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# The Area of a Bicentric Quadrilateral

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**Abstract**. We review and prove a total of ten different formulas for the area of a bicentric quadrilateral. Our main result is that this area is given by

$$K = \left| \frac{m^2 - n^2}{k^2 - l^2} \right| kl$$

where $m, n$ are the bimedians and $k, l$ the tangency chords.

## 1. The formula $K = \sqrt{abcd}$

A bicentric quadrilateral is a convex quadrilateral with both an incircle and a circumcircle, so it is both tangential and cyclic. It is well known that the square root of the product of the sides gives the area of a bicentric quadrilateral. In [12, pp.127–128] we reviewed four derivations of that formula and gave a fifth proof. Here we shall give a sixth proof, which is probably as simple as it can get if we use trigonometry and the two fundamental properties of a bicentric quadrilateral.

**Theorem 1.** *A bicentric quadrilateral with sides $a, b, c, d$ has the area*

$$K = \sqrt{abcd}.$$

*Proof.* The diagonal $AC$ divide a convex quadrilateral $ABCD$ into two triangles $ABC$ and $ADC$. Using the law of cosines in these, we have

$$a^2 + b^2 - 2ab\cos B = c^2 + d^2 - 2cd\cos D. \tag{1}$$

The quadrilateral has an incircle. By the Pitot theorem $a + c = b + d$ [4, pp.65–67] we get $(a - b)^2 = (d - c)^2$, so

$$a^2 - 2ab + b^2 = d^2 - 2cd + c^2. \tag{2}$$

Subtracting (2) from (1) and dividing by 2 yields

$$ab(1 - \cos B) = cd(1 - \cos D). \tag{3}$$

In a cyclic quadrilateral opposite angles are supplementary, so that $\cos D = -\cos B$. We rewrite (3) as

$$(ab + cd)\cos B = ab - cd. \tag{4}$$

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The area $K$ of a convex quadrilateral satisfies $2K = ab\sin B + cd\sin D$. Since $\sin D = \sin B$, this yields

$$2K = (ab + cd)\sin B. \tag{5}$$

Now using (4), (5) and the identity $\sin^2 B + \cos^2 B = 1$, we have for the area $K$ of a bicentric quadrilateral

$$(2K)^2 = (ab + cd)^2(1 - \cos^2 B) = (ab + cd)^2 - (ab - cd)^2 = 4abcd.$$

Hence $K = \sqrt{abcd}$. □

**Corollary 2.** *A bicentric quadrilateral with sides $a$, $b$, $c$, $d$ has the area*

$$K = ac\tan\frac{\theta}{2} = bd\cot\frac{\theta}{2}$$

*where $\theta$ is the angle between the diagonals.*

*Proof.* The angle $\theta$ between the diagonals in a bicentric quadrilateral is given by

$$\tan^2\frac{\theta}{2} = \frac{bd}{ac}$$

according to [8, p.30]. Hence we get

$$K^2 = (ac)(bd) = (ac)^2\tan^2\frac{\theta}{2}$$

and similar for the second formula. □

**Corollary 3.** *In a bicentric quadrilateral $ABCD$ with sides $a$, $b$, $c$, $d$ we have*

$$\tan\frac{A}{2} = \sqrt{\frac{bc}{ad}} = \cot\frac{C}{2},$$
$$\tan\frac{B}{2} = \sqrt{\frac{cd}{ab}} = \cot\frac{D}{2}.$$

*Proof.* A well known trigonometric formula and (3) yields

$$\tan\frac{B}{2} = \sqrt{\frac{1 - \cos B}{1 + \cos B}} = \sqrt{\frac{cd}{ab}} \tag{6}$$

where we also used $\cos D = -\cos B$ in (3). The formula for $D$ follows from $B = \pi - D$. By symmetry in a bicentric quadrilateral, we get the formula for $A$ by the change $b \leftrightarrow d$ in (6). Then we use $A = \pi - C$ to complete the proof. □

Therefore, not only the area but also the angles have simple expressions in terms of the sides.

The area of a bicentric quadrilateral also gives a condition when a tangential quadrilateral is cyclic. Even though we did not express it in those terms, we have already proved the following characterization in the proof of Theorem 9 in [12]. Here we give another short proof.

