s)) \right\} ds$$

$$\text{FVA}^{(i,ii)} = V^{(i,ii)} - V^0.$$

Funding valuation: Black Scholes (only funding) III



Funding valuation: Black Scholes I

We now include credit & collateral. We consider a discrete probability distribution of default. Both “I” and “C” can only default at year 1 or year 2. The localized joint default probabilities are in the D matrix.

The rows denote τ^I values and columns denote τ^C . For example, in D the event $(\tau_I = 2yr, \tau_C = 1yr)$ has a 3% probability

Simultaneous defaults are introduced and we determine the close-out entity by a random draw from a standard uniform distribution wrt 0.5.

$$D = \begin{array}{c} \\ 1yr \\ 2yr \\ n.d. \end{array} \begin{array}{ccc} 1yr & 2yr & n.d. \\ \left(\begin{array}{ccc} 0.01 & 0.01 & 0.03 \\ 0.03 & 0.01 & 0.05 \\ 0.07 & 0.09 & 0.70 \end{array} \right), & \tau_K(D) = 0.21 & (24)$$

where $n.d.$ means no default and τ_K denotes the rank correlation as measured by Kendall's tau.

Funding valuation: Black Scholes II

Note also that the distributions are skewed in the sense that the counterparty has a higher default probability than the investor.

The loss given default is 50% for both the investor and the counterparty and the loss on any posted collateral is considered the same.

The collateral rates \tilde{c} are chosen to be equal to r so $LVA = 0$.

We assume that the collateral account is equal to the risk-free price at each margin date, $C_t = V_t^0$. This is reasonable as the dealer and client will be able to agree on this price. Also, choosing the collateral this way has the added advantage that the collateral account C works as a control variate, reducing the variance of the least-squares Monte Carlo estimator of the deal price.

CVA $(V - C)^+$ & DVA $-(V - C)^+$ are small due to collateral but they can make V higher or lower than V^0 & trigger costs.

Table: Total value. Standard errors are given in parentheses. ^a Changes in one funding rate while keeping the other fixed to 100 bps. ^b. $V_0^0 = 28.9$

Funding ^a	Long	Short
<i>Borrowing rate f^+</i>		
0 bps	29.36 (0.12)	-26.20 (0.17)
100 bps	28.70 (0.15)	-28.72 (0.15)
200 bps	28.05 (0.21)	-31.37 (0.32)
300 bps	27.38 (0.29)	-34.26 (0.55)
400 bps	26.67 (0.38)	-37.24 (0.86)
<i>Lending rate f^-</i>		
0 bps	26.17 (0.18)	-29.38 (0.11)
100 bps	28.70 (0.15)	-28.72 (0.15)
200 bps	31.37 (0.32)	-28.07 (0.22)
300 bps	34.28 (0.55)	-27.41 (0.30)
400 bps	37.28 (0.88)	-26.69 (0.39)

Funding valuation: Black Scholes & re-hypothecation I

Finally, assuming cash collateral, we consider rehypothecation & allow parties to use collateral as a funding source.

If the collateral is posted to the investor, this means it effectively reduces his costs of funding the delta-hedging strategy. As the payoff of the call is one-sided, the investor only receives collateral when he holds a long position in the call option. But as he hedges this position by short-selling the underlying stock and lending the excess cash proceeds, the collateral adds to his cash lending position and increases the funding benefit of the deal.

Analogously, if the investor has a short position, he posts collateral to the counterparty and a higher borrowing rate would increase his costs of funding the collateral he has to post as well as his delta-hedge.

The following Table reports the results for the short and long positions in the call option when rehypothecation is allowed.

Table: Total value with re-hyp. Std errs are given in parentheses. ^a Changes in one funding rate while keeping the other fixed to 100 bps. $V_0^0 = 28.9$

Funding ^a	Long	Short
<i>Borrowing rate f^+</i>		
0 bps	29.33 (0.12)	-25.56 (0.22)
100 bps	28.70 (0.15)	-28.73 (0.15)
200 bps	28.07 (0.22)	-32.14 (0.36)
300 bps	27.42 (0.30)	-35.93 (0.68)
400 bps	26.75 (0.41)	-39.95 (1.14)
<i>Lending rate f^-</i>		
0 bps	25.53 (0.22)	-29.36 (0.11)
100 bps	28.70 (0.15)	-28.73 (0.15)
200 bps	32.14 (0.37)	-28.10 (0.22)
300 bps	35.94 (0.69)	-27.45 (0.31)
400 bps	39.99 (1.17)	-26.77 (0.42)

Nonlinear valuation: Black Scholes I

More generally, go back to the r -independent formula EQFund5.

$$\bar{V}_t = \int_t^T \mathbb{E}^h \{ D(t, u; \tilde{f}) [\pi_u + \lambda_u(\theta_u - \bar{V}_u) + (\tilde{f}_u - \tilde{c}_u)C_u + Z_u] | \mathcal{F}_t \} du \quad \text{EQFund5}$$

- Write this last eq as a BSDEs by completing the martingale term. Add and subtract \int_0^t , then notice that one term becomes \int_0^T and its E_t is a martingale M_t . Use the martingale representation theorem (see B. and Pallavicini [36], JFE 1, pp 1-60 for details).

$$d\bar{V}_t - [\tilde{f}_t \bar{V}_t + (\tilde{f}_t - \tilde{c}_t)C_t + \pi_t + \lambda_t(\theta(C_t, \bar{V}_t) - \bar{V}_t) - (r - \tilde{h})H_t + Z_t]dt = dM_t,$$

$$\bar{V}_t = H_t + F_t + C_t, \quad \varepsilon_t = \bar{V}_t \text{ (replacement closeout), } \bar{V}_T = 0.$$

Recall that \tilde{f} depends on \bar{V} nonlinearly, and so does \tilde{c} on C and \tilde{h} on H . M is a martingale under the pre-default filtration.

Nonlinear valuation: Black Scholes II

- Assume a Markovian vector of underlying assets S (pre- credit and funding) with diffusive generator $\mathcal{L}^{r,\sigma}$ under \mathbb{Q} . Let this be associated with brownian W under \mathbb{Q} .

$$dS = rSdt + \sigma(t, S)SdW_t, \quad \mathcal{L}^{r,\sigma} u(t, S) = rS\partial_S u + \frac{1}{2}\sigma(t, S)^2 S^2 \partial_S^2 u$$

- Use Ito's formula on $\bar{V}(t, S)$ and match dt (and dW) terms from BSDE: obtain PDE (& explicit representation for BSDE term ZdW). Details are given in the Pallavicini Perini and B. (2011, 2012) reports.

Nonlinear valuation: Black Scholes III

This leads to the following PDE with terminal condition $\bar{V}_T = 0$.

$$(\partial_t - \tilde{f}_t - \lambda_t + \mathcal{L}^{r,\sigma}) \bar{V}_t + (\tilde{f}_t - \tilde{c}_t) C_t + \pi_t + \lambda_t \theta(C_t, \bar{V}_t) - (r - \tilde{h}) H_t + Z_t = 0 \quad [\text{NPDE1}]$$

$$\bar{V}_t = H_t + F_t + C_t, \quad \varepsilon_t = \bar{V}_t \quad (\text{replacement closeout})$$

Alternatively, the funding/credit risk free price can be used for closeout (risk free closeout), simplifying calculations.

