# The Mathematics of the Astrolabe and Its History

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## Abstract

In this article we trace the scientific and cultural history of the astrolabe, a mechanical instrument used in the past for astronomical measurements and navigational purposes. The story of the astrolabe is interesting from several points of view, since it intertwines mathematical developments, geographical explorations, changing worldviews, and different cultures and civilizations. In our explorations, we move from the early understanding of the world due to the Greeks, to the loss of their work, its rediscovery, the reception of Arab thinkers in Western natural philosophy, and, finally, to the new European culture that emerged with the end of the Middle Ages and the beginning of the early Renaissance. We present a mathematical analysis of how the Celestial Sphere can be represented on the plane through the stereographic projection: this is the consequence of some geometrical results of Apollonius, together with later refinements, whose first traces appear in the work of Al-Farghānī.

**Key words:** astrolabe; celestial sphere; stereographic projections.

*Strepsiades* (discovering a variety of mathematical instruments):
Why, what is this, in the name of heaven? Tell me.
*Disciple:* This is Astronomy.
*Strepsiades:* But what is this?
*Disciple:* Geometry.
*Strepsiades:* What then is the use of this?
*Disciple:* To measure out the land.
*Strepsiades:* What belongs to an allotment?
*Disciple:* No, but the whole earth.
*Strepsiades:* You tell me a clever notion; for the contrivance is democratic and useful.

–Aristophanes, *Clouds*, edited by W.J. Hickie (London: Bohn, 1853), 180-220

## 1. Introduction

In this article we attempt to trace the scientific and cultural history of the astrolabe (Figure 1). This history is interesting from several points of view, since it intertwines mathematical developments, geographical explorations, changing worldviews, and different cultures and civilizations. We will move from the early understanding of the world due to the Greeks, to the loss of their work, its rediscovery, the reception of Arab thinkers in Western natural philosophy, and, finally, to the new European culture that emerged with the end of the Middle Ages and the beginning of the early Renaissance.



**Figure 1:** Astrolabe, maker Christoph Schissler, Augsburg, 1560 - © Museo Galileo, Firenze, inv. 1114 - http://www.museogalileo.it.

In Section 2, we explore the way in which the Greeks perceived the shape of the universe, and how this influenced their early attempts to devise a suitable instrument for astronomical observations and position finding. If, for a moment, we forget about current technology (that flattens our perception of the complexity of the world around us) it should be clear how difficult this must have been. This section traces some of those difficulties, and the first ideas that led to the construction of the astrolabe.

The success of the astrolabe relies on a fiction, but a useful one. The fiction is that the earth can be considered like a point in the center of an immense (but finite) sphere, on which the stars are glued. When looking at this sphere (what is usually called the Celestial Sphere), we see that some of the stars actually move in different fashion since, as already the Greeks understood, these moving stars are not stars but planets (the word planets deriving from the Greek πλάνητες ἀστέρες, meaning wanderer stars). Our Section 3 is therefore devoted to a brief description of this Celestial Sphere, in the way in which it was understood and described several centuries before Christ. This description will show how challenging it is to conceive the possibility to represent such a sphere on a flat instrument! The fact that one can do so is a marvelous consequence of a series of deep mathematical discoveries at least initially due to Apollonius of Perga (262-200 BCE).

The incredible work of Apollonius, as well as the following developments that place the construction of the astrolabe on sound grounds, are the subject of our Section 4, which contains materials that we do not believe have been collected in a single place before. In this section, we describe in detail the remarkable properties of the stereographic projection, which allows a planar representation of the Celestial Sphere. We show how we can use these properties to encode the geometry of the spherical world on a flat object, and how the astrolabe enables the user to decode them. Here is where Greek and Islamic cultures come together, to give us the mathematical foundations for the astrolabe.

Section 5 is devoted to the history of the astrolabe in medieval Islam. While it is impossible to do full justice to the contributions of Islamic scholars, travelers, and geographers, we felt it was important to offer at least a pathway to further study in this direction.

We finally conclude with Section 6, where we show how the mathematical techniques described in Section 4 can be pulled together to build a simple

yet powerful instrument. We present a step-by-step process that goes beyond the pure mechanical construction, and maintains the focus on the geometrical ideas behind the astrolabe.

## 2. "new Heaven, new earth"

> "Then must thou needs find out new Heaven, new earth".
>
> Shakespeare, *Anthony and Cleopatra*, AC 1, i, 17).

The observation of the sky and the need to measure periods, seasons and days certainly belonged to peoples since the earliest antiquity. As for the Mediterranean basin, one should point out the astronomical knowledge of Sumerians and Babylonians. The demands of astronomical knowledge were combined with astrological beliefs, since knowing the position of the moon and planets and foretelling eclipses was considered a necessary knowledge for predicting inauspicious events. Although they did not have precision instruments at their disposal, these people were able to define synodic periods, draw up ephemeris tablets, lunations, and determine the zodiacal band and the ecliptic.

Later, the poems of Homer (VIII Century BCE) and the writings of Hesiod (VIII-VII Century BCE) bear witness to the desire to represent that world that, to them, was still a flat disk, surrounded by the river Oceanus, that eventually flows inside the earth to generate rivers and lakes. But it is with the pre-Socratic philosophers (VI-V Century BCE) that humankind begins to translate celestial observations into the invention of instruments for the measurement and the definition of cosmological principles. In the city of Miletus (in today's Turkey), thinkers like Thales, Anaximander, and Anaximenes were interested in searching the original causes and principles for the universe.

While some of their answers appear fanciful to the modern spirit, legend claims that Thales knew enough about the universe to be able to approximate the apparent diameters of the Sun and the Moon and to predict a solar eclipse, that Anaximander measured the height of the Sun and the Moon and invented the gnomon, that he imagined a cosmology with infinite worlds and first described the universe as a celestial sphere. The lack of sources makes it difficult to confirm these stories, but what is commonly accepted is the fact that this was the time when the idea of earth as a sphere, the presence of planets and moon, and finally the first glimpse of heliocentrism appear.[1]

We are therefore indebted to the ancient Greek astronomers, and in particular to Eudoxus of Knidos (c.390-340 BCE) for the description of the complex system of homocentric spheres that through a (somewhat artificial) geometrical-kinematical model, was able to offer a reproduction of the movement of the celestial bodies.[2] Regretfully, we do not have direct witnesses for Eudoxus' ideas, nor we do have any remains of his works. What we know is due to some fragmentary references to his theories that later authors

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[1]These ideas were subjected to discussions and verification, and they eventually developed, for example, in the work of Aristarchus of Samos (c.310-c.230 BCE) who was the first to suggest a heliocentric system, although only fragmentary descriptions of his idea survive. Aristarchus found that, by placing the Sun to the center of the universe, the motions of celestial bodies had a simpler explanation (even if not yet perfect due to the lack of the imagination of elliptic orbits). He also suggested that the Earth is rotating around a tilted axis, thus explaining the existence of the four seasons.

[2]See [77]. There is a very large bibliography on ancient (and in particular Greek) mathematics and astronomy. In addition to the works quoted in this essay, we refer the reader to [17, 50, 71, 39, 10].

left us, mainly through Aratus (c.315-c.240 BCE), and the commentaries of Hipparchus (c.190-c.126 BCE) on the works of Eudoxus and Aratus. In particular, the ideas of Eudoxus on homocentric spheres have reached us through some short passages of Aristotle [6, Book XII, ch. VIII, 1073b17–1074a15] and Simplicius [68]. Aratus based his poem *Phaenomena* on the homonymous treatise of Eudoxus, and appropriated his geographical and astronomical theories, which he suggested to use for navigation:

> Now the one men call by name Cynosura [Ursa Minor] and the other Helice [Ursa Major]. It is by Helice that the Achaeans [Greeks] on the sea divine which way to steer their ships, but in the other the Phoenicians put their trust when they cross the sea. But Helice appearing large at earliest night, is bright and easy to mark; but the other is small, yet better for sailors: for in a smaller orbit wheel all her stars. By her guidance, then, the men of Sidon [Phoenicians] steer the straightest course. [5]

The Greek astronomer and navigator Pytheas of Massalia (c.350-c.310/306 BCE) wrote in his lost book "On the Ocean" (Περὶ του Ὠκεανου) about his travels that led him to circumnavigate Europe, reaching up to what is now Great Britain, the Baltic Sea, and even the Scandinavian peninsula. What made this feat possible were the development of new instruments that allowed a new reading of the sky, and therefore new paths across unexplored worlds. We now believe that Pytheas was a student of Eudoxus,[3] and it was probably thanks to Eudoxus' astronomical observations that Pytheas was able to leave the safe harbors of the Greek islands and the landviews of Northern Africa, to pursue new routes to the West and to the North. In order to navigate for days and nights, without stopping in harbors, leaving the ancient city of Tyrus for Cyprus, and then Crete and Malta, and even further to Carthago, it was not sufficient anymore to be a good sailor, able to navigate in the night by looking at the stars. Pytheas' annotations relative to the computations of latitudes made in different places, and the calculations of distances covered during the travel, constitute the draft of a net of meridians and parallels later adopted by Eratosthenes and Hipparchus [42, pages 15-16]. Indeed, additional instruments must have been necessary to solve the problem of position finding. Among these, the arachne, a circular object in

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[3]Bibliographical references on Pytheas are available in [42].

which a web of time-lines and celestial circles imitates the web of a spider (in Greek αραχνη or arachni), and whose purpose was to act as a solar watch; it is possible that such a device was indeed a precursor of the tympan, one of the fundamental components of the astrolabe. Its construction, according to Vitruvius (c.80–c.15 BCE) (see [44, 38]) was due to Eudoxus (or possibly to Apollonius) [43]: "Eudoxus the astronomer, or according to some Apollonius [invented] the arachne."