**Theorem 4.** *A tangential quadrilateral with sides $a, b, c, d$ is also cyclic if and only if it has the area $K = \sqrt{abcd}$.*

*Proof.* The area of a tangential quadrilateral is according to [8, p.28] given by

$$K = \sqrt{abcd}\sin\frac{B+D}{2}.$$

It's also cyclic if and only if $B + D = \pi$; hence a tangential quadrilateral is cyclic if and only if it's area is $K = \sqrt{abcd}$. □

This is not a new characterization of bicentric quadrilaterals. One quite long trigonometric proof of it was given by Joseph Shin in [15] and more or less the same proof of the converse can be found in the solutions to Problem B-6 in the 1970 William Lowell Putnam Mathematical Competition [1, p.69].

In this characterization the formulation of the theorem is important. The tangential and cyclic quadrilaterals cannot change roles in the formulation, nor can the formulation be that it's a bicentric quadrilateral if and only if the area is given by the formula in the theorem. This can be seen with an example. A rectangel is cyclic but not tangential. Its area satisfy the formula $K = \sqrt{abcd}$ since opposite sides are equal. Thus it's important that it must be a tangential quadrilateral that is also cyclic if and only if the area is $K = \sqrt{abcd}$, otherwise the conclution would be that a rectangle also has an incircle, which is obviously false.

## 2. Other formulas for the area of a bicentric quadrilateral

In this section we will prove three more formulas for the area of a bicentric quadrilateral, where the area is given in terms of other quantities than the sides. Let us first review a few other formulas and one double inequality for the area that can be found elsewhere.

In [12], Theorem 10, we proved that a bicentric quadrilateral has the area

$$K = \sqrt[4]{efgh}(e+f+g+h)$$

where $e, f, g, h$ are the tangent lengths, that is, the distances from the vertices to the point where the incircle is tangent to the sides.

According to Juan Carlos Salazar [14], a bicentric quadrilateral has the area

$$K = 2\,MN\sqrt{EQ\cdot FQ}$$

where $M, N$ are the midpoints of the diagonals; $E, F$ are the intersection points of the extensions of opposite sides, and $Q$ is the foot of the normal to $EF$ through the incenter $I$ (see Figure 1). This is a remarkable formula since the area is given in terms of only three distances. A short proof is given by "pestich" at [14]. He first proved that a bicentric quadrilateral has the area

$$K = 2\,MN\cdot IQ$$

which is even more extraordinary, since here the area is given in terms of only two distances!

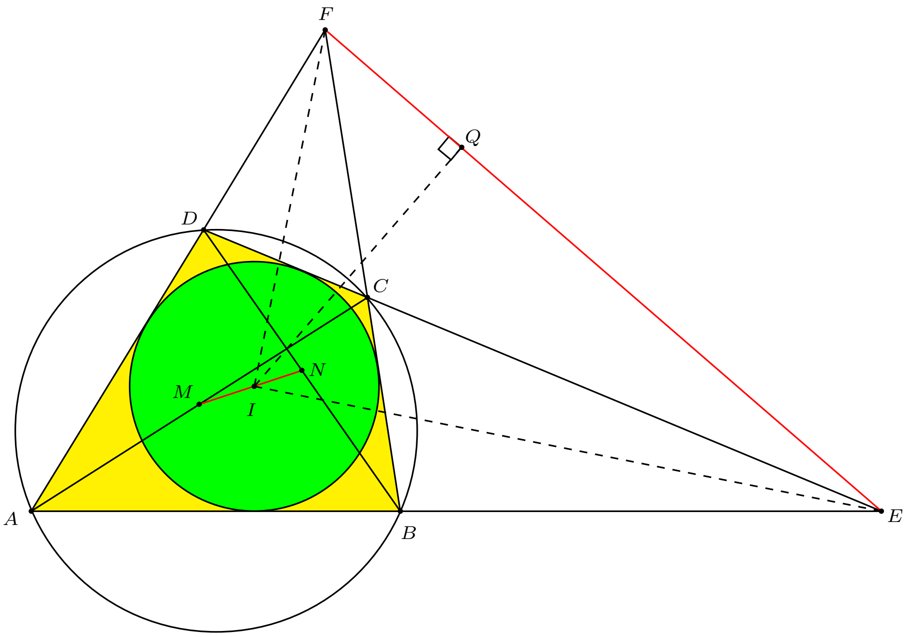


Figure 1. The configuration of Salazar's formula

The angle $EIF$ (see Figure 1) is a right angle in a bicentric quadrilateral,[1] so we also get that the area of a bicentric quadrilateral is given by

$$K = \frac{2\,MN \cdot EI \cdot FI}{EF}$$

where we used the well known property that the product of the legs is equal to the product of the hypotenuse and the altitude in a right triangle. The last three formulas are not valid in a square since there we have $MN = 0$.