Nonlinear valuation: Black Scholes IV

The above PDE can be simplified further by assuming Delta Hedging:

$$H_t = S_t \frac{\partial \bar{V}_t}{\partial S} \text{ (delta hedging), leading to}$$

$$(\partial_t - \tilde{r}_t - \lambda_t + \mathcal{L}^{\tilde{h}, \sigma}) \bar{V}_t + (\tilde{r}_t - \tilde{c}_t) C_t + \pi_t + \lambda_t \theta(C_t, \bar{V}_t) + Z_t(F_t) = 0, \text{ [NPDE2]}$$

This PDE is NON-LINEAR not only because of θ , but also because \tilde{r} depends on F , and \tilde{h} on H , and hence both on \bar{V} itself.

IMPORTANT: Again invariance theorem.

PDE DOES NOT DEPEND ON r .

This is good, since r is a theoretical rate that does not correspond to any market observable.

IMPORTANT: If valuing from the trading desk point of view, we should take out the Z term (treasury CVA_F and DVA_F).

Nonlinear valuation: Black Scholes V

We now try to bring this PDE closer to the classical Black Scholes PDE. Assume collateral is a variable fraction $\alpha_t > 0$ of mark to market, with α_t being \mathcal{F}_t adapted, typically non-negative and smaller than one. Recall that we assume $\tilde{h} = \tilde{f}$ and

$$\tilde{f}_t = f_+ 1_{F \geq 0} + f_- 1_{F \leq 0}, \quad \tilde{c}_t = c_+ 1_{V_t \geq 0} + c_- 1_{V_t \leq 0}, \quad f_{+,-} \text{ and } c_{+,-} \text{ constants.}$$

$$\begin{aligned} \partial_t V - (f_+ - s^I)(V - S_t \partial_S V_t - \alpha V)^+ + (f_- - s^F)(-V + S_t \partial_S V_t + \alpha V)^+ - \lambda_t V + \\ + \frac{1}{2} \sigma^2 S^2 \partial_S^2 V - c_+ \alpha_t (V_t)^+ + c_- \alpha_t (-V_t)^+ + \pi_t + \lambda_t \theta_t(V_t) = 0 \end{aligned}$$

NONLINEAR PDE (SEMILINEAR).

Nonlinear valuation: Black Scholes VI

$$\partial_t V - (f_+ - s^I)(V - S_t \partial_S V_t - \alpha V)^+ + (f_- - s^F)(-V + S_t \partial_S V_t + \alpha V)^+ - \lambda_t V + \frac{1}{2} \sigma^2 S^2 \partial_S^2 V - c_+ \alpha_t (V_t)^+ + c_- \alpha_t (-V_t)^+ + \pi_t + \lambda_t \theta_t (V_t) = 0$$

λ is the first to default intensity, π is the ongoing dividend cash flow process of the payout, θ are the complex optional contractual cash flows at default including CVA and DVA payouts after collateral. c_+ and c_- are the borrowing and lending rates for collateral, $s^{I,F} = \lambda^{I,F} L_{GD,I,F}$, spread of investment bank & funder from Z (treasury CVA and DVA).

We can use Lipschitz coefficients results to investigate $\exists!$ of viscosity solutions. Classical solutions may also be found but require much stronger assumptions and regularizations.

None of this is much applicable in practical situations.

The Black Scholes Benchmark Case I

$$\begin{aligned} \partial_t V - (f_+ - s^l)(V - S_t \partial_S V_t - \alpha V)^+ + (f_- - s^F)(-V + S_t \partial_S V_t + \alpha V)^+ - \lambda_t V + \\ + \frac{1}{2} \sigma^2 S^2 \partial_S^2 V - c_+ \alpha_t (V_t)^+ + c_- \alpha_t (-V_t)^+ + \pi_t + \lambda_t \theta_t (V_t) = 0 \end{aligned}$$

Notice that

- if $f_+ = f_- = r$ (symmetric risk free borrowing and lending),
- $\alpha = 0$ (no collateral),
- $\lambda = 0$ (no credit risk),

then we get back the Black Scholes LINEAR (parabolic) PDE.

$$\partial_t V + r S_t \partial_S V_t + \frac{1}{2} \sigma^2 S^2 \partial_S^2 V - rV + \pi = 0.$$

In Theory: Nonlinearities due to funding I

So what is the THEORY telling us?

We know that NONLINEAR PDEs cannot be solved as Feynman Kac expectations.

Backward Stochastic Differential Equations (BSDEs)

For NPDEs, the correct translation in stochastic terms are BSDEs. The equations have a recursive nature and simulation is quite complicated. Or we keep the PDE.

BSDEs due to asymmetric rates had been briefly introduced in El Karoui, Peng and Quenez (1997). We added credit gap risk & collateral processes, adding more nonlinearity into the picture.

In Theory: Nonlinearities due to funding II

Aggregation–dependent and asymmetric valuation

Worse, the valuation of a portfolio is aggregation dependent and is different for the two parties in a deal. In the classical pricing theory a la Black Scholes, if we have 2 or more derivatives in a portfolio we can price each separately and then add up. Not so with funding and replacement closeout at default. Moreover, without funding the price to one entity is minus the price to the other one. This is no longer true.

Aggregation levels decided a priori and somewhat arbitrarily.

In Theory: Nonlinearities due to funding III

Consistent global modeling across asset classes and risks

Once the level of aggregation is set, the funding valuation problem is non-separable. An holistic approach is needed and consistent modeling across trading desks and asset classes is needed. Internal competition in banks does not favour this.

Furthermore, the classical transaction-independent arbitrage free price is lost, now the price depends on the specific entities trading the product and on their policies (λ, f, ℓ) . Recall \mathbb{E}^h and PDE coefficients depending on \bar{V} nonlinearly.

In Theory: Nonlinearities due to funding IV

The end of Platonic pricing?

There is no Platonic measure \mathbb{Q} in the sky to price all derivatives with an expectation where all assets have the risk free return r . Now the pricing measure is product dependent, and every trade will have a specific measure. This is an implication of the PDE non-linearity.

When basic financial sense leads to complex mathematics

Notice that, in theory, adherence to real banking policies does not make the problem "boring, purely accounting-like and trivial". Rather, valuation becomes aggregation dependent and holistic. We need BSDEs rather than expected values, or nonlinear PDEs rather than linear ones.

In Theory: Nonlinearities due to funding V

This would open many problems of operational efficiency and efficiency of implementation.

However, in practice things are implemented quite differently, as we'll see in a minute...

Before looking at that, now that we have seen how to compute funding costs, a fundamental question.

Price of Value?

Why should the client pay for our funding policy choices?

Again recall entity specific (λ, f, ℓ) , \mathbb{E}^h and PDE coefficients depending on \bar{V} .

Each entity computes a different funding adjusted price for the same product

and “prices” change with aggregation.

The funding adjusted “price” is not a price in the conventional sense. We may use it for cost/profitability analysis or to pay our treasury, but can we charge it to a client?

Can the client charge us too as she has funding costs?

Price of Value?

Accessibility of valuation parameters

How can the client check our price is fair if she has no access to our funding policy (less transparent than credit standing) and vice versa?

It is more a "value" than a "price".

Provocative question. Why do not we charge an Electricity Bill Valuation Adjustment (EBVA)?