In several pages of his works and in particular in *Metaphysica* and *De Caelo*, Aristotle (384-322 BCE) adopted, modified and reformulated the theory of Eudoxus.

> Eudoxus held that the motion of the sun and moon involves in either case three spheres, of which the outermost is that of the fixed stars, the second revolves in the circle which bisects the zodiac, and the third revolves in a circle which is inclined across the breadth of the zodiac; but the circle in which the moon moves is inclined at a greater angle than that in which the sun moves. And he held that the motion of the planets involved in each case four spheres; and that of these the first and second are the same as before (for the sphere of the fixed stars is that which carries round all the other spheres, and the sphere next in order, which has its motion in the circle which bisects the zodiac, is common to all the planets); the third sphere of all the planets has its poles in the circle which bisects the zodiac, and the fourth sphere moves in the circle inclined to the equator of the third. In the case of the third sphere, while the other planets have their own peculiar poles, those of Venus and Mercury are the same. [6, Book XII, ch. VIII, 1073b, pages 156-157]

In order to explain the increasingly accurate astronomical observations, and to preserve Eudoxus' geocentric system, Aristotle was forced to propose an intricate system of fifty-five homocentric spheres, all animated by the motion of the most external sphere containing the fixed stars, which in turn was animated by Aristotle's unmoved mover.

> But if all the spheres in combination are to account for the phe-nomena, there must be for each of the other planets other spheres,

one less in number than those already mentioned, which counteract these and restore to the same position the first sphere of the star which in each case is next in order below. In this way only can the combination of forces produce the motion of the planets. Therefore, since the forces by which the planets themselves are moved are 8 for Jupiter and Saturn, and 25 for the others, and since of these the only ones which do not need to be counteracted are those by which the lowest planets is moved, the counteracting spheres for the first two planets will be 6, and those of the remaining four will be 16; and the total number of the spheres, both those which move the planets and those which counteract these, will be 55. If we do not invest the moon and the sun with the additional motions which we have mentioned, there will be 47 spheres in all. [6, Book XII, ch. VIII, 1074a, pages 158-159]

Ptolemy, in the second century AD, elaborated this cosmology by means of a complex mathematical structure. The Ptolemy's cosmology was accepted almost universally and constituted an essential point of view for the Medieval Scholastic until the Early Modern Age.

As we follow the documents through the ages, we see that the Romans incorporated, as well, the Greek navigational techniques. For example, Lucan (39-65 AD), in his *Pharsalia*, describes the dialogues of Pompeus with his steersman as follows:

> [Pompeus] questioned the steersman concerning all the stars; by what stars does he mark the land? what rule and measure for cleaving the sea does the sky afford? By what stars does he keep a course to Syria? Or which of the seven stars in the Wain is a sure guide to Libya? The skilled watcher of the silent sky replied to him thus: "All those lights which move and glide through the starry heavens mislead the hapless seaman, because the sky is ever shifting; to them we pay no heed; but the pole-star, which never sets or sinks beneath the waves, the brightest stars in the two Bears, he it is that guides our course. When I see him mount ever towards the Zenith, and when the Little Bear rises above the towering yard, then we face towards the Bosporus and the Black Sea that hollows the Scythian shore. But whenever Bootes sinks from the topmast and the Little Bear moves nearer the horizon,

the ship is making for the ports of Syria. Next after that comes
Canopus, a star that shuns the North and limits its wanderings to
the southern sky; if you keep it on the left and sail on past Pharos,
your vessel will strike the Syrtis in mid-ocean. But whither do
you bid me shape our course, and with which sheet shall the
canvas be stretched?" [41, Book VIII, pages 448-451, (167-186)]

This particular quote clearly indicates that, in the Roman age, the seamen
were well aware that the stars were sliding on the Celestial Sphere, with the
exception of the North Star, which appears to be fixed on this sphere.

If the arachne was indeed an intermediate step, the instrument that truly
changed navigation was the astrolabe, so named from the Greek ἄστρον or
star, and λαμβάνω or to take, a small circular instrument, made of three
superposed plates, designed to allow position finding by looking at the sun
and the stars. The novelty of the astrolabe, as we describe in detail in the
next sections, is that it allows the navigators to utilize the North Star and
other stars as well, stars that up to that point were considered unreliable.
We need to point out that in antiquity the term astrolabe was also used to
indicate several armillary instruments, often very different from each other,
used for the detection of the positions of stars and for the description of
the orbits of planets, like the armillary astrolabe of Ptolemy (110-170 AD)[4],
as called by Egnatio Danti in 1578 ([15, pages 231ff]; see also [78]). The
planar astrolabe however occupied a particularly important position in this
framework, because it turned out to be the most versatile and convenient
tool to perform astronomical computations of high complexity, and to help
navigators.

The invention of the astrolabe is often (but controversially; see [73, 69])
attributed to Hipparchus, astronomer of the second century BCE[5] as we
read for example in the letter that Synesius of Cyrene (c.370-413 AD) writes
to Peonius.[6] The letter supposedly accompanied an actual astrolabe, though

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[4]Plato (c.429-347 BCE), in the *Timaeus* attributes to the Demiurgus the mythological
construction of the dwelling of the gods, a globe in the shape of armillary spheres, that
clearly reminds us of the system of Eudoxus. See [57, page 72ff] and [76].

[5]The astrolabe has inspired many anecdotes. A particularly charming one, from the
Arabic medieval tradition, tells that Ptolemy, while riding a donkey, dropped a celestial
sphere and the donkey trod on it, originating an astrolabe. See [34].

[6]The title "Sermo de dono astrolabii" was added by Migne to the letter; see [47].

some scholars have expressed doubts as to whether this was indeed the case. However, in the *Commentary on the Phaenomena of Eudoxus and Aratus*, Hipparchus used a mathematical projection to build the disk of an "anaphoric clock", which, even if related, is far from being an astrolabe [26]. As to actual constructions of astrolabes, it is believed that such a process may have begun in the late Greek period, even though the first explicit descriptions actually come from the first centuries of the Christian era. One should note here that the bibliography regarding the early diffusion, the circulation, the construction, and the utilization of the astrolabe is extremely rich, and we cannot even attempt to summarize it.[7]

The very first such description is due to John Philoponus of Alexandria (c.490-c.570), whose detailed presentation, dated around 530 AD, has been preserved.[8] It contains not only the description of all the components of the astrolabe, but also its use, both during the day and during the night. The author teaches, in this treatise, how to use the astrolabe for the observation of the sun and the fixed stars as well as for the determination of the time and the calculation of the height, in the Zodiac, of sun, stars, and planets. Philoponus also describes how to use the astrolabe to identify the season, and the altitude of the sun in every day of the year, as well as many other astronomical calculations.

## 3. The Celestial Sphere

The astronomic culture and knowledge that led to the invention of the astrolabe was distinctively Greek. As we have hinted to in the previous pages, the creation of the astrolabe originated by the necessity, for astronomers and navigators, to represent on a plane, what we call the Celestial Sphere.

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[7]In addition to the texts we have already cited, see [46, 19, 58, 72, 55, 70, 20]. There are also many studies dedicated to specific artifacts of museological interest, for example see [28].

[8]Both Synesius, formerly a student of Hypatia, and Philoponus claim that the plane astrolabe was known by Theon of Alexandria (335-405 CE), father of Hypatia. Neugebauer, in his [51], reconstructs the complex of testimonials of Synesius and of Severus Sebokt (575-667 CE). Sebokt, in his treatise on the astrolabe, originally written in Syriac, seems to have contributed to the propagation of a version of the homonymous work of Theon. See also [49, 33]. As to the influence of Hypatia on Synesius knowledge of the astrolabe, see [16].

The representation had to be easily usable by navigators, even though it might be what mathematicians call a local representation, namely a representation only of portions of the Sphere. The idea itself of the Celestial Sphere is very natural, since the observation of the sky gives the impression that the stars, the planets, the moon, and the sun move on the background of a spherical object. What is interesting, is the fact that these objects appear to move following different rules, as if they, themselves, were on concentric spheres within the larger sphere.