In [2, p.64] Alsina and Nelsen proved that the area of a bicentric quadrilateral satisfy the inequalities

$$4r^2 \leq K \leq 2R^2$$

where $r, R$ are the radii in the incircle and circumcircle respectively. We have equality on either side if and only if it is a square.

Problem 1 on Quiz 2 at the China Team Selection Test 2003 [5] was to prove that in a tangential quadrilateral $ABCD$ with incenter $I$,

$$AI \cdot CI + BI \cdot DI = \sqrt{AB \cdot BC \cdot CD \cdot DA}.$$

The right hand side gives the area of a bicentric quadrilateral, so from this we get another formula for this area. It is easier to prove the following theorem than solving the problem from China,[2] since in a bicentric quadrilateral we can also use that opposite angles are supplementary angles.

[1]This is proved in Theorem 5 in [13], where the notations are different from here.

[2]One solution is given by Darij Grinberg in [9, pp.16–19].

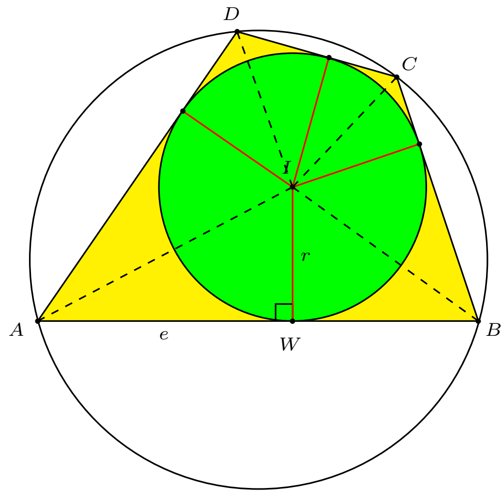


Figure 2. Partition of a bicentric quadrilateral into kites

**Theorem 5.** *A bicentric quadrilateral $ABCD$ with incenter $I$ has the area*

$$K = AI \cdot CI + BI \cdot DI.$$

*Proof.* The quadrilateral has an incircle, so $\tan \frac{A}{2} = \frac{r}{e}$ where $r$ is the inradius (see Figure 2). It also has a circumcircle, so $A+C = B+D = \pi$. Thus $\cot \frac{C}{2} = \tan \frac{A}{2}$ and $\sin \frac{C}{2} = \cos \frac{A}{2}$. A bicentric quadrilateral can be partitioned into four right kites by four inradii, see Figure 2.

Triangle $AIW$ has the area $\frac{er}{2} = \frac{r^2}{2\tan \frac{A}{2}}$. Thus the bicentric quadrilateral has the area

$$K = r^2\left(\frac{1}{\tan \frac{A}{2}} + \frac{1}{\tan \frac{B}{2}} + \frac{1}{\tan \frac{C}{2}} + \frac{1}{\tan \frac{D}{2}}\right).$$

Hence we get

$$\begin{aligned}
K &= r^2\left(\frac{1}{\tan \frac{C}{2}} + \frac{1}{\tan \frac{A}{2}} + \frac{1}{\tan \frac{D}{2}} + \frac{1}{\tan \frac{B}{2}}\right) \\
&= r^2\left(\left(\tan \frac{A}{2} + \cot \frac{A}{2}\right) + \left(\tan \frac{B}{2} + \cot \frac{B}{2}\right)\right) \\
&= r^2\left(\frac{1}{\sin \frac{A}{2} \cos \frac{A}{2}} + \frac{1}{\sin \frac{B}{2} \cos \frac{B}{2}}\right) \\
&= \frac{r^2}{\sin \frac{A}{2} \sin \frac{C}{2}} + \frac{r^2}{\sin \frac{B}{2} \sin \frac{D}{2}} \\
&= AI \cdot CI + BI \cdot DI
\end{aligned}$$

where we used that $\sin \frac{A}{2} = \frac{r}{AI}$ and similar for the other angles. $\square$