Should funding costs be zero?

In a number of papers, Hull and White argued that there should be no funding costs.

They invoked the Modigliani Miller theorem. A folk version of the theorem is this:

“If market price processes follow random walks, and there are no

- taxes,
- bankruptcy costs,
- agency costs,
- asymmetric information

and if the market is efficient *then* the value of a firm does not depend on how the firm is financed.

Should funding costs be zero?

However the above assumptions do not hold in practice.

The very presence of liquidity bases ℓ violates the assumptions.

However we saw in the above calculations that if $\ell = 0$ then there are indeed no funding costs. For example, in the EFB framework

$$V = V_0 - CVA + DVA + LVA \underbrace{-FCA}_{-DVA_F - FCA_\ell} + \underbrace{FBA}_{CVA_F + FBA_\ell} + DVA_F - CVA_F$$

we see that if $\ell = 0$ and $\tilde{c} = r$ we end up with

$$V = V_0 - CVA + DVA$$

and there are no funding costs indeed.

Should funding costs be zero?

So it is a matter of qualifying the assumptions in the Modigliani Miller theorem.

Market imperfections such as the bases ℓ , among others, may make the theorem not valid and hence funding costs become relevant.

We now go back to the implications of nonlinearities of aggregation dependent values and nonlinear valuation. We analyzed the theoretical implications. But are banks taking those into account?

Nonlinearities in theory. What about practice?

... in practical implementation, in many cases one forces symmetries and linearization so as to go back to a linear setting and have either funding included as simple discounting or a linear pricing problem. This is not accurate in general but allows the quick calculation of a funding valuation adjustment (FVA).

In our earlier formulas for the Reduced Borrowing Benefit (RBB) case:

$$V_0 = \int_t^T \mathbb{E} \left\{ D(t, u; r) \pi_u | \mathcal{F}_t \right\} du, \quad LVA = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) (r_u - \tilde{c}_u) C_u | \mathcal{F}_t \right\} du$$

$$-FCA (= -DVA_F - FCA_\ell) = - \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) (s_I(u) + \ell^+(t)) (\bar{V}_u - C_u)^+ | \mathcal{F}_t \right\} du$$

$$FBA (= DVA + FBA_\ell) = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[(s_I + \ell^-) (-(\bar{V}_u - C_u))^+ \right] | \mathcal{F}_t \right\} du$$

$$-CVA = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[-s_C (\bar{V}_u - C_{u-})^+ \right] | \mathcal{F}_t \right\} du$$

$$DVA = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) \left[s_I (-(\bar{V}_u - C_{u-}))^+ \right] | \mathcal{F}_t \right\} du$$

$$DVA_F = \int_t^T \mathbb{E} \left\{ D(t, u; r + \lambda) s_I(u) (\bar{V}_u - C_u)^+ | \mathcal{F}_t \right\} du$$

$$-CVA_F = 0;$$

One of the two DVA must go.

In Theory: Nonlinearities due to funding

Here if we assume $\ell^+ \approx \ell^-$, and closeout term is the risk free price $V^0(\tau)$ rather than the replacement value $\bar{V}(\tau)$, then the problem becomes linear and is much more manageable. In practice everyone assumes this and applies a posteriori corrections if needed.

NVA

In the recent paper <http://ssrn.com/abstract=2430696> we introduce a **Nonlinearity Valuation Adjustment** (NVA), which analyzes the double counting involved in forcing linearization. Our numerical examples for simple call options show that NVA can easily reach 2 or 3% of the deal value even in relatively standard settings.

Equity call option (long or short), $r = 0.01$, $\sigma = 0.25$, $S_0 = 100$, $K = 80$, $T = 3y$, $V_0 = 28.9$ (no credit risk or funding/collateral costs). Precise credit curves are given in the paper. No ψ (value for Desk)

$$NVA = \bar{V}_0(\text{nonlinear}) - \bar{V}_0(\text{linearized})$$

NVA

Table: NVA with default risk and collateralization

Funding Rates bps			Default risk, low ^a		Default risk, high ^b	
			Long	Short	Long	Short
f^+	f^-	\hat{f}				
300	100	200	-3.27 (11.9%)	-3.60 (10.5%)	-3.16 (11.4%)	-3.50 (10.1%)
100	300	200	3.63 (10.6%)	3.25 (11.8%)	3.52 (10.2%)	3.13 (11.3%)

The percentage of the total call price corresponding to NVA is reported in parentheses.

^a Based on the joint default distribution D_{low} with low dependence.

^b Based on the joint default distribution D_{high} with high dependence.

NVA

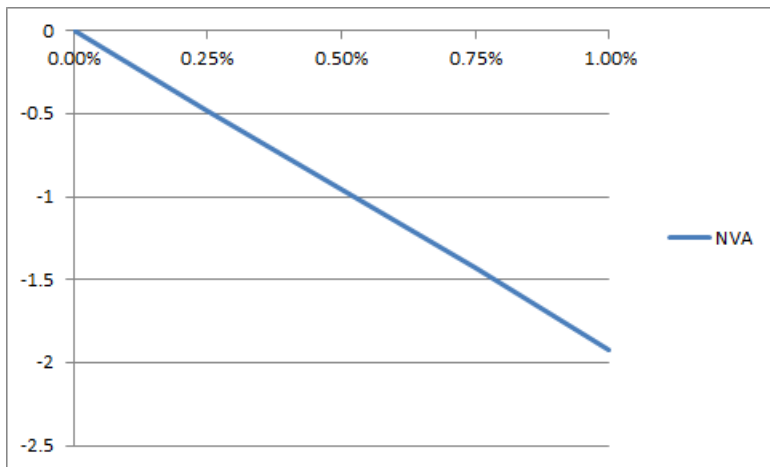
Table: NVA with default risk, collateralization and rehypothecation

Funding Rates bps			Default risk, low ^a		Default risk, high ^b	
			Long	Short	Long	Short
f^+	f^-	\hat{f}				
300	100	200	-4.02 (14.7%)	-4.45 (12.4%)	-3.91 (14.0%)	-4.35 (12.0%)
100	300	200	4.50 (12.5%)	4.03 (14.7%)	4.40 (12.2%)	3.92 (14.0%)

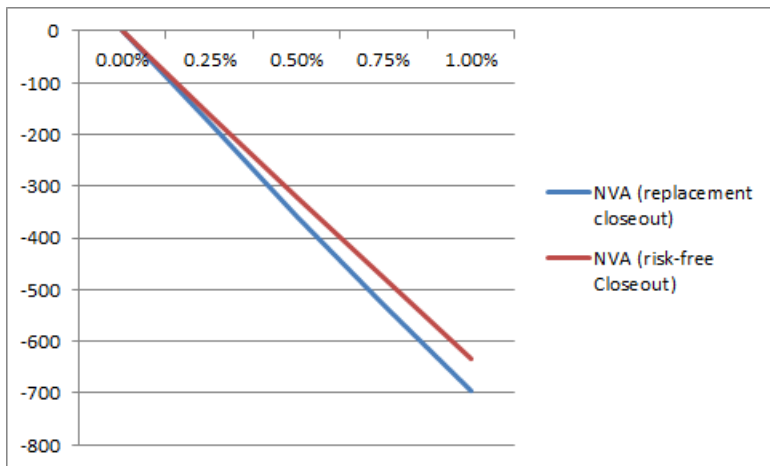
The percentage of the total call price corresponding to NVA is reported in parentheses.