With a typical approximation process, the earth was thought to be so small with respect to the Celestial Sphere, that one could identify it with a point, coinciding with the eye of the observer. This is an important point to which we will come back later. On the Celestial Sphere, the eye of the observer sees all the stars and planets, including the sun and the moon (Figure 2). This is what is nowadays represented—usually in portions—in a Planetarium.
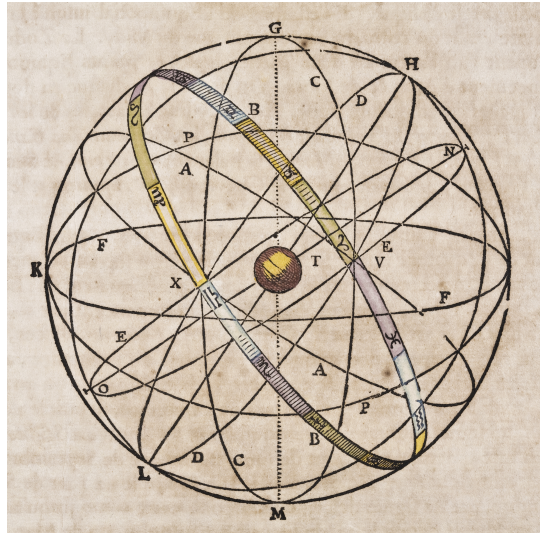


**Figure 2:** The celestial sphere. Willem Janszoon Blaeu, *Le grand atlas, ou, Cosmographie Blaviane: en laquelle est exactement descripte la Terre, la mer et le ciel*, Amsterdam, 1667 - Museo Galileo, Firenze.

To the observer, the Celestial Sphere appears to rotate around the earth through an east-west rotation around the north-south axis of the earth. (Of course, we now know that it is the earth that rotates around its axis, in the direction from west to east, and that induces the apparent motion of the

Celestial Sphere.) The axis of rotation of the Celestial Sphere is the pro-
longation of the earth axis and is called Celestial Axis, and its intersections
with the Celestial Sphere are called North and South Celestial Poles. One
can then define the Celestial Equator as the maximal circle equidistant from
the two Poles. The Zenith and Nadir of a point P on the earth are classi-
cally defined as the two intersection points of the Celestial Sphere with the
straight line passing through the center of the earth and the point P. The
Zenith of P is that point of intersection that stays on the side of P (exactly
above the head of an observer standing in P on the earth).

To define the Celestial Tropics, we proceed as follows. Consider the precise
moment in which the summer solstice takes place, i.e. the moment in which
the sun reaches its maximal north latitude. There is a unique point C on
the earth whose Zenith the sun occupies in exactly that moment. We now
consider the unique circle on the earth containing C and parallel to the
Equator, and we call it Tropic of Cancer. Its central projection (namely
the projection of this circle from the center of the earth) to the Celestial
Sphere is called Celestial Tropic of Cancer. Analogously, the point D on
the earth whose Zenith is occupied by the sun in the precise moment of the
Winter solstice (i.e. the moment in which the sun reaches its maximal south
latitude) identifies a unique circle on the earth, containing D and parallel
to the Equator, called Tropic of Capricorn. Its central projection to the
Celestial Sphere is called Celestial Tropic of Capricorn. The Celestial Tropics
are therefore the two circles of the Celestial Sphere that are described by
the sun on the summer solstice (Tropic of Cancer, north of the Equator)
and of the winter solstice (Tropic of Capricorn, south of the Equator). The
counterparts (or central projections) of the terrestrial meridians and parallels
onto the Celestial Sphere are called, respectively, circles of declination and
parallels of declination. They constitute a first web on the Celestial Sphere
(Figure 4a).

The so-called Ecliptic determines a second important web on the Celestial
Sphere. Consider the intersection of the plane of the (apparent) orbit of the
sun with the surface of earth: this gives a great circle, whose projection on
the Celestial Sphere is called the Ecliptic. The Ecliptic contains exactly all
those points of the Celestial Sphere that are occupied by the sun at some
moment of the year (Figure 3).
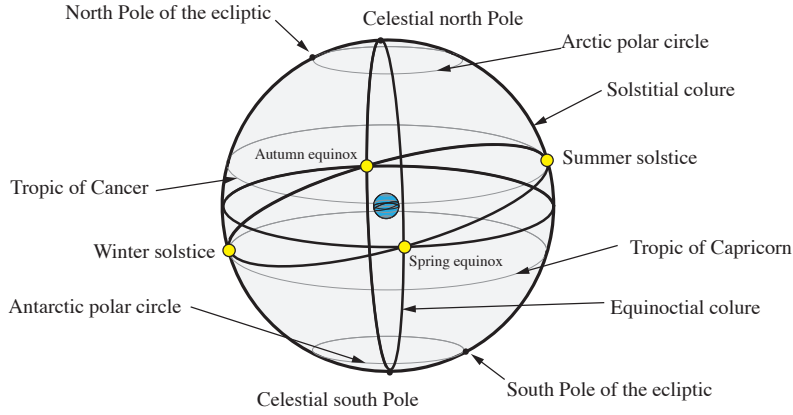
**Figure 3:** The Ecliptic, the Equator, the solstitial and equinoctial colures.

The intersection between the Celestial Equator and the Ecliptic (as all the intersections between any two maximal circles of a sphere) consists of two antipodal points, that correspond in this case to the positions of the sun at the two moments of the spring equinox (this point is called point $\gamma$) and the autumn equinox. On the other hand, the highest and lowest points of the Ecliptic, by definition, correspond to the position of the sun on, respectively, the summer solstice and the winter solstice and belong to the Celestial Tropics of cancer and Capricorn, respectively. By this process, we associate four dates to these four points according to the days in which the sun occupies them: March 21, June 21, September 23, and December 21. This divides the Ecliptic in four arcs. What we have described explains expressions like "the sun enters the Lion", meaning that in a certain day the sun enters a portion of the Ecliptic that is superimposed to the constellation named Lion, on the surface of the Celestial Sphere.

Since the Ecliptic is a great circle, we can consider it as the Equator of the sphere, and thus we determine two Ecliptic poles on the Celestial Sphere: the North and the South Poles of the Ecliptic. Similarly, we can now draw the Ecliptic web made by the parallels of latitude (to the Ecliptic) and circles of latitude, as in Figure 4b. The circle of declination that passes through the two equinoctial points is called Equinoctial Colure, the one that passes through the two solstitial points is called Solstitial Colure (see Figure 3).
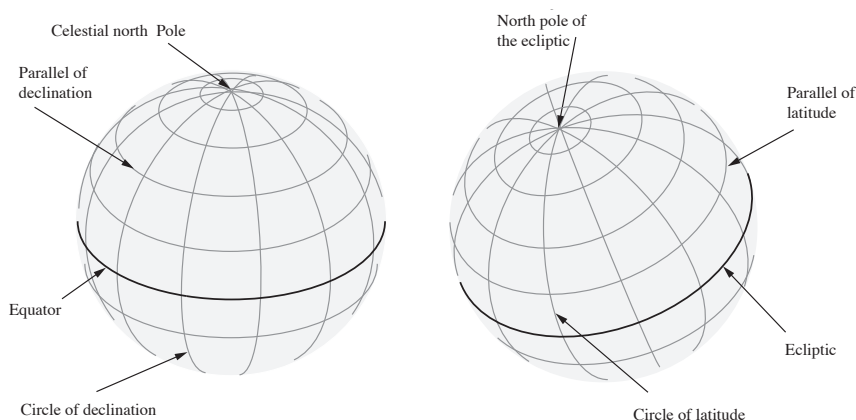
**Figure 4:** The celestial north-south web and the web of the ecliptic.

A third and final web on the Celestial Sphere is the celestial web connected to a given observer. As we know, the observer determines two antipodal points of the Celestial Sphere, his Zenith and his Nadir, which in turn determine an equidistant maximal circle called, for natural reasons, his Celestial Horizon. The circles of the Celestial Sphere that are parallel to the Celestial Horizon are called almucantars (from the Arabic word for the arch) and measure the elevation of stars and planets with respect to the Horizon of the observer, see Figure 7a. We call verticals the counterpart of meridians, with respect to the Zenith and Nadir of the observer. Figure 5, due to Andreas Cellarius (1596-1666), illustrates how the ancient imagined these three webs of the Celestial Sphere.