**Corollary 6.** *A bicentric quadrilateral $ABCD$ has the area*

$$K = 2r^2\left(\frac{1}{\sin A} + \frac{1}{\sin B}\right)$$

*where $r$ is the inradius.*

*Proof.* Using one of the equalities in the proof of Theorem 5, we get

$$K = r^2\left(\frac{1}{\sin\frac{A}{2}\cos\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}\cos\frac{B}{2}}\right) = r^2\left(\frac{1}{\frac{1}{2}\sin A} + \frac{1}{\frac{1}{2}\sin B}\right)$$

and the result follows. □

Here is an alternative, direct proof of Corollary 6:

In a tangential quadrilateral with sides $a, b, c, d$ and semiperimeter $s$ we have $K = rs = r(a + c) = r(b + d)$. Hence

$$\begin{aligned} K^2 &= r^2(a + c)(b + d) \\ &= r^2(ad + bc + ab + cd) \\ &= r^2\left(\frac{2K}{\sin A} + \frac{2K}{\sin B}\right) \end{aligned}$$

since in a cyclic quadrilateral $ABCD$, the area satisfies $2K = (ad + bc)\sin A = (ab + cd)\sin B$. Now factor the right hand side and then divide both sides by $K$. This completes the proof.

From Corollary 6 we get another proof of the inequality $4r^2 \leq K$, different form the one given in [2, p.64]. We have

$$K = 2r^2\left(\frac{1}{\sin A} + \frac{1}{\sin B}\right) \geq 2r^2(1 + 1) = 4r^2$$

for $0 < A < \pi$ and $0 < B < \pi$.

In [12], Theorem 11, we proved that a bicentric quadrilateral with diagonals $p, q$ and tangency chords [3] $k, l$ has the area

$$K = \frac{klpq}{k^2 + l^2}. \tag{7}$$

We shall use this to derive another beautiful formula for the area of a bicentric quadrilateral. In the proof we will also need the following formula for the area of a convex quadrilateral, which we have not found any reference to.

**Theorem 7.** *A convex quadrilateral with diagonals $p, q$ and bimedians $m, n$ has the area*

$$K = \tfrac{1}{2}\sqrt{p^2q^2 - (m^2 - n^2)^2}.$$

[3] A tangency chord is a line segment connecting the points on two opposite sides where the incircle is tangent to those sides in a tangential quadrilateral.

*Proof.* A convex quadrilateral with sides $a, b, c, d$ and diagonals $p, q$ has the area

$$K = \tfrac{1}{4}\sqrt{4p^2q^2 - (a^2 - b^2 + c^2 - d^2)^2} \tag{8}$$

according to [6, p.243], [11] and [16]. The length of the bimedians [4] $m, n$ in a convex quadrilateral are given by

$$m^2 = \tfrac{1}{4}(p^2 + q^2 - a^2 + b^2 - c^2 + d^2), \tag{9}$$

$$n^2 = \tfrac{1}{4}(p^2 + q^2 + a^2 - b^2 + c^2 - d^2). \tag{10}$$

according to [6, p.231] and post no 2 at [10] (both with other notations). From (9) and (10) we get

$$4(m^2 - n^2) = -2(a^2 - b^2 + c^2 - d^2)$$

so

$$(a^2 - b^2 + c^2 - d^2)^2 = 4(m^2 - n^2)^2.$$

Using this in (8), the formula follows. □

The next theorem is our main result and gives the area of a bicentric quadrilateral in terms of the bimedians and tangency chords (see Figure 3).

**Theorem 8.** *A bicentric quadrilateral with bimedians $m, n$ and tangency chords $k, l$ has the area*

$$K = \left|\frac{m^2 - n^2}{k^2 - l^2}\right| kl$$

*if it is not a kite.*

*Proof.* From Theorem 7 we get that in a convex quadrilateral

$$(m^2 - n^2)^2 = (pq)^2 - 4K^2. \tag{11}$$

Rewriting (7), we have in a bicentric quadrilateral

$$pq = \frac{k^2 + l^2}{kl}K.$$

Inserting this into (11) yields

$$\begin{aligned}(m^2 - n^2)^2 &= \frac{(k^2 + l^2)^2}{k^2l^2}K^2 - 4K^2\\ &= K^2\left(\frac{(k^2 + l^2)^2 - 4k^2l^2}{k^2l^2}\right)\\ &= K^2\left(\frac{(k^2 - l^2)^2}{k^2l^2}\right).\end{aligned}$$

Hence

$$\left|m^2 - n^2\right| = K\frac{\left|k^2 - l^2\right|}{kl}$$

and the formula follows.