^a Based on the joint default distribution D_{low} with low dependence.

^b Based on the joint default distribution D_{high} with high dependence.



NVA for long call as a function of $f^+ - f^-$, with $f^- = 1\%$, f_+ increasing over 1% and \hat{f} increasing accordingly. NVA expressed as an additive price component on a notional of 100, risk free option price 29. Risk free closeout. For example, $f^+ - f^- = 25\text{bps}$ results in $NVA = -0.5$ circa, $50\text{bps} \Rightarrow NVA = -1$



NVA for long call as a function of $f^+ - f^-$, with $f^- = 1\%$, f_+ increasing over 1% and \hat{f} increasing accordingly. NVA expressed as a percentage (in bps) of the linearized \hat{f} price. For example, $f^+ - f^- = 25\text{bps}$ results in $\text{NVA} = -100\text{bps} = -1\%$ circa, replacement closeout relevant (red/blue) for large $f^+ - f^-$

The Role of Capital and Kapital Valuation Adjustment

The industry is now dealing with the role of capital in pricing derivative contracts and the related Kapital Valuation Adjustment (KVA). There are mainly two ways in which capital affects trading

- Capital can be used as a source of funding (then related to FVA)
 - How much capital is used?
 - Capital requirements as proxy? (\mathbb{P} vs \mathbb{Q} modeling)
 - Does it really matter? (Modigliani Miller strikes again)
- Capital requirements as a limit for the amount of possible trades per unit of capital
 - Treasury penalizes capital-expensive deals with ad-hoc funding policies and charges to trading desks, then passed to clients?
 - Target Return On Investment? Utility (??) based approach?
- Problems: (i) Capital not traded asset, can't be inserted in the replica. Even more dubious than for D/FVA.
- (ii) KVA should consider capital requirements on CVA VaR (Basel III). A Valuation adjustment built on another one. Adding up?

The Role of Capital and Kapital Valuation Adjustment

KVA should be investigated and managed differently than other valuation adjustments, given its link with a non-tradable entity like capital.

Most authors (in the industry) treat KVA as a tradable quantity and compute it with a replication approach. We have seen how dubious this is for C/D/FVA. For KVA it becomes almost impossible to believe.

If there has to be an explicit KVA charge, this appears to be very different in nature wrt other VA's, and the idea of simply adding it up to C/D/FVA's is debatable to say the least.

A lot of fundamental work remains to be done in this respect.

Multiple Interest Rate Curves

Derive interest rate dynamics consistently with credit, collateral and funding costs as per the above master valuation equations.

- We use our market based (no r_t) master equation to price OIS & find OIS equilibrium rates. Collateral fees will be relevant here, driving forward OIS rates.
- Use master equation to price also one period swaps based on LIBOR market rates. LIBORs are market given and not modeled from first principles from bonds etc. Forward LIBOR rates obtained by zeroing one period swap and driven both from primitive market LIBOR rates and by collateral fees.
- We'll model OIS rates and forward LIBOR/SWAP jointly, using a mixed HJM/LMM setup
- In the paper we look at non-perfectly collateralized deals too, where we need to model treasury funding rates.
- See <http://ssrn.com/abstract=2244580>

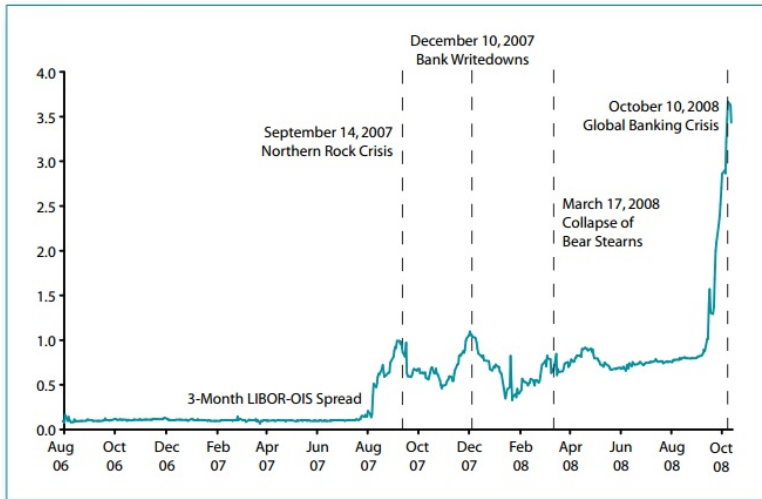


Figure: Spread between 3 months Libor and 3 months ONIA (OIS) swaps. Plotting $t \mapsto L(t, t + 3m) - L^O(t, t + 3m)$ (proxy of credit and liquidity risk). From Economic Synopses 2008, Number 25, FRB of St Louis

Multiple Interest Rate Curves I

We now briefly introduce multiple curves and then connect them to the above analysis of credit and funding costs and collateralization.

From summer 2007 and especially Sept-Oct 2008, market quotes of forward LIBOR rates and zero-coupon (OIS) bonds began to violate standard no-arbitrage relationships.

$$F_t^{\text{LIBOR}}(T_0, T_1) \neq \frac{1}{T_1 - T_0} \left(\frac{P_t^{\text{OIS}}(T_0)}{P_t^{\text{OIS}}(T_1)} - 1 \right) = F_t^{\text{OIS}}(T_0, T_1).$$

We will now derive interest rate dynamics consistently with credit, collateral and funding costs as per the above master valuation equations.

Collateralization and Gap Risk – I

- All liquid market quotes on the money market are daily collateralized at overnight rate e_t .
- We assume that daily collateralization can be considered as a perfect collateralization, and, in particular, *we disregard gap risk*.
 - See B., Capponi, Pallavicini, Papatheodorou (2011) for a discussion on the impact of partial collateralization on interest-rate derivatives.
 - See B., Capponi and Pallavicini (2011) for gap-risk analysis for credit derivatives.
- Since we can disregard gap risk for the interest-rate derivatives, we can assume that derivative mark-to-market is continuous in time.
 - A different approximation, which allows for gap risk and accounts for Initial Margins (CCPs. Standard CSA) as well is in B. and Pallavicini (2014).

Effective Discount Approximation – I

- Hence, we can follow Pallavicini and B. (2013), to introduce a particular form for collateral and close-out prices.

$$G_t \doteq \alpha_t \bar{V}_t, \quad \varepsilon_\tau \doteq \bar{V}_\tau$$

where $\alpha_t \geq 0$ is the collateral fraction.

- We have some special cases:
 - no collateralization: $\alpha_t = 0$, e.g. IRS with a corporate;
 - partial collateralization: $0 < \alpha_t < 1$, e.g. IRS with asymmetric CSA;
 - perfect collateralization: $\alpha_t = 1$, e.g. standard IRS;
 - over-collateralization: $\alpha_t > 1$, e.g. IRS with a CCP requiring initial margin. This is a rough way to include initial margins. A more realistic approach is given in B. and Pallavicini (2014).