Notwithstanding the fact that the cartographers always represented the earth as a sphere, even the first astronomers knew that the earth had to be considered as a single point when compared to the dimensions of the Celestial Sphere itself.[9] Indeed, the portion of the Celestial Sphere visible to an ob-

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[9] "Another clear indication that this is so is that the planes drawn through the observers lines of sight at any point [on earth], which we call 'horizons', always bisect the whole heavenly sphere. This would not happen if the earth were of perceptible size in relation to the distance of the heavenly bodies; in that case only the plane drawn through the center of the earth could bisect the sphere, while a plane through any point on the surface of the earth would always make the section [of the heavens] below the earth greater than the section above it." [60, I. 6, page 43].
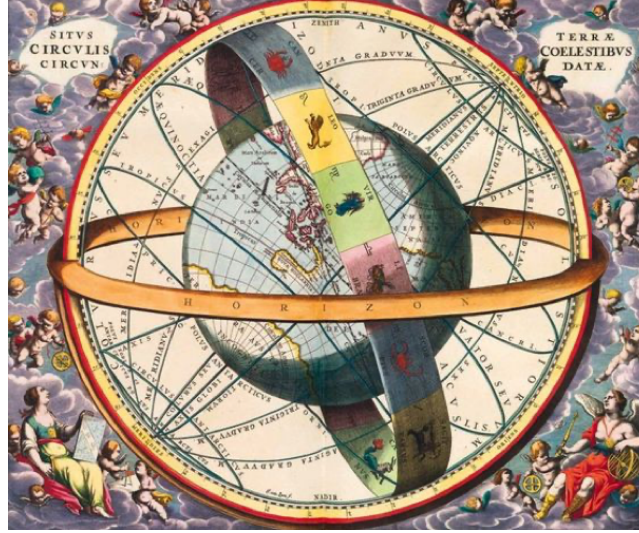
**Figure 5:** Andreas Cellarius, *Harmonia Macrocosmica*, Amsterdam,1660.

server placed at a point A on the earth is that portion of the sphere that stays above (i.e. on the side of the Zenith of A) the tangent plane $\pi$ to the earth at the point A. The intersection of $\pi$ with the Celestial Sphere gives the Celestial Horizon. If the radius r of the earth were non-zero, then—by an easy geometrical reasoning—the Celestial Horizon could not be a great circle of the Celestial Sphere. Instead, the Celestial Horizon is always described and represented as the great circle equidistant from the two points Zenith and Nadir. One deduces that the horizon is the intersection of the tangent plane $\pi$ with the Celestial Sphere, as r approaches zero (see Figure 6).

We describe this situation in Figure 6, where we have drawn both the horizon corresponding to a positive radius r, as well as the horizon corresponding to a punctiform earth. Note however that it would be impossible to build this third web if one began directly with a zero-radius earth, since in that case we could not identify where the Zenith and the Nadir are located.

## 4. The Stereographic Projection

We will now see how an observer, located at a point of given latitude on the earth, can represent the Celestial Sphere on a plane. The first step is to consider the celestial web connected to the observer, as in Figure 7a.
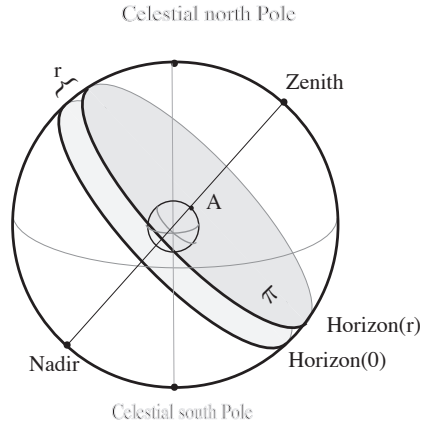
**Figure 6:** The dimension of the Earth in the celestial sphere.

The next step in the construction of the astrolabe would require the use of the stereographic projection that allows us to translate the spherical world of the Celestial Sphere to the plane of the astrolabe. To do so, we define the notion of stereographic projection (see Figure 7b).
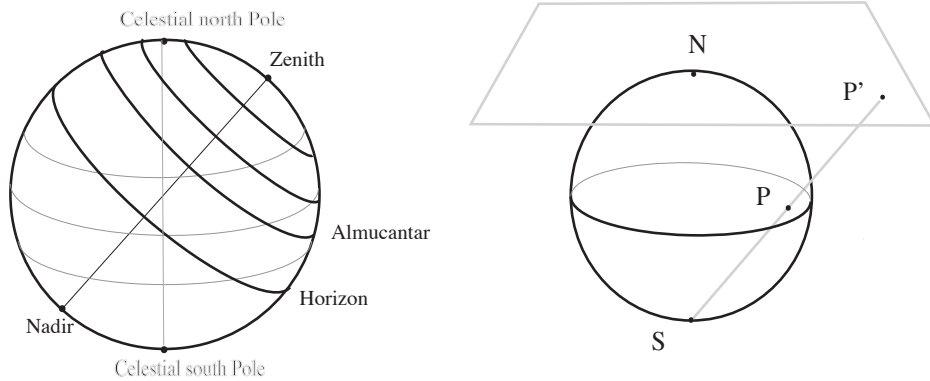


**Figure 7:** The celestial web of the observer, and the stereographic projection.

Take a point S (projection point) and the plane $\pi$ which is tangent to the sphere on the point N antipodal to S. The stereographic projection P' of a point P on the sphere is the intersection of the straight line through P and S with the plane $\pi$. The stereographic projection enjoys a remarkable property,

which makes the construction of the astrolabe possible, and which the following theorem describes.

**Theorem SP.** *The stereographic projection maps circles on the sphere to circles on the plane (except when the circles on the sphere pass through the point of projection, in which case they are mapped to straight lines).*

The property of the stereographic projection to be circle-preserving—stated in Theorem SP—is illustrated in Figure 8.



**Figure 8:** The stereographic projection is circle-preserving.

Commentators often attribute this theorem to Apollonius [75, page 183], even though in *Conicorum Liber quatuor* [3], he proves only the first (crucial) part of it, Proposi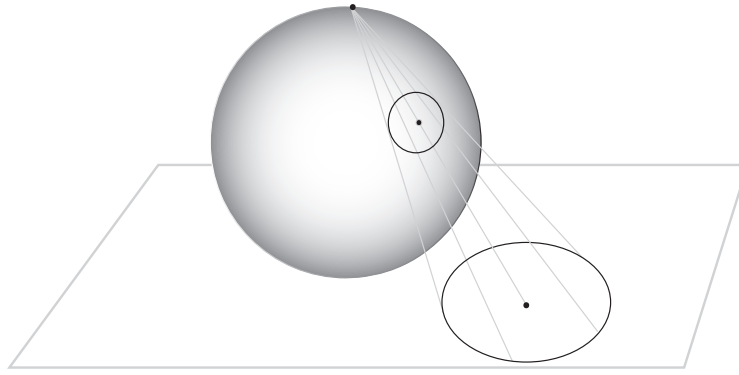tion V in book I, dedicated to the establishment of a peculiar property of oblique cones, as it appears (Figure 9) in Latin in the historical edition of Apollonius' *Conics* by Federicus Commandinus.

It is obvious that if one considers an oblique cone with circular basis, all planes that are parallel to the basis will cut the cone along circles. However Apollonius realized that there exists indeed a second family (and, in fact, only one such family) of parallel planes, which will cut the cone along circular sections. More precisely, Apollonius states and proves the following result:

> Proposition I.V. If an oblique cone is cut by a plane through the axis at right angles to the base, and is also cut by another plane on the one hand at right angles to the axial triangle, and on the
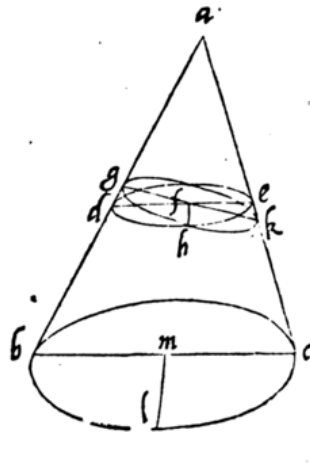
CONICORVM LIB. I.                    10

### THEOREMA V. PROPOSITIO V.

SI conus fcalenus plano per axem fecetur ad rectos angulos ipfi ba-
fi; feceturq; altero plano ad triangulum per axem recto, quod ex uer-
ticis parte triangulum abfcindat fimile ei, quod per axem, fubcontra-
rie uero pofitum: fectio circulus erit. uocetur autem huiufmodi fectio
fubcontraria.

Sit conus fcalenus, cuius uertex a punctum; bafis circulus b c: & fecetur plano A
per axem, ad circulum b c recto, quod faciat fectionem triangulum a b c. fecetur au- B
tem, & altero plano ad rectos angulos ipfi a b c, quod ex parte a triangulum abfcin-
dat a g k triangulo a b c fimile, fub contrarie uero pofitum; ut uidelicet angulus a k g
æqualis, fit a b c angulo: & faciat fectionem in fuperficie lineam g h k. Dico ipfam
g h k circulum effe. Sumantur enim in lineis g h k, b c puncta quæpiam h l: à qui-
bus ad planum, quod per triangulum a b c tranfit, perpendiculares ducantur, ca-       38. unde.
dent hæ in communes planorum fectiones. cadant ut h f, l m. æquidiftans eft igitur    6. undec4.
h f ipfi l m. ducatur autem per f ipfi b c æquidi-
ftans d f e. ergo planum, quod per f h, d e tranfit                                    15. unde.
æquidiftans eft bafi ipfius coni: & idcirco fectio                                     4. huius
d h e circulus erit, cuius diameter d e. æquale eft                                    8. & 17.fe
igitur rectangulum d f e quadrato f h. Quòd cum                                        xti.
æquidiftet d e ipfi b c, angulus a d e æqualis eft
angulo a b c.& ponitur angulus a k g angulo a b c                                      19. primi
æqualis. ergo & a k g ipfi a d e æqualis erit. funt
autem & qui a d f anguli æquales, quòd fint ad                                         15.primi.
uerticem. quare d f g triangulum fimile eft trian-
gulo k f e. & ut e f ad f k, ita g f ad f d. rectangu-                                 4. fexti
lum igitur e f d æquale eft rectangulo k f g. Sed                                      16. fexti
rectangulum e f d demonftratum eftæquale qua-                                         C
drato f h. ergo & k f g eidem æquale erit. fimili
quoque ratione demonftrabuntur & omnes, quæ
à linea g h k ad ipfam g k perpendiculares ducun-
tur, poffe æquale ei, quod partibus ipfius g k conti                                   2. lemm.
netur.fectio igitur circulus eft, cuius diameter g k.                                  pappi
                                                                                       D

**Figure 9:** *Apollonii Pergaei Conicorum libri quatuor* [3], Bologna, 1566.

other cutting off on the side of the vertex a triangle similar to
the axial triangle and lying subcontrariwise, then the section is
a circle, and let such a section be called subcontrary. [4, pages
607–608]

We give here a faithful presentation of the original proof, to which we only
added more details and illustrations, to help the modern reader to appreciate
the beauty, the importance and the depth of the result.

