Kruskal's Count

James Grime

We present a magic trick that can be performed anytime and without preparation. This trick may be perform to one individual or to a whole audience, and involves the spectators counting through a pack of cards until they reach a final chosen card. Yet, despite this seemingly random choice of cards, the magician is still able to predict the spectator's chosen card. The trick is known as 'Kruskal's Count' and was invented by the American mathematician and physicist, Martin Kruskal [R] [W] and described by Martin Gardner [FG] [G]. Although this trick will not work everytime, we will show that the probability of success is around 85%.

The Trick

A spectator is invited to shuffle a pack of cards as many times as they like. The spectator is then asked to secretly pick a number between 1 and 10 and to count along as cards from the deck are displayed. The magician may choose to display the cards one at a time, or he may choose to display all 52 cards together. The magician explains that the card in the position of the spectator's secret number becomes the spectator's first chosen card. The spectator is then told to use the value of that chosen card as his new number, and to repeat the process until the magician runs out of cards. Here, aces are worth 1; Jack, Queen, King are worth 5; and all other cards take their face value.
Yet, despite this seemingly random path through a shuffled pack of cards, the magician is able to predict the spectator's last chosen card. Watch and interact with a video of the trick being performed here [Gr].

The Secret

How is this done? Well, unknown to the spectator, the magician also picks an initial number between 1 and 10, and proceeds to go through the same process. He might be doing this as he displays the cards. And although the magician may not have picked the same number as the spectator, there is a high probability they will land on the same final card. This is because, even though the magician and the spectator begin on different paths, there will come a point, simply by coincidence, when the two players land on the same card. And from that point on the two paths will become synchonised, meaning both players end on the same final card.

Don't believe me? Then I invite you to try this out for yourself. Grab a pack of cards and use a counter to mark the position of each player, or try this online version [MF1]. You will find that, more often than not, each player will land on the same final card.
In fact we will show that, if we assume the initial numbers are equally likely to be chosen, then the probability of success is 84%. And we can increase that chance slightly, to 85%, if the magician chooses 1 as his initial number.

Furthermore, we will now show that, if N is the number of cards, and x is the mean average card value, then the probability of success may be approximated with the simple formula

$$\mathrm{P}(\text{success}) = 1 - \left(\frac{x^2-1}{x^2}\right)^N$$

The Probability of Success

Calculating the probabilty of success is not an easy thing to do. So we will simplify the problem in a number of ways. The following is based on [LRV].

First, we will assume the cards are labelled with a number from $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$; and specifically for Kruskal's Count, from the numbers 1 to 10; and that these labels are written independently. This assumption would not be true for a real deck of cards as the probability of a card's label will depend on which cards have already been revealed.

Secondly, we assume each card is labelled with values determined by a geometric distribution, and that each player chooses their initial number with the same distribution. In other words, the probability that a card is labelled with the number k is given by $p_k = (1-p)p^{k-1}$, for some $0 \leqslant p \leqslant 1$. If x is the expected card value, it is a standard result for the geometric distribution that $x = \frac{1}{1-p}$. In other words, $p = \frac{x-1}{x}$, and we may now write $p_k = \frac{1}{x}\left(\frac{x-1}{x}\right)^{k-1}$. For Kruskal's Count, face cards are worth 5, so we have the average card value of $x = 70/13$.

The use of the geometric distribution rather than the uniform distribution, as one might more reasonably expect, simplifies calculation while still giving us an excellent approximation of the true probability. This is due to the Law of Large Numbers and the fact that we are giving the geomettric distribution the same expected value as the uniform distribution.

Now consider a deck of N cards. Let t be the 'coupling time', i.e. the position in the deck when the paths of magician and spectator first coincide. For example, $\mathrm{P}[t = 1]$ is the probability that coupling happens on the first card. This would be the probability that both players choose an initial value of 1, and they would each do so with a geometric distribution, so;

$$\mathrm{P}[t = 1] = p_1^2 = \frac{1}{x^2}.$$

Kruskal's Count fails if the coupling time is greater than the number of cards. That means the probability of failure is $\mathrm{P}[t > N]$, and;

$$\begin{aligned}
\mathrm{P}[t > N] &= \mathrm{P}[t > N | t = 1]\mathrm{P}[t = 1] + \mathrm{P}[t > N | t \neq 1]\mathrm{P}[t \neq 1] \\
&= \left(0 \times \frac{1}{x^2}\right) + \mathrm{P}[t > N | t \neq 1]\left(1 - \frac{1}{x^2}\right) \\
&= \mathrm{P}[t > N - 1]\left(\frac{x^2-1}{x^2}\right)
\end{aligned}$$

Here we use the fact that $\mathrm{P}[t > N | t = 1] = 0$, and the memoryless property of a Markov chain of geometric distributions which means $\mathrm{P}[t = k | t \geqslant l] = \mathrm{P}[t = k - l]$.