Effective Discount Approximation – II

- Recall (leaving out term Z_u for Treasury CVA_F and DVA_F)

$$\bar{V}_t = \int_t^T \mathbb{E}^{\tilde{h}} \{ D(t, u; \tilde{f}) [\pi_u + \lambda_u(\theta_u - \bar{V}_u) + (\tilde{f}_u - \tilde{c}_u)C_u] | \mathcal{F}_t \} du \quad \text{EQFund5}$$

- In the special case $C = \alpha \bar{V}$ this specializes to

$$\bar{V}_t = \int_t^T du \mathbb{E}^{\tilde{h}} \left[\pi_u D(t, u; \tilde{f} + \tilde{\xi}) | \mathcal{F} \right]$$

where $(\lambda_t^{C<I} dt = \mathbb{Q}(\tau_C \in dt, \tau_C < \tau_I | \tau_C \geq t)$ etc)

$$\begin{aligned} \tilde{\xi}_t &:= (1 - \alpha_t)^+ \left(\lambda_t^{C<I} \text{LGD}_C \mathbf{1}_{\{\bar{V}_t > 0\}} + \lambda_t^{I<C} \text{LGD}_I \mathbf{1}_{\{\bar{V}_t < 0\}} \right) \\ &- (\alpha_t - 1)^+ \left(\lambda_t^{I<C} \text{LGD}_I \mathbf{1}_{\{\bar{V}_t > 0\}} + \lambda_t^{C<I} \text{LGD}_C \mathbf{1}_{\{\bar{V}_t < 0\}} \right) \\ &- \alpha_t (\tilde{f}_t - \tilde{c}_t) \end{aligned}$$

Effective Discount Approximation – III

- Since the rates $\tilde{\xi}$, \tilde{f} and \tilde{h} depend on the derivative price \bar{V} , we must resort in the general case to numerical simulations to calculate the prices (nonlinearities, as discussed earlier).

Effective Discount Approximation – IV

- In case of over-collateralization the second term contributing to $\tilde{\xi}$ come from the assumption of re-hypothecation.
 - Thus, if re-hypothecation is forbidden the over-collateralization case is simply: $\tilde{\xi}_t \doteq -\alpha_t(\tilde{f}_t - \tilde{c}_t)$.
- The discount factor appearing in the above pricing equation depends on the hedging strategy through the choice of hedging instruments.
 - We imply the growth rate \tilde{h}_t from market quotes, so that we obtain different prices only if the market is segmented.
- In the case of interest-rate derivatives we assume as hedging instruments liquid instruments quoted on the money market: e.g. OIS and IRS.
 - All such instruments are daily collateralized at overnight rate e_t .
 - We will use such (collateralized) instruments also to hedge non-perfectly-collateralized products.
 - In this case $\tilde{h}_t = e_t$, even though we keep writing $\mathbb{E}^{\tilde{h}}$.

Overnight and OIS Rates – I

- Since we are going to price OIS under the assumption of perfect collateralization, namely we are assuming that daily collateralization may be viewed as done on a continuous basis, we approximate also daily compounding in the OIS floating leg with continuous compounding
- Thus, we can price one-period OIS contracts on a generic rate K as given by

$$\mathbb{E}_t^{\tilde{h}} \left[\left(1 + xK - \exp \left\{ \int_{T-x}^T du e_u \right\} \right) D(t, T; e) | \mathcal{F} \right]$$

Overnight and OIS Rates – II

- One-period OIS rates are implicitly defined as the rates making the corresponding OIS contract fair, so that

$$E_t(T, x; e) := \frac{1}{x} \left(\frac{P_t(T - x; e)}{P_t(T; e)} - 1 \right)$$

where is useful to introduce the collateralized zero-coupon bonds as

$$P_t(T; e) := \mathbb{E}_t^{\tilde{h}} [D(t, T; e) | \mathcal{F}]$$

- One-period OIS rates $E_t(T, x; e)$, along with multi-period ones, are actively traded on the market.
 - We can bootstrap collateralized zero-coupon bonds from such quotes.

LIBOR Rates – I

- We introduce the (par) fix rates $F_t(T, x; e)$ which makes the one-period IRS contracts fair. They are implicitly defined as

$$\mathbb{E}_t^{\tilde{h}} \left[(xF_t(T, x; e) - xL_{T-x}(T)) D(t, T; e) | \mathcal{F} \right] = 0$$

leading to

$$F_t(T, x; e) := \frac{1}{P_t(T; e)} \mathbb{E}_t^{\tilde{h}} \left[L_{T-x}(T) D(t, T; e) | \mathcal{F} \right]$$

- We assume that LIBOR forward rates are quoted on the market, so that we do not derive them from non-arbitrage relationships as in single-curve approaches. In other terms, now L_{T-x} is a primitive market rate.

LIBOR Rates – II

- We can simplify the above expression by considering the following Radon-Nikodym derivative

$$\left. \frac{d\mathbb{Q}^{\tilde{h}, T; e}}{d\mathbb{Q}^{\tilde{h}}} \right|_{\mathcal{F}_t} := \mathbb{E}_t^{\tilde{h}} [D(0, T; e) | \mathcal{F}] = D(0, t; e) P_t(T; e)$$

which is a positive $\mathbb{Q}^{\tilde{h}}$ -martingale.

- For any payoff ϕ_T , perfectly collateralized at overnight rate, we can switch to the collateralized T -forward (equivalent) measure $\mathbb{Q}^{\tilde{h}, T; e}$, and we get

$$\mathbb{E}_t^{\tilde{h}} [\phi_T D(t, T; e) | \mathcal{F}] = P_t(T; e) \mathbb{E}_t^{\tilde{h}, T; e} [\phi_T | \mathcal{F}]$$

LIBOR Rates – III

- In particular, we can write the forward LIBOR rate as

$$F_t(T, x; e) = \mathbb{E}_t^{\tilde{h}, T; e} [L_{T-x}(T) | \mathcal{F}]$$

- One-period forward rates $F_t(T, x; e)$, along with multi-period ones (swap rates), are actively traded on the market.
 - Once collateralized zero-coupon bonds are derived, we can bootstrap forward rate curves from such quotes.
 - See, for instance, Ametrano and Bianchetti (2009) or Pallavicini and Tarenghi (2010) for a discussion on bootstrapping algorithms.

Modelling Constraints – I

- Our aim is to setup a multiple-curve dynamical model starting from:

→ collateralized zero-coupon bonds $P_t(T; e)$, and

→ collateralized forward LIBOR rates $F_t(T, x; e)$.

- Thus, we can define collateralized zero-coupon bonds under $\mathbb{Q}^{\tilde{h}}$ as

$$\frac{dP_t(T; e)}{P_t(T; e)} = e_t dt - \alpha_t(T) \cdot dW_t^{\tilde{h}}$$

and forward LIBOR rates under $\mathbb{Q}^{\tilde{h}, T; e}$ as

$$dF_t(T, x; e) = \gamma_t(T, x) \cdot dZ_t^{\tilde{h}, T; e}$$

where W_t and Z_t are correlated standard Brownian motions with correlation matrix ρ , and the volatility processes may depend on bonds and rates themselves.

Modelling Constraints – II

- Starting from collateralized zero-coupon bonds we can define instantaneous forward rates $f_t(T; e)$ as given by

$$f_t(T; e) := -\frac{\partial}{\partial T} \log P_t(T; e)$$

- We can derive instantaneous forward-rate dynamics by Itô lemma, and we get under $\mathbb{Q}^{\tilde{h}, T; e}$ measure

$$df_t(T; e) = \sigma_t^*(T) \cdot dW_t^{\tilde{h}, T; e}, \quad \sigma_t(T) := \frac{\partial}{\partial T} \alpha_t(T)$$

- Furthermore, we have $e_t = f_t(t; e)$.