**Proof.** Consider a circle $\gamma$ on a plane $\pi$ and a point A outside the plane $\pi$ and not belonging to the axis of the circle. We call oblique cone the cone $\Gamma$ with vertex A and intersecting the plane $\pi$ exactly at $\gamma$ (Figure 10a).
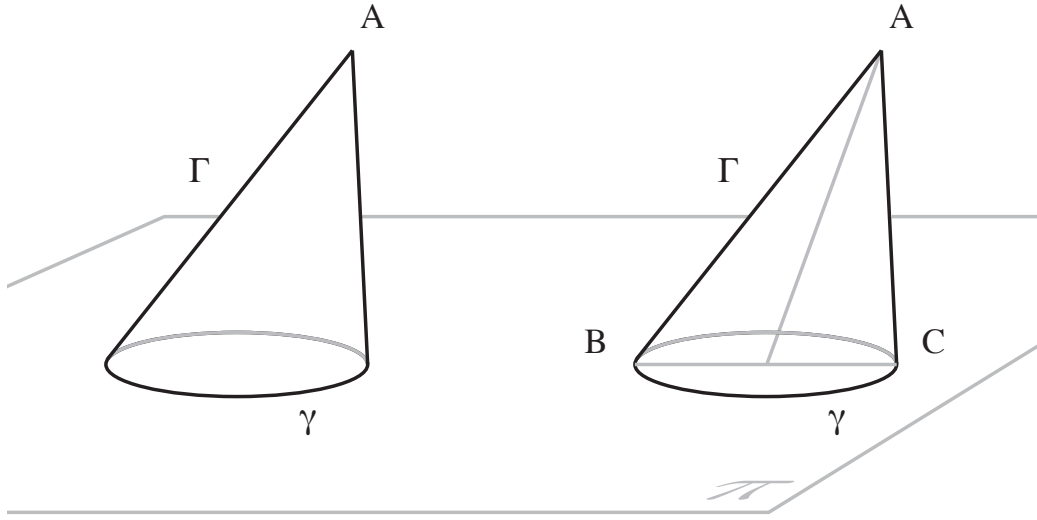


**Figure 10:** Oblique cones with a circular section.

As pointed out before, all planes parallel to $\pi$ will intersect the given oblique cone $\Gamma$ along circles, since the parallel sections are similar to the original one. However, there exists a second family of parallel planes that intersect the oblique cone $\Gamma$ along circles, and we will now proceed to find it. Consider the unique plane $\sigma$, orthogonal to $\pi$ and containing the axis of the cone. This plane, called the axial plane, will contain a diameter BC of the circle $\gamma$ (Figure 10b). Inside the triangle ABC we construct a "subcontrary" triangle AGK, similar to ABC but with opposite orientation (Figure 11a).

Consider now the unique plane $\rho$ orthogonal to the axial plane and passing through G and K. We will show that the intersection between $\rho$ and $\Gamma$ is a circle. To do this take a generic point H of the intersection of $\Gamma$ with $\rho$ and consider the orthogonal projection F of H onto the axial plane $\sigma$, and the circle $\gamma'$ parallel to $\gamma$ passing through H (Figure 11b). The circle $\gamma'$ will intersect the axial plane in the two points D and E. Since $\gamma'$ is a circle, we know that (by Proposition VIII in book VI of Euclid's *Elements*)

$$(DF)(FE) = (HF)^2.$$

**Figure 11:** Subcontrary sections.

On the other hand, since inside the similar mutually "subcontrary" triangles ADE and AGK the triangles GDF and FKE are similar (Figure 12), we have that (DF)(FE)=(GF)(FK). Therefore, we get $(GF)(FK) = (HF)^2$.



**Figure 12:** Subcontrary triangle.

This, by the converse of the already mentioned result of Euclid,[10] implies that the generic point H belongs to the circle on the plane $\rho$ with diameter GK. Thus, the intersection of $\Gamma$ and $\rho$ is a circle as well. We say that this circle is a subcontrary section of $\Gamma$. This concludes the proof of Apollonius' Proposition I.V.                                                                           □

This result, of course, is not sufficient, by itself, to establish **Theorem SP**, namely that the stereographic projections of circles on the sphere are still circles on the equatorial plane (or any other plane parallel to it), even though, as we already pointed out, many commentators did attribute this theorem to Apollonius. Thomas Heath [25, Volume II, pages 292-293] objected, and conjectured that this attribution was probably due to the intuition that Proposition I.V is indeed its logical precursor; in support of his position, Heath pointed out that even Ptolemy (while using the same method) did not mention Apollonius. In agreement with this view, Fried and Unguru say that "there is no evidence this application was anywhere near Apollonius' thoughts" [21, page 77, note 33] and in fact a textual examination of all the remaining work of Apollonius supports this opinion.

So, how can we establish who is responsible for applying Proposition I.V to obtain **Theorem SP**? In ancient science historiography the second step, which leads to the full statement of the theorem, is often attributed to Hipparchus, though no actual evidence for this is available. The first proof of Theorem **SP** that we can find in the Western canon is due to Jordanus de Nemore (c.1220-c.1260) (see [30, pages 275-294], [31, ff. 26-38]). More specifically Jordanus proves from the beginning (and without explicit reference to Apollonius) that the result stated in Proposition I.V by Apollonius holds for any oblique cone used in the stereographic projection, and hence concludes that any circle on the sphere is projected onto a circle of the plane.

We now present the proof of **Theorem SP** in detail, following (and simplifying) the proof given by Jordanus de Nemore [32, pages 86–96].

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[10]This converse is not explicitly mentioned by Euclid, and Apollonius uses it without providing a specific reference (or proof). Pappus, in his edition of the work of Apollonius, which is contained in [53] justifies this step as a Lemma that he proves. Later on, Commandinus in his edition of *Apollonii Pergaei Conicorum libri quatuor* [3] will make reference to Pappus' Lemma II as the justification for this step.

**Proof.** As illustrated in Figure 13 below, we will show that the stereographic projection, from a point A of a sphere $\Sigma$ onto the plane $\pi$ tangent to the sphere at the antipodal point P, is such that any circle on the sphere (not containing A) and its stereographic projection on the plane $\pi$ are subcontrary circular sections of a same cone.
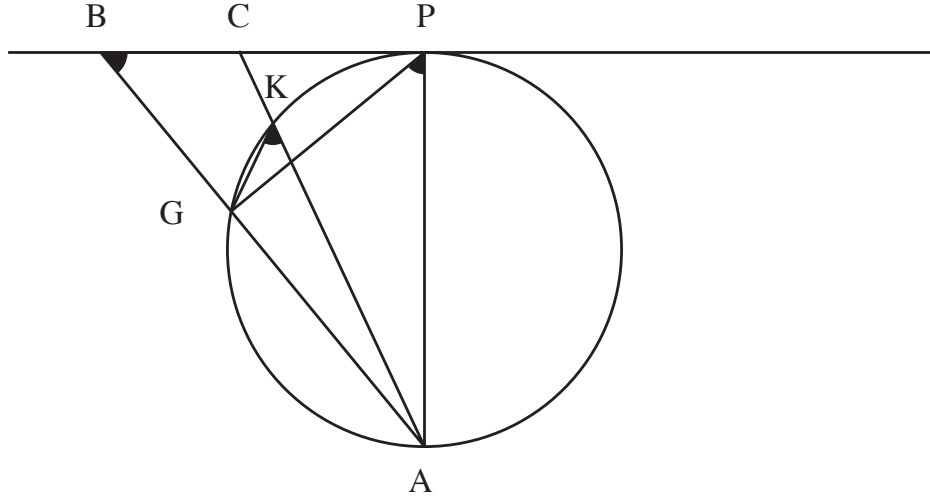


**Figure 13:** Subcontrary sections in the stereographic projection.

Figure 13 is obtained by cutting the sphere $\Sigma$ with a plane containing both the diameter AP of $\Sigma$ and a diameter GK of a circle $\gamma$ on the sphere. Now, the angle in K equals the angle in P since their vertices belong to a same disc and they insist on the same arc GA. On the other hand, being GAP and BGP both right-angled triangles, the angle in P equals the angle in B. Therefore, the angle in B and the one in K are equal, and hence the triangles BAC and GAK are similar subcontrary triangles of the same oblique cone with vertex in A passing exactly through the circle $\gamma$ of $\Sigma$. Therefore, Apollonius' Proposition I.V implies that the stereographic projection of the circle $\gamma$ of the sphere $\Sigma$ is a circle of the plane $\pi$.                                  $\square$

A similar proof appears in a lesser known demonstration of **Theorem SP**, written three centuries earlier by Al-Farghānī (820-861 CE) [2]; this proof is technically more complicated than the one of Jordanus, because it relies exclusively on Euclid's results on similar triangles.