[4] A bimedian is a line segment connecting the midpoints of two opposite sides in a quadrilateral.

It is not valid in two cases, when $m = n$ or $k = l$. In the first case we have according to (9) and (10) that

$$-a^2 + b^2 - c^2 + d^2 = a^2 - b^2 + c^2 - d^2 \quad \Leftrightarrow \quad a^2 + c^2 = b^2 + d^2$$

which is a well known condition for when a convex quadrilateral has perpendicular diagonals. The second case is equivalent to that the quadrilateral is a kite according to Corollary 3 in [12]. Since the only tangential quadrilateral with perpendicular diagonals is the kite (see the proof of Corollary 3 in [12]), this is the only quadrilateral where the formula is not valid. [5] □

In view of the expressions in the quotient in the last theorem, we conclude with the following theorem concerning the signs of those expressions in a tangential quadrilateral. Let $m = EG$ and $n = FH$ be the bimedians, and $k = WY$ and $l = XZ$ be the tangency chords in a tangential quadrilateral, see Figure 3.

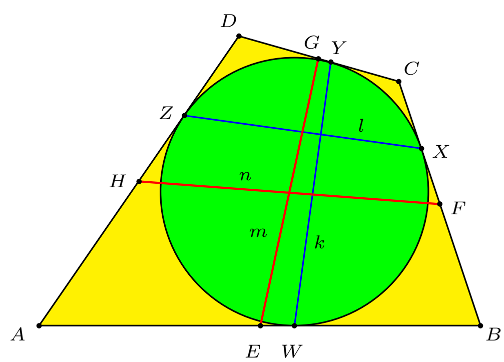


Figure 3. The bimedians $m, n$ and the tangency chords $k, l$

**Theorem 9.** *Let a tangential quadrilateral have bimedians $m, n$ and tangency chords $k, l$. Then*

$$m < n \quad \Leftrightarrow \quad k > l$$

*where $m$ and $k$ connect the same pair of opposite sides.*

*Proof.* Eulers extension of the parallelogram law to a convex quadrilateral with sides $a, b, c, d$ states that

$$a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4v^2$$

where $v$ is the distance between the midpoints of the diagonals $p, q$ (this is proved in [7, p.107] and [3, p.126]). Using this in (9) and (10) we get that the length of the bimedians in a convex quadrilateral can also be expressed as

$$m = \tfrac{1}{2}\sqrt{2(b^2 + d^2) - 4v^2},$$
$$n = \tfrac{1}{2}\sqrt{2(a^2 + c^2) - 4v^2}.$$

[5] This also means it is not valid in a square since a square is a special case of a kite.

Thus in a tangential quadrilateral we have

$$\begin{aligned}
& m < n \\
\Leftrightarrow \quad & b^2 + d^2 < a^2 + c^2 \\
\Leftrightarrow \quad & (f+g)^2 + (h+e)^2 < (e+f)^2 + (g+h)^2 \\
\Leftrightarrow \quad & fg + he < ef + gh \\
\Leftrightarrow \quad & (e-g)(h-f) < 0
\end{aligned}$$

where $e = AW, f = BX, g = CY$ and $h = DZ$ are the tangent lengths.

In [12], Theorem 1, we proved that the lengths of the tangency chords in a tangential quadrilateral are

$$k = \frac{2(efg + fgh + ghe + hef)}{\sqrt{(e+f)(f+h)(h+g)(g+e)}},$$
$$l = \frac{2(efg + fgh + ghe + hef)}{\sqrt{(e+h)(h+f)(f+g)(g+e)}}.$$

Thus

$$\begin{aligned}
& k > l \\
\Leftrightarrow \quad & (e+f)(f+h)(h+g)(g+e) < (e+f)(f+h)(h+g)(g+e) \\
\Leftrightarrow \quad & eh + fg < ef + gh \\
\Leftrightarrow \quad & (e-g)(h-f) < 0.
\end{aligned}$$

Hence in a tangential quadrilateral

$$m < n \quad \Leftrightarrow \quad (e-g)(h-f) < 0 \quad \Leftrightarrow \quad k > l$$

which proves the theorem. □

We also note that the bimedians are congruent if and only if the tangency chords are congruent. Such equivalences will be investigated further in a future paper.

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