Multiple-Curve HJM Models I

- Hence, we obtain the dynamical framework by Pallavicini and Brigo (2013) under $\mathbb{Q}^{\tilde{h}, T; e}$ measure as given by

$$df_t(T; e) = \sigma_t(T) \cdot dW_t^{\tilde{h}, T; e}, \quad dF_t(T, x; e) = \gamma_t(T, x) \cdot dZ_t^{\tilde{h}, T; e}$$

- This is where the *multiple curve picture* finally shows up:
 - We have LIBOR curve based forward rates $F_t(T, x; e)$: collateral adjusted expectation of LIBOR market rates $L_{T-x}(T)$ we take as primitive rates from the market, driven both by LIBOR and by collateral fees,
 - and we have instantaneous OIS-based forward rates $f_t(T; e)$ driven by collateral fees
- We wish to stress an important property of the above dynamics, namely that *it does not depend on unobservable rates such as the risk-free rate r_t .*

Multiple-Curve HJM Models II

- Now the framework for multiple curves is ready and we may start populating the specific dynamics with modeling choices. We refer to our paper <http://ssrn.com/abstract=2244580> for the details.

Multiple curves with Non-Perfect -Collateralization

- Independently of the collateralization policy, when we trade an interest-rate derivative contract, we can hedge interest-rate risks by trading money-market liquid instruments (e.g. OIS, IRS).
- On the money market the liquid contracts are usually collateralized on a daily basis at overnight rate e_t , while, in general, the collateral accrual rate \tilde{c}_t of the deal may be different.
- Thus, whatever the collateralization policy of the contract is, we assume to implement the hedging strategy by means of overnight collateralized contracts.
- In the following we study a partially collateralized IRS

Funding Rates and Liquidity Policies – I

- When we consider non-perfectly-collateralized contracts, we must model the Treasury funding rates \tilde{f} .
 - Funding rates are determined by the Treasury department according to the Bank funding policies.
- Thus, a term structure of funding rates is known, but it is far from being unambiguously derived from market quotes.
 - For instance, long maturities in the term structure of funding rates are calculated by rolling over short-term funding positions, and not by entering into a long-term funding position.
 - The Treasury department may implement a maturity transformation policy along with a fund transfer price (FTP) process.
- It is very difficult to forecast the future strategies followed by the Treasury.
 - The term structure of funding rates is model-depend.
 - The option market (e.g. contingent funding derivatives) is missing.

Funding Rates and Liquidity Policies – II

- A tempting possibility is using the LIBOR rates as a proxy of funding rates.
 - This choice is widespread, but problematic, since it implies that the funding policies of the Treasury department are based on inter-bank deposits (not to mention frauds in LIBOR published rates).
 - After the crisis only a small part of funding comes from this source.
- The main source of funding for an investment Bank is the collateral portfolio which is mainly driven by the credit spreads of the underlying names.

Funding Rates and Liquidity Policies – III

- Here, by following Pallavicini and Brigo (2013), we wish to select a sensible choice for the dynamics of funding and investing rates to perform numerical simulations.

$$f_t^- := e_t + w^-(t) + w^P(t)\lambda_t^P$$

and

$$f_t^+ := e_t + w^+(t) + w^P(t)\lambda_t^P + w^I(t)\lambda_t^I$$

where the w 's are deterministic functions of times. The w 's are factor weights that also help modelling dependence between f^+ , f^- , e_t and λ 's building on the initial dependence between λ 's and e 's brownian motions.

- We define also λ_t^I as the investor's default intensity and λ_t^P as the average default intensity of the names of the collateral portfolio.
 - The w 's can be calibrated to Treasury data, since they represent the Treasury liquidity policy.
- Here, we do not try to model the dynamics of liquidity policies.

Convexity Adjustments for LIBOR Rates – I

- As first example, we price a non-perfectly collateralized IRS, namely

$$1_{\{\tau > t\}} \mathbb{E}_t^{\tilde{h}} \left[(xK - xL_{T-x}(T)) D(t, T; \tilde{f} + \tilde{\xi}) | \mathcal{F} \right]$$

- The above definition can be simplified by moving to $\mathbb{Q}^{\tilde{h}, T; e}$ measure

$$1_{\{\tau > t\}} P_t(T; e) \mathbb{E}_t^{\tilde{h}, T; e} \left[(xK - xL_{T-x}(T)) D(t, T; \tilde{q}) | \mathcal{F} \right]$$

where we define the effective dividend rate

$$\tilde{q}_t := \tilde{f}_t + \tilde{\xi}_t - e_t$$

which includes the effects of credit risk, funding, and the mismatch in collateralization between the exotic deal and the instruments used in its hedging strategy.

Convexity Adjustments for LIBOR Rates – II

- We define for $\tau > t$ the non-perfectly-collateralized (credit-risk adjusted) zero-coupon bond as

$$\bar{P}_t(T; e) := P_t(T; e) \mathbb{E}_t^{\tilde{h}, T; e} [D(t, T; \tilde{q}) | \mathcal{F}]$$

- We define for $\tau > t$ also the LIBOR non-perfectly-collateralized (credit-risk adjusted) forward rate as

$$\bar{F}_t(T, x; e) := \frac{\mathbb{E}_t^{\tilde{h}, T; e} [L_{T-x}(T) D(t, T; \tilde{q}) | \mathcal{F}]}{\mathbb{E}_t^{\tilde{h}, T; e} [D(t, T; \tilde{q}) | \mathcal{F}]}$$

- It is possible to check that the above rate is the par rate of the corresponding non-perfectly-collateralized IRS contract.
- Indeed, we price the non-perfectly collateralized IRS contract as

$$1_{\{\tau > t\}} \bar{P}_t(T; e) (xK - x\bar{F}_t(T, x; e))$$

Convexity Adjustments for LIBOR Rates – III

- Furthermore, it is possible to express $\bar{F}_t(T, x; e)$ in term of $F_t(T, x; e)$ by means of a convexity adjustment as shown in Pallavicini and Brigo (2013).
- We can derive the result by direct calculation, and we have

$$\bar{F}_t(T, x; e) = F_t(T, x; e)(1 + \gamma_t(T, x; e))$$

where $\gamma_t(T, x; e)$ is the partial-collateralization convexity adjustment given by

$$\gamma_t(T, x; e) := \frac{\text{Cov}_t^{\tilde{h}, T; e}[F_{T-x}(T, x; e), D(t, T; \tilde{q})]}{F_t(T, x; e)\bar{P}_t(T; e)}.$$

Convexity Adjustments for LIBOR Rates – IV

- We obtain that non-perfectly-collateralized zero-coupon bonds and forward rates depend on the price process of the contract paying them.
 - They have different values for different contracts.
 - We can interpret them respectively as a *per-contract* Z-spread-adjusted bonds and convexity-adjusted forward rates.
- Thus, when collateralization is not perfect, we obtain that each contract has its own curve, in line with our earlier general master equation in presence of funding costs.
- This seems to point to banks developing both "objective" curves for charging prices to the client, or to compute mid market prices, and "subjective" curves to address costs and profitability analysis.