In general, the circulation of a first translation, usually an inadequate one, of a work does not guarantee the actual transmission and the deep understanding of its content. Such an inaccurate translation existed for the *Conics* of Apollonius, since it appears that an anonymous author had translated into Arabic the first books of Apollonius. During the caliphate of A-Ma'mum (786–833), who provided significant stimulus for the translation of the original classical works of the Greeks, Thābit bin Qurra (836–901) revised this version. However, according to some authors [27, pages 35-38], the transmission of Apollonius' work took place before these translations. In fact the structure of Al-Farghānī's proof [66] of **Theorem SP** seems to indicate that Al-Farghānī himself must have been exposed to the work of Apollonius.

In the introduction to [2], Al-Farghānī is quite explicit. He claims to have written this work in order to "demonstrate the correctness of the form of the astrolabe as made by the ancients, its theoretical basis, the truth of what it shows, the obtaining of the size of all the circles that are formed on the astrolabe to replace the spheres of the Heavens" [2, Introduction, pages 23-25]. Unlike many of Al-Farghānī's astronomical works, which saw several translations during the early modern era, his work on the astrolabe was not very well known, it was never translated to Latin, and its impact was limited to the Arab literature on the astrolabe [2, Introduction, page 5].

A different proof of **Theorem SP** is attributed to Maslamah ibn Ahmad Al-Majrītī (c.950-c.1007). Just like Al-Farghānī's, this new proof is based on the same principles of Euclidean geometry, but it is less linear of the previous one, and does not use the idea of subcontrary sections. This demonstration began to circulate as a manuscript in Europe, towards the end of the tenth century. At the beginning of the twelfth century, it garnered further attention thanks to an Eastern Arab translation of the Ptolemy's *Planisphaerium* to which it appears as a note to its eighteenth chapter,[11] and ultimately it circulated in the manuscript translation of the *Planisphaerium* due to Hermann of Carinthia.[12] The work of Maslamah, and the school around him, was extremely influential in the wealth of writings that developed on the astrolabe in Spanish, French, and in English.

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[11]For a reconstruction of the manuscripts, and the diffusion of the works of Maslamah see [59, 22, 37, 40, 67, 64].

[12]On his manuscript translation, see [11].

Before we close this mathematical section, we want to point out another very significant property of stereographic projections that, as it will be immediately clear, is of great practical importance. To state (and prove) the theorem we have in mind, we will need to restrict our attention to curves, in space, with well-defined tangents (in other words, no angles or cusps are allowed). When two such curves intersect, we can use their tangents at the point of intersection to define the angle between the curves as the angle formed by their tangents at that point. The extraordinary thing about stereographic projections is the fact that the angles between two curves on the sphere is equal to the angle between the stereographic projections of the two curves on the plane. In mathematical terms, we say that the stereographic projection is a conformal map (a fancy way to say that it preserves the angles).

With this introduction, we can state our next result:

**Theorem SPC.** *The stereographic projection is conformal. In other words, the angle between any two intersecting curves on the sphere and the angle between the stereographic projections of the two intersecting curves on the plane are equal.*

**Proof.** We will follow here the original, and elegant, proof of this fairly deep result, attributed to Thomas Harriot.[13]

Let us consider any point P of the sphere, different from the south pole S (the point of projection). If two curves on the sphere intersect at P, then their tangent vectors belong to the plane $\pi$ tangent to the sphere at P. Therefore, to prove the assertion, one has to prove that the stereographic projection preserves all angles of $\pi$ with vertex at P. It is of course enough to prove the result for angles with vertex at P and with one of the sides tangent to the meridian at P: the general case will follow by applying this result separately to each of the two sides of any given angle with vertex at P.

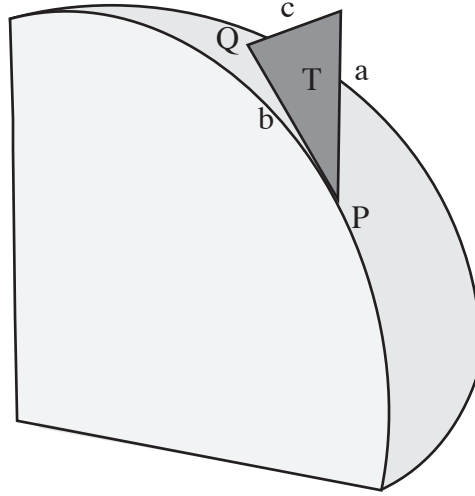Let us consider the triangle T on the plane $\pi$ tangent to the sphere at P, with (see Figure 14):

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[13]Thomas Harriot (1560–1621) was an English astronomer, mathematician, translator. The first mathematical proof of the conformality of the stereographic projection was published in 1695 by Edmond Halley (1656–1741), an English astronomer, geophysicist, mathematician, meteorologist, and physicist.

- one vertex at P,

- one of the sides exiting from P, let us say b, tangent to the meridian at the vertex P,

- the side opposite to the vertex P, let us say c, lying on the plane of projection, i.e., lying on the plane $\tau$ tangent to the sphere at the North pole N,

- the second side exiting from P, let us say a, making an angle $\alpha$ with the side b.

Since the side c of T belongs to both planes $\pi$ and $\tau$, which are orthogonal to the plane containing the entire meridian of P, then c is orthogonal to the plane containing the entire meridian of P, and hence to b. Therefore, the triangle T will be right-angled at the vertex Q of b different from P.



**Figure 14:** The meridian triangle T.

Consider now the stereographic projection T′ of the triangle T made from the south pole S to the plane $\tau$ tangent to the sphere at N (see Figure 15). The triangle T′ is clearly "attached" to T along the side c opposite to P (the side c belongs to $\tau$). Since c is orthogonal to the plane containing the entire meridian of P, then T′ as well is right-angled at the vertex Q.
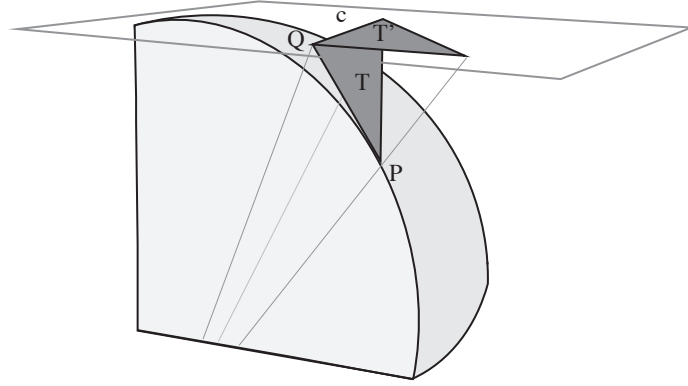
**Figure 15:** The triangle T'.

The triangle T'' of Figure 16 is isosceles with respect to the vertex Q: indeed (see Figure 17), we cut the sphere and the entire construction along the plane containing the meridian of P, and obtain that the triangle T'' and the triangle T''' of figure 17 are similar. Now, since triangle T''' is isosceles in R, so is the triangle T'' in Q.
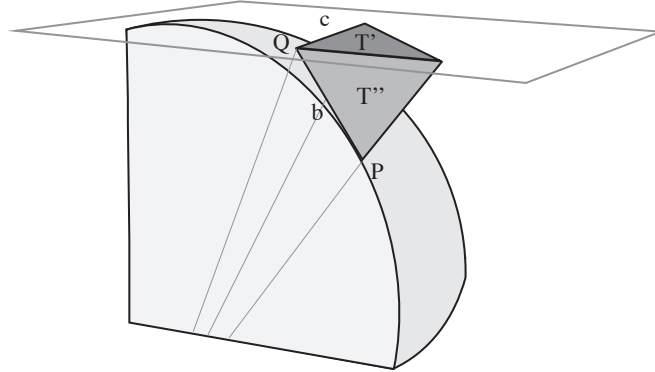


**Figure 16:** The triangle T''.

To conclude, we notice that T and T' are equal triangles: they are both right-angled at Q, they have the side b in common, and two other corresponding sides equal (the side c of T and its correspondent in T'). In particular the angle of T at the vertex P is equal to the angle of T' at the stereographic projection of the vertex P. This proves the assertion.                □
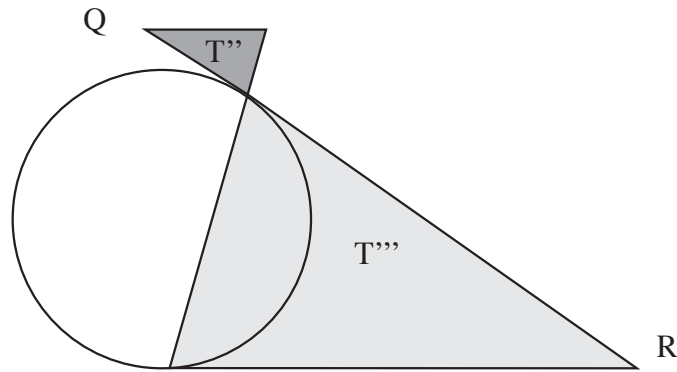
**Figure 17:** The triangle T$'''$.