Continuing in this way we find $\mathrm{P}[t > N] = \left(\frac{x^2-1}{x^2}\right)^N$. In other words,

$$\mathrm{P}(\text{success}) = 1 - \left(\frac{x^2-1}{x^2}\right)^N .$$

Applying this result to Kruskal's Count, where $x = 70/13$, and we find $\mathrm{P}(\text{success}) \approx 83.88\%$.

If, instead of choosing an initial value randomly, the magician chooses an initial value of 1, a similar calculation will show that;

$$\mathrm{P}(\text{success}) = 1 - \left(\frac{x-1}{x}\right)\left(\frac{x^2-1}{x^2}\right)^{N-1} ;$$

which in the case of Kruskal's Count would give $\mathrm{P}(\text{success}) \approx 86.41\%$.

We can verify these results with a Monte Carlo simulation of a shuffled deck of cards. Imagine the magician chooses the first card while the spectator picks their initial value uniformly between 1 and 10. Then, for 10^6 trials, the proportion of decks where $n/10$ initial values end on the same card as the magician is;

n	Proportion
10	58.39%
9	7.98%
8	7.82%
7	6.95%
6	5.96%
5	4.95%
4	3.79%
3	2.64%
2	1.38%
1	0.14%

giving an average probability of success of 85.35%. A difference of 1.06% from the geometric distribution approximation.

Note, this is a value of success averaged over all possible decks. In fact we can see some decks are even more successful than this, with 58.39% of decks having every initial choice land on the same final card. Which means, if you perform this trick to an audience, every single person in that audience will land on the same card!

Expected Coupling Time

We may also calculated the expected coupling time; that is to say, the expected value of t. Under the assumptions of a geometric distribution we showed that $\mathrm{P}[t > N] = \left(\frac{x^2-1}{x^2}\right)^N$. So now it is a simple matter to calculate the more specific probability $\mathrm{P}[t = k]$ to be;

$$
\begin{aligned}
\mathrm{P}[t = k] &= \mathrm{P}[t > k-1] - \mathrm{P}[t > k] \\
&= \left(1 - \frac{x^2 - 1}{x^2}\right)\left(\frac{x^2 - 1}{x^2}\right)^{k-1} \\
&= \frac{1}{x^2}\left(\frac{x^2 - 1}{x^2}\right)^{k-1}
\end{aligned}
$$

We use the standard result that $\sum_{k=1}^{\infty} kq^{k-1} = \frac{1}{(1-q)^2}$ for $0 \leqslant q \leqslant 1$, to calculate the expectation of t, giving the following simple answer;

$$
\begin{aligned}
E[t] &= \sum_{k=1}^{\infty} k\mathrm{P}[t = k] \\
&= \frac{1}{x^2}\sum_{k=1}^{\infty} k\left(\frac{x^2 - 1}{x^2}\right)^{k-1} \\
&= \frac{1}{x^2}\left(1 - \frac{x^2 - 1}{x^2}\right)^{-2} \\
&= x^2.
\end{aligned}
$$

These calculations show that the coupling time t is also a geometric distribution with $p = \frac{x^2-1}{x^2}$ and expectation x^2. So in Kruskal's Count, with $x = 70/13$, we see that the expected coupling time $E[t] \approx 29$.

How Many Final Cards

If we placed a counter on each of the ten initial starting positions, then followed Kruskal's counting procedure for each counter, how many final positions would we end up with? As we have seen by simulation, 58.39% of decks have all ten initial cards end on the same final card. In fact, for a regular deck of cards, we may have no more than six final placements. Here are the proportions of decks, from simulation, for each number of final placements:

# Final Placements	Proportion
1	58.39%
2	39.51%
3	2.10%
4	0.005%
5	0
6	0

As you can see by this simulation, 5 are more final cards becomes extremely rare.