Pricing under Initial Margins: SCSA and CCPs I

CCPs: Default of Clearing Members, Delays, Initial Margins...

Our general theory can be adapted to price under Initial Margins, both under CCPs and SCSA.

The type of equations is slightly different but quantitative problems are quite similar.

See B. and Pallavicini (2014) for details. See also “Brigo, D. and A. Pallavicini (2014). Nonlinear consistent valuation of CCP cleared or CSA bilateral trades with initial margins under credit, funding and wrong-way risks. *Journal of Financial Engineering* 1 (1), 1-60.” Here we give a summary.

Pricing under Initial Margins: SCSA and CCPs II

So far all the accounts that need funding have been included within the funding netting set defining F_t .

If additional accounts needed, for example segregated initial margins, as with CCP or SCSA, their funding costs must be added.

Initial margins kept into a segregated account, one posted by the investor ($N_t^I \leq 0$) and one by the counterparty ($N_t^C \geq 0$):

$$\begin{aligned} \varphi(t, u) &:= \int_t^u dv (r_v - f_v) F_v D(t, v) - \int_t^u dv (f_v - h_v) H_v D(t, v) \\ &+ \int_t^u dv (f_v^{NC} - r_v) N_v^C + \int_t^u dv (f_v^{NI} - r_v) N_v^I, \end{aligned} \quad (25)$$

with f_t^{NC} & f_t^{NI} assigned by the Treasury to the initial margin accounts.
 $f^N \neq f$ as initial margins not in funding netting set of the derivative.

Pricing under Initial Margins: SCSA and CCPs III

$$\dots + \int_t^u dv(f_V^{NC} - r_V)N_V^C + \int_t^u dv(f_V^{NI} - r_V)N_V^I$$

Assume for example $f > r$. The party that is posting the initial margin has a penalty given by the cost of funding this extra collateral, while the party which is receiving it reports a funding benefit, but only if the contractual rules allow to invest the collaterals in low-risk activity, otherwise $f = r$ and there are no price adjustments.

We can describe the default procedure with initial margins and delay by assuming that at 1st default τ the surviving party enters a deal with a cash flow ϑ , at maturity $\tau + \delta$ (DELAY!).

δ 5d (CCP) or 10d (SCSA).

Pricing under Initial Margins: SCSA and CCPs IV

For a CCP cleared contract priced by the clearing member we have $N_{\tau-}^I = 0$, whatever the default time, since the clearing member does not post the initial margin.

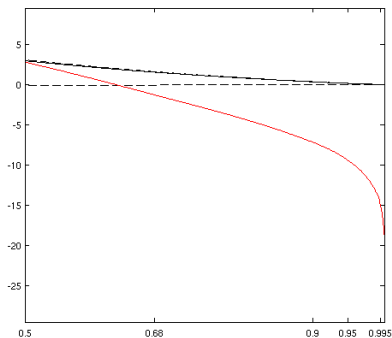
We assume that each margining account accrues continuously at collateral rate c_t .

We may further

- include funding default closeout and also
- define the Initial Margin as a percentile of the mark to market at time $\tau + \delta$.

This is done explicitly in the paper.

Now a few numerical examples:



Ten-year receiver IRS traded with a CCP.

Prices are calculated from the point of view of the CCP client. Mid-credit-risk for CCP clearing member, high for CCP client.

Initial margin posted at various confidence levels q .

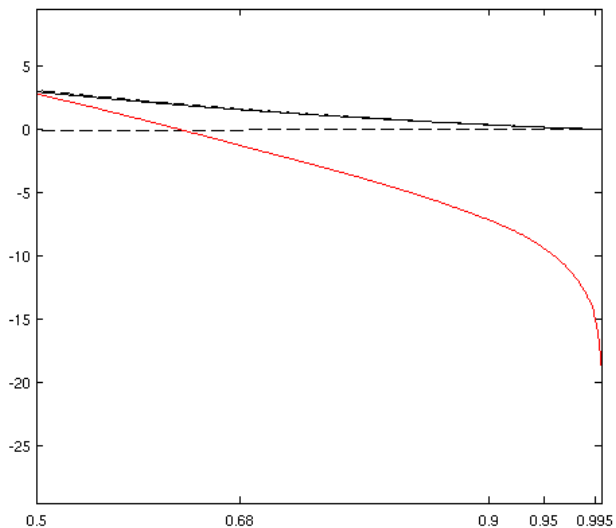
Prices in basis points with a notional of one Euro

Black continuous line: price inclusive of residual CVA and DVA after margining but not funding costs

Dashed black lines represent CVA and the DVA contributions.

Red line is the price inclusive both of credit & funding costs.

Symmetric funding policy. No wrong way correlation overnight/credit.



CCP Pricing: Tables (see paper for WWR etc)

Table: Prices of a ten-year receiver IRS traded with a CCP (or bilaterally) with a mid-risk parameter set for the clearing member (investor) and a high-risk parameter set for the client (counterparty) for initial margin posted at various confidence levels q . Prices are calculated from the point of view of the client (counterparty). Symmetric funding policy. WWR correlation $\bar{\rho}$ is zero. Prices in basis points with a notional of one Euro.

q	Receiver, CCP, $\beta^- = \beta^+ = 1$				Receiver, Bilateral, $\beta^- = \beta^+ = 1$			
	CVA	DVA	MVA	FVA	CVA	DVA	MVA	FVA
50.0	-0.126	3.080	0.000	-0.1574	-2.1317	4.3477	0.0000	-0.0842
68.0	-0.066	1.605	-2.933	0.1251	-1.1176	2.2613	-4.1389	0.2491
90.0	-0.015	0.357	-8.037	0.5492	-0.2578	0.4997	-11.3410	0.7924
95.0	-0.007	0.154	-10.316	0.7205	-0.1149	0.2151	-14.5561	1.0250
99.0	-0.001	0.025	-14.590	1.0290	-0.0204	0.0346	-20.5869	1.4544
99.5	-0.001	0.013	-16.154	1.1402	-0.0107	0.0176	-22.7947	1.6107
99.7	-0.000	0.008	-17.233	1.2165	-0.0070	0.0114	-24.3164	1.7184
99.9	-0.000	0.004	-19.381	1.3684	-0.0035	0.0056	-27.3469	1.9326

XVA Desk?

We now move to a general discussion on the CVA/FVA (XVA?) desk and of its role in the bank.

FVA Desk or CVA Desk, or both? XVA Desk?

First recall the role of the CVA Desk.

How do banks price and trade/hedge CVA?

The idea is to move Counterparty Risk management away from classic asset classes trading desks by creating a specific counterparty risk trading desk, or "CVA desk".

Under simplifying assumptions, this would allow "classical" traders to work in a counterparty risk-free world in the same way as before the counterparty risk crisis exploded.

CVA Desks and "Best practices"

What lead to CVA desks?

Roughly, CVA followed this historical path:

- Up to 1999/2000 no CVA. Banks manage counterparty risk through rough and static credit limits, based on exposure measurements (related to Credit VaR: Credit Metrics 1997).
- 2000-2007 CVA was introduced to assess the cost of counterparty credit risk. However, it would be charged upfront and would be managed mostly statically, with an insurance based approach.
- 2007 on, banks increasingly manage CVA dynamically. Banks become interested in CVA monitoring, in daily and even intraday CVA calculations, in real time CVA calculations and in more accurate CVA sensitivities, hedging and management.
- CVA explodes after 7[8] financials defaults occur in one month of 2008 (Fannie Mae, Freddie Mac, Washington Mutual, Lehman, [Merrill] and three Icelandic banks).