It seems that the conformality of the stereographic projection was neither proved nor pointed out until the beginning of the seventeenth century. This is probably due to the fact that the stereographic projection became crucial in the cartography of navigation only when routes became oceanic, and the conformality property had to play its fundamental role.

The stereographic projection played a central role in the understanding of how to represent the celestial sphere on a flat surface. This is particularly evident in the work of Claudius Ptolemy (100-170 CE). While in his *Almagest*, Ptolemy uses what he called the armillary astrolabe (spherically shaped and totally different from the planar astrolabe) to expose his astronomical vision of the universe that unifies astronomical theories, empirical observations, and the metaphysical approach of Aristotle, his *Planisphaerium* [61, in particular ff. 1v, 7, 9v, 17v, 22] relies on the method of the stereographic projection from the South Pole, even without giving any specific proofs, nor any historical background and references. Indeed, here Ptolemy limits himself to some particular cases, such as the circular shape of the projection of the Tropics, of the Equator, and of the Ecliptic, and takes instead the general method for granted. Still in the *Planisphaerium* Ptolemy mentions a horoscopic instrument endowed of a rete; it is not known whether it was actually an astrolabe, but it is believed that most likely he was using the terms ἀστρολάβος and ἀστρολάβον ὄργανον to indicate a three-dimensional instrument, namely an armillary sphere. A similar use of these terms can be found in Pappus,

a commentator of the *Almagest*, as well as in Proclus and finally in Nikolaos Sophianus who devotes a treaty, in the sixteenth century, to the description of the armillary sphere, despite the title of his work that points out to the construction and use of the astrolabe; see [54].

Not even in the *Cosmographia* ([62], also see [1, in particular chapters XXI-XXIV, pages 60-64]) Ptolemy gives any proofs or mathematical justifications. Once again, though, he uses the stereographic projection as a tool for the cartographic construction. We should point out that in the *Cosmographia* one can find, in peaceful coexistence, both the viewpoint of the spherical cartography (with technical suggestions on how to construct maps on the sphere) and the viewpoint of the planar cartography.

## 5.  The Astrolabe in Medieval Islam

We have seen in Section 4 the importance of Arab mathematicians in the transmission, and development, of some of the Greek ideas that are foundational for the construction of the astrolabe. This said, we should begin this section by pointing out that there have been many important contributions to the history of the astrolabe in Islam, and that we are only offering here a pathway for further study of this important area of scholarship.[14] The Arab translation of the *Almagest*, and more generally the influence of the Greek-Hellenistic knowledge, reached the Arab-Islamic world through Alexandria and Byzantium.[15] Partly because of the influence of the scientific knowledge from India and Persia, the Arab astronomers already had an ample knowledge of the sky, even in pre-Islamic times. In particular they knew a large number of stars, which they used to orient themselves in the desert, and they possessed a deep knowledge of the calendar to direct their agricultural lives and their religious ceremonies. The names we still use for many of the stars (Aldebaran in the Pleiades, Betelgeuse in Orion, Deneb in the Swan, Fomalhaut in the Pisces, just to quote a few) remind us of this deep and ancient knowledge.

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[14]The interested reader should definitely consult [8, 63, 65]. See also the expositions in Bibliothèque nationale, Paris, available at http://expositions.bnf.fr/monde-en-spheres/la-sphere-en-terres-dislam/index.html and in British Library, London, available at https://blog.britishmuseum.org/seeing-stars-astrolabes-and-the-islamic-world/, both last accessed on January 27, 2020.

[15]See [56], [24], [18, in particular pages 55-67], [29].

For a while the astrolabe was a wooden object, built for religious purposes by the astronomer of the mosque, in order to use the sun and the stars to identify the qibla, or the prayer direction of the Mecca, and the time for prayer. The five daily prayers that Muslims observe, are all depending on the location of the sun, but the importance of facing Mecca made it necessary to be able to determine such a direction from any location, day or night. Concurrently, the Islamic calendar (hijri calendar, as it is known in recognition of its beginning with the migration from Mecca to Medina of the Prophet Muhammad in the years 622) is a lunar calendar, and the necessity to develop accurate mathematical methods to calculate such calendars were an important stimulus for the development of Islamic mathematics and astronomy. This is a significant observation, because it indicates a different reason for the construction of the instrument, in addition to the customary geographical explorations. The Quran often mentions natural and especially astronomical phenomena as the creation of God, and challenges the faithful to pursue the discovery of the universe as a way to come close to God.[16] This is a perspective that is definitely lacking in the rest of the astrolabe tradition, and offers a different way of thinking about its role in society.

Later on, astrolabes were sometimes built with precious metals and stones, when designed for royal and noble families. These developments were concurrent with the writing of several manuals (first manuscripts, and later on in print): these manuals were mostly devoted to practical explanations, rather than detailed mathematical discussions on the geometry that makes the astrolabe possible,[17] like for example the treatise on the plane astrolabe by the Jewish Persian astronomer Mâshâllâh (c.740-815).[18] Moreover, it was in the period of the strongest development of Arab-Islamic science, between the end of the ninth and the beginning of the tenth century, that one could witness the refinement of the construction of the astrolabe, and its adjustment to the

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[16]One example among many: "[God] is the One who has set out for you the stars, that you may guide yourselves by them through the darkness of the land and of the sea. We have detailed the signs for people who know", Quran 6:97.

[17]An exception to this praxis was Andal di Negro, Pacini editore, Pisa, 2005, who merged these two aspects and, in the period between the end of 1200 and the beginning of 1300 composed the *Trattato sull'astrolabio*, first published in Ferrara in 1475.

[18]On Mâshâllâh see [74] and [35, in particular chapter X]. On the proliferation of the Arabic treatises on how to operate the astrolabe, see the interesting analysis of Taro Mimura in [48].

various latitudes for the Mediterranean Sea and the Middle East.[19]

Among scholars of Islam, it is usually accepted that the Persian mathematician Muhammad al Fazari (who lived in the eighth century and died around the year 800) was the first to build an astrolabe in the Islamic world, while there are many sources indicating that by the middle of the ninth century there was a very active production of astrolabes; for example the name of Mariam Al-Astrolabiyy Al-Ijliya, who lived around 950 CE in what would be now Northern Syria, and that of her father are also indicated as names of a family of astrolabe makers and it is clear that during this period the design of the astrolabe was undergoing important improvements. Among the many important design changes that Islam brought to the construction of this instrument is the introduction of angular scales as well as additional circles on the horizon, as documented in the work of J.L. Berggren [9].

The Islamic culture was responsible, as well, for the introduction of the spherical astrolabe, whose early description dates back to the end of the nineteenth century. Not much is known about this instrument, except that it is somewhat similar to the armillary sphere, but rather than being composed of a framework of rings representing celestial longitude, latitude, and the ecliptic, it actually contains a rete, like the actual astrolabe, which is not a plane, but rather a spherical cage that rotates around a globe that it contains. The rete in a spherical astrolabe plays the same role as the rete of the plane astrolabe, as it includes several stars, as illustrated in Figure 18 below that describes one such astrolabe currently in the collection of the History of Science Museum at the University of Oxford.

We conclude this section by closing the circle, so to speak, and by noticing that the many popular Arab writings on the constructions on the astrolabe (some of which mistakenly attributed to the Persian Jewish astrologist and astronomer Mâshâllâh; see for instance [36, 45]) were extremely influential in Western literature, Spanish, French, and English, including the surprising work of Geoffrey Chaucer (c. 1340-1400), who wrote "A Treatise on the Astrolabe", the oldest known technical manual in English.

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[19]An abbreviated but helpful historical reconstruction of the first descriptions of this instrument is given in [50, pages 278ff]. As far as the descriptions and constructions within the Arab Islamic tradition, see [12]. A thorough discussion of the manuscripts in Greek, Arab, and Latin is in [35], see in particular chapter VIII.

**Figure 18:** Spherical astrolabe, maker Musa, Eastern Islamic, 1480/81 — ©
History of Science Museum, University of Oxford, inv. 49687.

Chaucer says in the Preface that he wrote the treatise for his ten-year-old
son Lewis, "... I aperceyve wel by certeyne evydences thyn abilite to lerne
sciences touching nombres and proporciouns".[20] The Treatise became in fact
a way to teach the young Lewis the foundations of the astronomical culture
of the time, by means of the description of a real object, the astrolabe, which
this culture subsumed.