It is easy to show that we may have no more than six final placements, as described by Pollard in [P1]. Place a counter on one of the initial positions and start the counting procedure. Let the sum of the card values on which it lands be the length of its path, this includes the final card which would move it

beyond the end of the pack. For a regular deck of cards, the shortest possibly path length for each starting position would be if the counter ended on card 53. And the shortest sum of seven path lengths would be using starting positions 4 to 10. So, the shortest sum of seven path lengths is 322, but the total value of the whole pack is only 280. This means some counters must land on the same card before reaching card 53, meaning some of the seven counters become synchronised giving fewer than seven final positions.
For six piles, the shortest sum of six path lengths is 273, and there are arrangements that fit this value, with the remaining four cards having a total value of 7. Like Pollard, we will leave these to the interested reader to find.

The Final Card Placement

Clearly, the final chosen card will be one of the last ten cards. By simulation, the proportion of trials finishing in each position were found to be;

Position	Proportion
52	18.50%
51	17.15%
50	15.66%
49	14.25%
48	12.86%
47	7.15%
46	5.79%
45	4.31%
44	2.88%
43	1.43%

These figures can be approximated using Bayes' Theorem as follows:

Let a be the label of the card; each card may be labelled semi-uniformly with probabilities; 1/13, 1/13, 1/13, 1/13, 4/13, 1/13, 1/13, 1/13, 1/13, 1/13. Let b be the placement of the card, numbering the card placements from the end, with $b = 1$ being the last card. Assume b is chosen uniformly with probability 1/10.
A card will be a final card if its label exceeds its placement, i.e. $a \geqslant b$. The probability of this condition will be;

$$\begin{aligned} \mathrm{P}(a \geqslant b) &= \sum_{k=1}^{10} \mathrm{P}(a \geqslant k)\mathrm{P}(b = k) \\ &= \frac{1}{10}\left(1 + \frac{12}{13} + \frac{11}{13} + \frac{10}{13} + \frac{9}{13} + \frac{5}{13} + \frac{4}{13} + \frac{3}{13} + \frac{2}{13} + \frac{1}{13}\right) \\ &= \frac{7}{13}. \end{aligned}$$

By Bayes' Theorem:

$$P(b = n|a \geqslant b) = \frac{P(a \geqslant b|b = n)P(b = n)}{P(a \geqslant b)},$$

where

$$P(a \geqslant b|b = n) = \sum_{k=n}^{10} P(a = k);$$

giving;

$$\begin{aligned}
P(b = 1|a \geqslant b) = &\quad (1/10)/(7/13) &&= 13/70 = 18.57\%;\\
P(b = 2|a \geqslant b) = &\quad (12/130)/(7/13) &&= 12/70 = 17.14\%;\\
P(b = 3|a \geqslant b) = &\quad (11/130)/(7/13) &&= 11/70 = 15.71\%;\\
P(b = 4|a \geqslant b) = &\quad (10/130)/(7/13) &&= 10/70 = 14.29\%;\\
P(b = 5|a \geqslant b) = &\quad (9/130)/(7/13) &&= 9/70 = 12.86\%;\\
P(b = 6|a \geqslant b) = &\quad (5/130)/(7/13) &&= 5/70 = 7.14\%;\\
P(b - 7|a \geqslant b) = &\quad (4/130)/(7/13) &&= 4/70 = 5.71\%;\\
P(b = 8|a \geqslant b) = &\quad (3/130)/(7/13) &&= 3/70 = 4.29\%;\\
P(b = 9|a \geqslant b) = &\quad (2/130)/(7/13) &&= 2/70 = 2.86\%;\\
P(b = 10|a \geqslant b) = &\quad (1/130)/(7/13) &&= 1/70 = 1.43\%;
\end{aligned}$$

as expected.

Some Variations

Our formula for approximating the probability of success of Kruskal's Count was reasonably accurate, being within 1.06% of simulation. So let's consider some variations of the main trick.