CVA Desks and "Best practices"

CVA desk location in a bank

- Trading floor: PROS works with other trading desk, direct use of hedge trades (especially CDS).
CONS: competition and political problems.
- Treasury: PROS since it involves credit policy, collateral, good for coordination with funding. DVA as funding benefit.
CONS: interface w/ other desks needs to be managed carefully.
- Often CVA desk does systemically important operations for the bank. Should it be part of RISK / CRO? See how Goldman CVA desk may have saved the firm in the AIG case.^a Nonprofit desk, runs a service.
- Considerable operational implications too for the bank functioning.
COO?

^a "How Goldman's Counterparty Valuation Adjustment (CVA) Desk Saved The Firm From An AIG Blow Up"

<http://www.zerohedge.com/>, accessed on Dec 1, 2014

CVA Desks and "Best practices"

CVA desk and Classical Trading desks

The CVA desk charges classical trading desks a CVA fee in order to protect their trading activities from counterparty risk through hedging. This may happen also with collateral/CSA in place (Gap Risk, WWR, etc). The cost of implementing this hedge is the CVA fee the CVA desk charges to the classical trading desk. Often the hedge is performed via CDS trading.

CVA Desks and "Best practices"

CVA desk in the treasury department

Charging a fee is not easy and can make a lot of P&L sensitive traders nervous. That is one reason why some banks set the CVA desk in the treasury for example. Being outside the trading floor can avoid some "political" issues on P&L charges among traders.

—
Furthermore, given that the treasury often controls collateral flows and funding policies, this would allow to coordinate CVA and FVA calculations and charges after collateral.

CVA Desks and "Best practices"

How the CVA desk helps other trading desks

The CVA desk^a would free the classical traders from the need to:

- develop advanced credit models to be coupled with classical asset classes models (FX, equity, rates, commodities...);
- know the whole netting sets trading portfolios; traders would have to worry only about their specific deals and asset classes, as the CVA desk takes care of "options on whole portfolios" embedded in counterparty risk pricing and hedging;
- Hedge counterparty credit risk, which is very complicated.

^aSee for example "CVA Desk in the Bank Implementation", *Global Market Solutions* white paper

CVA Desks and "Best practices"

The CVA desk task looks quite difficult

The CVA desk has **little/no control** on inflowing trades, and has to:

- quote quickly to classical trading desks a "incremental CVA" for specific deals, mostly for pre-deal analysis with the client;
- For every classical trade that is done, the CVA desk needs to integrate the position into the existing netting sets and in the global CVA analysis in real time;
- related to pre-deal analysis, after the trade execution CVA desk needs to allocate CVA results for each trade ("marginal CVA")
- Manage the global CVA, and this is the core task: Hedge counterparty credit and classical risks, including credit-classical correlations (WWR), and check with the risk management department the repercussions on capital requirements.

CVA Desks and "Best practices"

CVA Desks effectiveness is often questioned

Of course the idea of being able to relegate all CVA(/DVA/FVA) issues to a single specialized trading desk is a little delusional.

- WWR makes isolating CVA from other activities quite difficult.
- In particular WWR means that the idea of hedging CVA and the pure classical risks separately is not effective.
- CVA calculations may depend on the collateral policy, which does not depend on the CVA desk or even on the trading floor.
- We have seen FVA and CVA interact

In any case a CVA desk can have different levels of sophistication and effectiveness.

CVA Desks and "Best practices"

Classical traders opinions

Clearly, being P&L sensitive, the CVA desk role is rather delicate. There are mixed feelings.

- Because CVA is hard to hedge (especially the jump to default risk and WWR), occasionally classical traders feel that the CVA desk does not really hedge their counterparty risk effectively and question the validity of the CVA fees they pay to the CVA desk.
- Other traders are more optimistic and feel protected by the admittedly approximate hedges implemented by the CVA desk.
- There is also a psychological component of relief in delegating management of counterparty risk elsewhere.

Including FVA. XVA Desks?

As we have seen funding costs are now an important component of the valuation process, and FVA is calculated for the bank deals.

This may be charged internally to classical trading desks, who pay the FVA desk for the funding costs, and in turn charge the cost to clients externally.

XVA Desk

Both CVA and FVA reference collateral importantly, so they should be managed together, especially given analogies in these quantities, given DVA as funding benefit and given that one would like to avoid double counting.

Ideally, the XVA desk should immunize classical trading desk from credit risk and funding costs, using mirror trades that isolate those risks

XVA Desks?

XVA Desk and Mirror Trades

Isolating Credit Risk and Funding Costs away from traditional trading desks is made difficult by wrong way risk, where dependence makes all risks connected. One can manage this by assigning risk reserves to deal with wrong way risk losses.

One more difficulty is the little transparency on the bases ℓ . They depend on CDS-Bond basis & the bank funding policy: maturity transformation, netting of funding sets, fund transfer pricing policy, etc.

XVA Desks?

Cross Gammas

In this sense quantities that are helpful are cross gammas: sensitivities of computed values to joint shocks in credit and underlying risk factor, and possibly sensitivity to bases ℓ and underlying risk factors.

As own credit risk and the bases ℓ are difficult to hedge, a reserve is set in place for these risks.

Charging FVA to clients?

Charging FVA to Clients

From what we understand, *most of the banks we cited earlier charge FVA to clients*. The classical trading desk pays the funding costs to the FVA desk but then charges the FVA to the client. However, this is controversial. The client often has no transparency on our funding policy. Why should we pay for our choices? And what if the client decides to charge us her funding costs? Can this be done bilaterally given the lack of transparency?

We also debated the price vs value aspect of FVA earlier.

Possible objections to FVA charge are due to the Modigliani Miller theorem. We addressed these earlier via market imperfections and bases ℓ . *Banks are now satisfied with charging clients with FVA. Hence a bank that does not do that risks to be inconsistent with the market.*

FVA Desks?

FVA separate desk?

Some tier-2 banks are considering creating a FVA desk apart from the CVA Desk. However this is not a popular option with tier-1 banks and most banks are trying to incorporate the FVA function in the already existing CVA desk, that becomes a XVA desk. This is what may be happening with all the banks we mentioned earlier.

The reason is that the split between credit and funding is not as clearcut as one may think. See our derivation of CVA, DVA, LVA, FCA, FBA, CVA_F , DVA_F and of all ways to recombine them.

All quantities are driven by s_I , s_C and ℓ^+, ℓ^- .

Recall also that in the full theory FVA and CVA are not really separable.

KVA in XVA Desks? The next XVAs?

We have hinted at the problems the capital valuation adjustment (KVA) raises.

KVA includes capital assessments for future CVA.

That KVA may be simply added to other VA's appears even less credible than for FVA.

In general this tendency of the industry to introduce more and more valuation adjustments every year, assuming their underlying risks are additive and non-overlapping, is worrying.

Waiting for the Electricity-bill Valuation Adjustment (EVA)....

... and for your local baker shop to charge you an explicit running water valuation adjustment (WVA) the next time you buy some bread...

Thank you for your attention!

Questions?

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