## 6. The Construction of the Astrolabe

We are now ready to use the stereographic projection to build the astrolabe.
This construction of course appears elsewhere (see for example [52]); however,
we give here a (hopefully) particularly understandable presentation, which
takes advantage of the study of the properties of the stereographic projection

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[20]In modern English: "... since I see some evidence that you have the ability to learn
science, numbers and proportions", from [14, page 641]. On Mâshâllâh and Chaucer see
[23, 13, 7].

undertaken in the previous sections. We begin by projecting the points of the Celestial Sphere from the South Pole onto the plane of the Equator (Figure 19a).
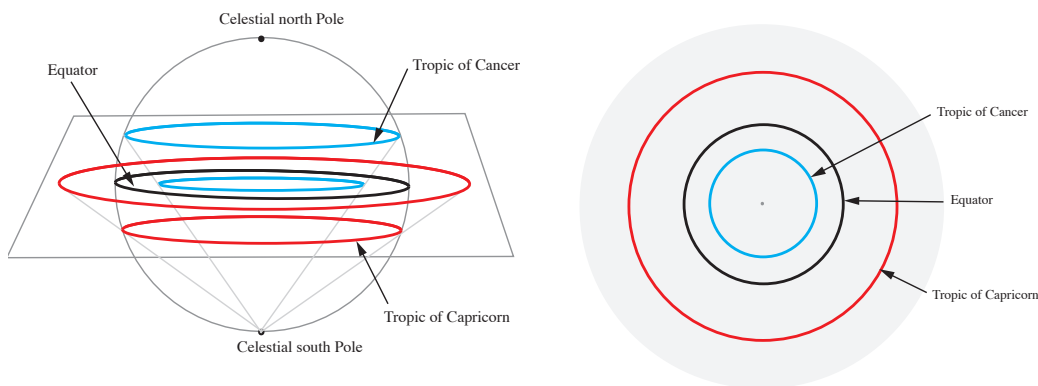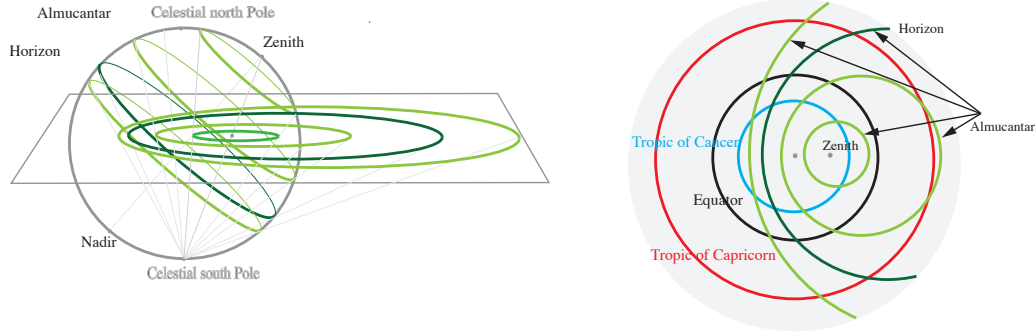


**Figure 19:** Stereographic projection of Equator and Tropics on the plane of the Equator.

The projection of the entire sphere, deprived of the South Pole, covers the entire plane of the Equator. For this reason, and for the reason that for many years the world was mainly boreal, the stereographic projection of the Celestial Sphere was done only up to a few degrees south of the Tropic of Capricorn. We call tympan the disc of the equatorial plane that contains the image of the projection of this region of the sphere. In Figure 19b, we represent the stereographic projections of the two Tropics and the Equator: they appear as three concentric circles; the largest circle, the nearest to the South Pole, is the projection of the Tropic of Capricorn.

We now project the Vertical and the Almucantars of the observer on the tympan, and we get what is roughly described in Figure 20a, where the special Almucantar called Horizon is the darkest one.

The tympan, containing the projections of the Tropics, Equator and the coordinates circles connected to the observer, i.e. the Zenith (possibly the Nadir depending on the position of the observer), several Almucantars and the Horizon looks like what we see in Figure 20b (the Horizon is the tick Almucantar).

**Figure 20:** Stereographic projection of Zenith, Horizon and several Almucantars connected to an observer.

In Figure 21 we find a tympan, in a different position, that also contains several vertical circles related to the observer. The tympan is the fixed part of the astrolabe, the part which represents the celestial coordinates; specifically, it contains some global coordinates such as the Equator, and the Tropics, as well as local coordinates, which depend on the latitude of the observer.



**Figure 21:** The tympan. The so-called "Chaucer Astrolabe", c. 1400 – © Museo Galileo, Firenze, inv. 3931.

The central point (appearing as a white hole in Figure 21) is the (projection of the) Celestial North Pole. The vertical diameter passing through the North Pole and the Zenith of the observer corresponds to (a portion of) the meridian on which the observer is located, and its upper extreme indicates the South of the observer. Note that as the Celestial Sphere moves, the projections of the Equator and the Tropics do not change. On the other hand, the stars, and the constellations on the Celestial Sphere move as time passes, and therefore need to be represented on a different disk in the astrolabe.

We now project the fixed stars, and the Ecliptic of the sun, on a new disc having the same dimensions of the tympan, called the rete (rete being the Italian word for web). In the rete, the sharp points carrying the names of the stars will represent the fixed stars, and the Ecliptic, being a circle of the Celestial Sphere (not containing the South Pole) will be a circle on the rete, because of **Theorem SP**. The rete usually contains (the projections of) thirty or forty stars, usually the most visible, and always the Ecliptic, on which, as we already pointed out, one can note the daily position of the Sun. The parts of the rete that do not contain any projections are hollowed out, so not to block the view of the tympan (see Figure 22).



**Figure 22:** The rete. The so-called "Chaucer Astrolabe", c. 1400 – © Museo Galileo, Firenze, inv. 3931.

The rete is a second piece of the astrolabe, the movable part, which rotates as the hours pass. In its center, like in the tympan, you see a white hole (see Figure 22), which corresponds to the Celestial North Pole, and around which the rete can rotate.

We now put these pieces together to construct our astrolabe. We need a base, called mater, to contain the two other pieces, the tympan and the rete. Figure 23 shows a representation of the mater. On the round boundary of the mater, the twenty-four hours of the day are engraved (A stands for I, B for II, and so on), starting from noon (South, at the top) to midnight (North, bottom). The explanation for the presence of the hours is because the diameters of the mater are (portions of) the stereographic projections of the Celestial meridians, and hence they determine the hour in the coordinates of the observer, starting from the meridian of the observer, which determines its noon.



**Figure 23:** The mater. The so-called "Chaucer Astrolabe", c. 1400 – © Museo Galileo, Firenze, inv. 3931.

One now inserts the tympan into the mater, in such a way that it cannot rotate (see the tooth on the top part of the tympan, and the hollow in the corresponding top part of the mater). Then insert the rete in the mater, over the tympan, in such a way that it can rotate freely around the central point representing the North Pole. Voila! You have yourself an astrolabe! (See Figure 24.)

**Figure 24:** The so-called 'Chaucer Astrolabe", c. 1400 – © Museo Galileo, Firenze, inv. 3931.

The rule, appearing in Figure 1 over the rete, (the rule is attached to the center of the mater, and can rotate freely around it) helps to point with precision, for example, the hour, when the position of the Sun or of some star is given.

The back of the mater of every astrolabe included an alidade, used for measuring the altitude of celestial objects while the astrolabe was suspended vertically (as in Figure 24). However, the measure of the altitude of celestial objects obtained in this fashion was too imprecise: for this reason, in this use, the sextant eventually replaced the astrolabe. As explained, when the observer changes latitude, to continue to use the astrolabe properly it is necessary to change the tympan with a new one, reflecting the new latitude, and containing the projections of the new coordinates.

Readers who have followed with interest this description may be tempted to build their own astrolabes, and even start using it to see how the ancient navigators must have felt. A great reference for the construction of the astrolabe is http://www.astrolabeproject.com; we refer the reader who wants to try the construction to that website. Alternatively, we have acquired a working (and cheap) astrolabe kit through the web via https://shop.sciencefirst.com. It is an inexpensive kit that requires some very easy assembly, and it also includes instructions.

It is not difficult now to start using the astrolabe. For example, if at night the observer locates in the sky the stars that are represented in the rete, and moves the rete until the stars are all placed in the right places of the web of coordinates of the observer, he can recover the exact hour of the night. On the other hand, if the observer measures the altitude A of the Sun at Midday, then moving the rete of the astrolabe until the Ecliptic touches the meridian of the observer at altitude A, he can determine the day of the year (actually there are always two possible dates with the exception of the solstices). Finally, if the observer knows the exact date, then estimating the real altitude S of the Sun, and moving the rete of the astrolabe until the position of the Sun on the Ecliptic touches the right altitude S, he can recover the hour of the day.

It is worth pointing out that the astrolabe is the product of a geocentric viewpoint, so one might wonder why a geocentric conception still manages to give an instrument that works correctly for the real world, which is heliocentric. The answer lies in the fact that the builders of the astrolabe, while driven by a geocentric viewpoint, in reality simply designed an instrument that describes the universe as seen when placing the reference system to the center of the earth. In other words, even though the earth is not at the center of the universe, it is in fact appropriate to choose the center of the earth as the origin of the reference system to interpret what the observer actually sees, and to be able to navigate.

Finally, we should point out that a simplified version of the astrolabe, known as mariner's (or nautical) astrolabe and used together with astronomical tables, became a fundamental instrument for navigation, until its replacement by the sextant and the mariner's chronometers in the eighteenth century.

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