In the standard version of Kruskal's Count, face cards were given a value of 5. If face cards were given a value of 10 instead, then the average card value is $x = 85/13$. And if the magician picks the first card, our formula will give the probability of success as 74.67% - a much lower probability of success. Simulation gives this value as 72.21%, a difference of 2.46%.
If the three face cards had values 11, 12 and 13 then $x = 7$. Assuming the magician picks the first card, our formula will give the probability of success as 70.05%. Simulation gives this value as 68.48%, a difference of 1.57%.

A variant with a much higher probability of success would be if we spelled out the names of the cards, so now an 'ace' would have a value of 3 as we spelled out A-C-E. 'Jack' would have a value of 4, 'three' would have a value of 5 and so on. In that case, the average card value is now 4. Assuming the magician picks first, our formula gives a probability of success of 97.21%. Simulation gives this value as 95.66%, a difference of 1.55%.

Note that these simulations work under the assumption that the spectator chooses his initial value with uniform probability. In real-life this would clearly not be the case, and when asked to pick a number between

1 and 10 the most common choice is 7. Which means, if the magician picks 7 as his initial value, he can increase his chances of success even more!

Another fun variation is to use a piece of text instead of cards. For example, pick a word from the first sentence from this piece of text from the hitch-hiker's guide to the galaxy:

> *'It is known that there are an infinite number of worlds, simply because there is an infinite amount of space for them to be in it. However, not every one of them is inhabited. Therefore, there must be a finite number of inhabited worlds. Any finite number divided by infinity is as near to nothing as makes no odds, so the average population of all the planets in the Universe can be said to be zero. From this it follows that the population of the whole Universe is also zero, and that any people you may meet from time to time are merely the products of a deranged imagination.'*

Use the length of the word as your value. Count through the paragraph until you reach your next word, and repeat until you reach your final chosen word. In this case, your final chosen word would be '*products*'.

Finally, more serious applications of these methods are found in cryptography, such as Pollard's rho Factorization Method [P2] and 'Kangaroo' (or 'lambda') method for solving the Discrete Logarithm problem: given the generator g of a cyclic group G, and an element $h \in G$, find x such that $g^x = h$. Here x is a secret message and h is the encrypted message, [MT]. The essential idea being; if we know the total of the jumps made by each participant, we can deduce the other participant's starting point. An online demonstration of this method can be found at [MF2].

With thanks to Colin Wright and Mathieu Nogaret for helping me with the simulations, and for their constructive conversations.

References

[FG] C. Fulves and M. Gardner, The Kruskal Principle, The Pallbearers Review, June 1975.

[G] M. Gardner, Mathematical Games, Sci. Amer. 238 (1978) No. 2 (February), 1932

[Gr] J. Grime, Maths Card Trick: Last to be chosen, http://www.youtube.com/watch?v=uRI4XtnJxXo

[LRV] The Kruskal Count, J C Lagarias, E Rains, Robert J Vanderbei, The Mathematics of Preference, Choice and Order. Essays in Honor of Peter J. Fishburn (Stephen Brams, William V. Gehrlein and Fred S. Roberts, Eds.), Springer-Verlag: Berlin Heidelberg 2009, pp. 371–391. (http://arxiv.org/abs/math/0110143)

[MF1] Kruskal's Count online demo, Ravi Montenegrom, Alexander Frieden, http://faculty.uml.edu/rmontenegro/research/kruskal_count/kruskal.html

[MF2] Pollard's Kangaroo Method Kangaroo demo, Ravi Montenegrom, Alexander Frieden, http://faculty.uml.edu/rmontenegro/research/kruskal_count/kangaroo.html

[MT] How long does it take to catch a wild kangaroo?, R. Montenegro, P. Tetali, Proc. of 41st ACM Symposium on Theory of Computing (STOC 2009). (http://arxiv.org/abs/0812.0789)

[P1] Kruskal's Card Trick, John M. Pollard, The Mathematical Gazette Vol. 84, No. 500 (Jul., 2000), pp. 265-267 (http://www.jstor.org/stable/3621657 and http://www4.ncsu.edu/∼singer/437/Kruskal_card.pdf)

[P2] Pollard rho Factorization Method, http://mathworld.wolfram.com/PollardRhoFactorizationMethod.html

[R] Martin Krukal memorial page, from Rutger University, http://www.math.rutgers.edu/docs/kruskal

[W] Martin Kruskal, Wikipedia, http://en.wikipedia.org/wiki/Martin_David_Kruskal